Bounds on the Energy Dissipation on the Magnetic Couette and Poiseuille (Hartmann) Shear Flow

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1 Introduction

We are going to study the stability and bounds on turbulent dissipation shear flows in a conducting fluid when a vertical (with respect to the flow) magnetic field is applied. More precisely we are going to investigate Couette flow and Poiseuile (Hartmann) flow in the presence of the magnetic field. First using integral inequalities we are going to estimate regions in the parameter space when the flow is energy stable. Then we are going to derive bounds on the dissipation valid even in the presence of turbulent flows.

2 Couette Flow

2.1 Preliminaries

First we consider we plane Couette flow. We consider two plates separated by a distance d (from $-d/2$ to $+d/2$) that move with respect to each other with velocity $\mathbf{i}U^*$. The unit vector $\mathbf{i}$ is one of the horizontal directions and $\mathbf{j}$ is the vertical. Between the plates there is a conducting liquid of density $\rho \equiv 1$, magnetic diffusivity η and viscosity ν. For the top and bottom boundary we use no-slip boundary conditions for the velocity and "line-tied" for the magnetic field, e.g. $(B = \mathbf{j}B_0)$ where B_0 is an externally imposed field. We assume periodic boundary conditions for the other directions.The setup is shown in figure(1).

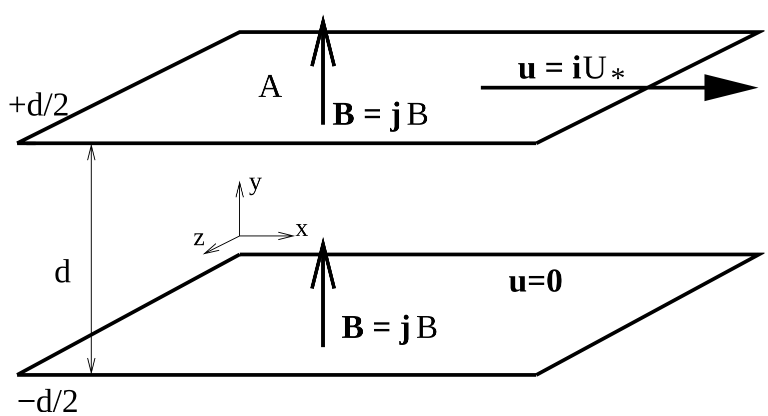

Figure 1: The setup for magnetic Couette flow

The equations of motion that govern this system are [1]

$$\begin{aligned} \partial_t u + u \cdot \nabla u &= -\nabla P + B \cdot \nabla B + \nu \nabla^2 u \\ \partial_t B + u \cdot \nabla B &= B \cdot \nabla u + \eta \nabla^2 B \\ \nabla \cdot u = 0 \quad &, \quad \nabla \cdot B = 0. \end{aligned} \tag{1}$$

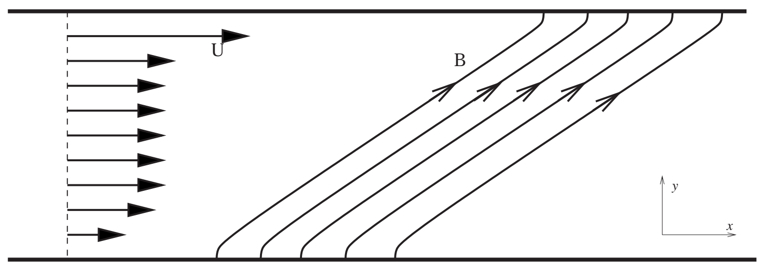

Figure 2: The laminar velocity profile and the magnetic field lines.

Where B is the magnetic field and u is the fluid velocity. There are three non-dimensional numbers that govern this system. Our choice is going to be the Reynolds number $\mathbf{Re}$, the Hartmann number $\mathbf{Q}$ and the Prandtl number $\mathbf{P}$ or alternatively the magnetic Reynolds number $\mathbf{R_M}$. Their definition is given bellow:

$$\mathbf{Re} = \frac{U^* d}{\nu}, \qquad \mathbf{Q} = \frac{B_2 d}{2\sqrt{\nu\eta}}, \qquad \mathbf{Pr} = \frac{\nu}{\eta} \quad \Big(\text{or } \mathbf{R_M} = \mathbf{RePr}.\Big)$$

The Hartmann number $\mathbf{Q}$ gives an estimate of how strong the magnetic field is when compared with the diffusive velocities $d/\sqrt{\nu\eta}$. In the limit $\mathbf{Q} \to 0$ we should obtain the non conductive fluid results. The energy dissipation of this system is given by

$$\mathcal{D} = \nu\langle|\nabla u|\rangle + \eta\langle|\nabla B_T|\rangle = \frac{U^{*3}}{d}\mathbf{D}. \tag{2}$$

$\mathbf{D}$ is a non-dimensional form of the dissipation and our principal aim is to estimate it as a function of the non-dimensional parameters mentioned before.

2.2 The Laminar State

The above set of equations allow for an exact laminar solution. Assuming homogeneity in the x and z direction and no time dependence we have $u = \mathbf{i}U(y)$, $B = \mathbf{i}B_1(y) + \mathbf{j}B_2$ and

$$\begin{aligned} 0 &= B_2 \cdot \partial_y B_1 + \nu\partial_y^2 U && (3)\\ 0 &= B_2 \cdot \partial_y U + \eta\partial_y^2 B_1 && (4)\\ B_2 &= \text{constant}. && (5) \end{aligned}$$

The last equation came from the solenoidal constraint on B. The above equations have the solution:

$$B_1 = \frac{1}{2}\sqrt{\frac{\nu}{\eta}}U^* \left[\frac{\cosh\left(\frac{B_2 d}{2\sqrt{\nu\eta}}\right) - \cosh\left(\frac{B_2 y}{\sqrt{\nu\eta}}\right)}{\sinh\left(\frac{B_2 d}{2\sqrt{\nu\eta}}\right)}\right], \qquad U = \frac{1}{2}U^*\left[1 + \frac{\sinh\left(\frac{B_2 y}{\sqrt{\nu\eta}}\right)}{\sinh\left(\frac{B_2 d}{2\sqrt{\nu\eta}}\right)}\right] \tag{6}$$

In the limit $B_2 d/\sqrt{\nu\eta} \to 0$ we return to plane Couette flow. The laminar solution for the velocity profile as well as the magnetic field lines is shown in figure (2).

Next we examine the energy dissipation $\mathcal{D}_{ls}$ of the laminar solution. The dissipation can be easily calculated from (2) and it gives

$$\begin{aligned}
\mathcal{D}_{ls} &= \mathcal{D}_{visc} + \mathcal{D}_{magn} \\
&= \sqrt{\frac{\nu}{\eta}}\frac{B_2 U^{*2}}{4d}\left[\coth\left(\frac{B_2 d}{2\sqrt{\nu\eta}}\right) + \frac{\frac{B_2 d}{2\sqrt{\nu\eta}}}{\sinh^2\left(\frac{B_2 d}{2\sqrt{\nu\eta}}\right)}\right] + \qquad (7)\\
&\quad \sqrt{\frac{\nu}{\eta}}\frac{B_2 U^{*2}}{4d}\left[\coth\left(\frac{B_2 d}{2\sqrt{\nu\eta}}\right) - \frac{\frac{B_2 d}{2\sqrt{\nu\eta}}}{\sinh^2\left(\frac{B_2 d}{2\sqrt{\nu\eta}}\right)}\right] \\
&= \frac{1}{2}\sqrt{\frac{\nu}{\eta}}\frac{B_2 U^{*2}}{d}\coth\left(\frac{B_2 d}{2\sqrt{\nu\eta}}\right) \qquad (8)
\end{aligned}$$

or in a non-dimensional form

$$\mathbf{D} = \mathbf{Re^{-1}Q}\coth(\mathbf{Q}) \qquad (9)$$

There are a few points we have to make for the above equation. We note first that the viscous dissipation is always bigger than the resistive dissipation although the difference is exponentially small for large $\mathbf{Q}$. Moreover for fixed magnetic field B_2 and velocity U the dissipation increases with the Prandtl number. In other words decreasing η increases the dissipation. In the limit $\nu \to \infty$, $\eta \to \infty$ keeping the Prandtl number fixed the dissipation goes to the finite limit $\frac{1}{2}\mathbf{Pr}B_2 U^{*2}/d$. Taking the limit $\mathbf{Q} = B_2 d/2\sqrt{\nu\eta} \to 0$ we obtain the plane Couette dissipation

$$\mathcal{D}_{ls} \simeq \nu\frac{U^{*2}}{d^2} \text{ and for large } \mathbf{Q} \text{ we obtain } \mathcal{D}_{ls} \simeq \frac{1}{2}\sqrt{\frac{\nu}{\eta}}\frac{B_2 U^{*2}}{d}.$$

2.3 Stability

Next we examine the energy stability of the above flow. Writing the magnetic and the velocity field as the laminar solution plus an arbitrary perturbation $u = U_{ls} + v$ and $B_{\text{total}} = B_{ls} + b$ we obtain from (1):

$$\begin{aligned}
\partial_t u + v\cdot\nabla v + U\cdot\nabla v + v\cdot\nabla U &= -\nabla P + B\cdot\nabla b + b\cdot\nabla B + b\cdot\nabla b + \nu\nabla^2 v \qquad (10)\\
\partial_t b + v\cdot\nabla B + b\cdot\nabla U + v\cdot\nabla b &= B\cdot\nabla v + b\cdot\nabla U + b\cdot\nabla v + \eta\nabla^2 b \qquad (11)\\
\nabla\cdot v = 0 \quad &, \quad \nabla\cdot b = 0. \qquad (12)
\end{aligned}$$

where we dropped the index ls for convenience. Multiplying the first one with v and the second one with b adding them and taking their space average we obtain

$$\frac{1}{2}\partial_t\langle v^2 + b^2\rangle = -\langle(v_1 v_2 - b_1 b_2)U'\rangle - \langle(b_1 v_2 - v_1 b_2)B_1'\rangle - \eta\langle|\nabla b|^2\rangle - \nu\langle|\nabla v|^2\rangle \qquad (13)$$

where the prime indicates a derivative with respect to y and many terms dropped out due to the boundary conditions. Using the inequalities:

$$\langle(v_1 v_2 - b_1 b_2)U'\rangle \leq \frac{1}{2}\langle(v_1^2 + v_2^2 + b_1^2 + b_2^2)\rangle \max|U'| \leq \frac{1}{2}\langle(v^2 + b^2)\rangle \max|U'|$$

$$\langle (b_1 v_2 - v_1 b_2) B_1' \rangle \leq \frac{1}{2}\langle (\tau v_1^2 + \frac{1}{\tau} b_2^2 + \tau v_2^2 + \frac{1}{\tau} b_1^2)\rangle \max|B_1'| \leq \frac{1}{2}\langle (\tau v^2 + \frac{1}{\tau} b^2)\rangle \max|B_1'|$$

(where τ is a free parameter) and the Poincare inequality

$$\langle |\nabla v|^2 \rangle \geq \frac{\pi^2}{d^2}[\langle |v|^2 \rangle$$

we end up with

$$\begin{aligned}\frac{1}{2}\partial_t \langle v^2 + b^2 \rangle &\leq -\left[\nu \frac{\pi^2}{d^2} - \frac{1}{2}\max|U'| - \frac{1}{2}\tau \max|B_1'|\right]\langle v^2 \rangle \qquad (14)\\ &- \left[\eta \frac{\pi^2}{d^2} - \frac{1}{2}\max|U'| - \frac{1}{2\tau}\max|B_1'|\right]\langle b^2 \rangle \qquad (15)\end{aligned}$$

The energy of the perturbation is going to decrease if each term in the square brackets is greater than zero. Eliminating τ and recalling that $\max|U'| = \frac{U^* B_2}{2\sqrt{\nu\eta}} \coth\left(\frac{B_2 d}{2\sqrt{\nu\eta}}\right)$ and $\max|B_1'| = \frac{B_2 U^*}{2\eta}$ We obtain that for stability:

$$\frac{B_2^2 U^{*2}}{4\eta^2} < \left[\frac{2\pi^2\nu}{d^2} - \frac{U^* B_2}{2\sqrt{\nu\eta}} \coth\left(\frac{B_2 d}{2\sqrt{\nu\eta}}\right)\right] \cdot \left[\frac{2\pi^2\eta}{d^2} - \frac{U^* B_2}{2\sqrt{\nu\eta}} \coth\left(\frac{B_2 d}{2\sqrt{\nu\eta}}\right)\right] \qquad (16)$$

where each term in the square brackets should be non-negative.
In dimensionless numbers

$$\mathbf{Q^2 Re^2 Pr} < \left[2\pi^2 - \mathbf{ReQ}\coth(\mathbf{Q})\right] \cdot \left[2\pi^2 \mathbf{Pr^{-1}} - \mathbf{ReQ}\coth(\mathbf{Q})\right] \qquad (17)$$

or

$$\mathbf{Q^2 R_M^2} < \left[2\pi^2 - \mathbf{ReQ}\coth(\mathbf{Q})\right] \cdot \left[2\pi^2 - \mathbf{R_M Q}\coth(\mathbf{Q})\right] \qquad (18)$$

For small $\mathbf{Q}$ we obtain that $\max\{\mathbf{Re}, \mathbf{R_M}\} \leq 2\pi^2$. For large $\mathbf{Q}$ the range of $\mathbf{R_M}, \mathbf{Re}$ decreases inversely proportional to $\mathbf{Q}$ (e.g. $\max\{\mathbf{Re}, \mathbf{R_M}\} \leq 2\pi\mathbf{Q}^{-1}$). Figure (3) summarizes our results. We note that the conditions we derived are sufficient for energy stability but their violation does necessarily not imply energy instability.

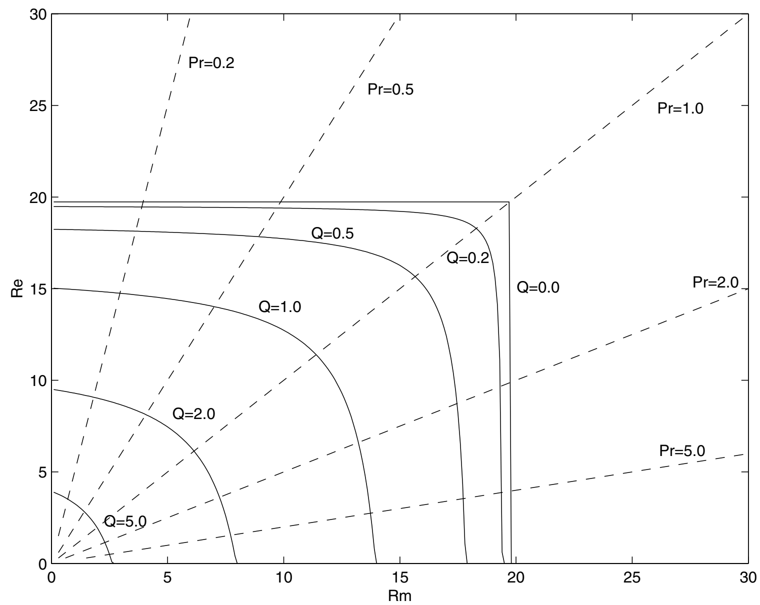

Figure 3: Energy stability reagions for magnetic Couette flow. The solid lines indicate the estimated stability boundaries for different values of Q. The dashed lines indicate constant Prandtl number

2.4 Background Method

Next we want to examine how the energy dissipation is modified when the flow is in a "turbulent" regime. We are going to use the Doering-Constantin background method [2] [3] to produce an upper bound on the dissipation. As in the energy stability method we are going to separate the flow to a background component $\mathbf{i}U(y), \mathbf{i}B_1(y) + \mathbf{j}B_2$ that we are going to leave undetermined and a fluctuating component v, b. Following the same procedure as in the energy method we obtain:

$$\frac{1}{2}\partial_t\langle v^2 + b^2\rangle = \langle v_1 B_2 B_1'\rangle + \langle b_1 B_2 U'\rangle - \langle (v_1 v_2 - b_1 b_2)U'\rangle - \langle (b_1 v_2 - v_1 b_2)B_1'\rangle$$

$$-\nu\langle|\nabla v|^2\rangle - \eta\langle|\nabla b|^2\rangle + \nu\langle v_1 U''\rangle + \eta\langle b_1 B_1''\rangle \qquad (19)$$

where the linear and constant terms in v and b appeared because the background profile is no longer a solution of the MHD equations (1). To eliminate some of them we are going to add half of the total dissipation:

$$\frac{1}{2}\mathcal{D} = +\frac{1}{2}\nu\langle|\nabla(U+v)|^2\rangle + \frac{1}{2}\eta\langle|\nabla(B+b)|^2\rangle$$

$$= \frac{1}{2}\nu\langle|\nabla v|^2\rangle + \frac{1}{2}\eta\langle|\nabla b|^2\rangle + \nu\langle U'\partial_y v_1\rangle + \eta\langle B_1'\partial_y b_1\rangle + \frac{1}{2}\nu\langle U'^2\rangle + \frac{1}{2}\eta\langle {B_1'}^2\rangle$$

by doing so, we obtain

$$\frac{1}{2}\partial_t\langle v^2+b^2\rangle+\frac{1}{2}\mathcal{D}=\langle v_1B_2B_1'\rangle+\langle b_1B_2U'\rangle-\langle(v_1v_2-b_1b_2)U'\rangle-\langle(v_1b_2-b_1v_2)B_1'\rangle$$

$$-\frac{1}{2}\nu\langle|\nabla v|^2\rangle-\frac{1}{2}\eta\langle|\nabla b|^2\rangle+\frac{1}{2}\nu\langle U'^2\rangle+\frac{1}{2}\eta\langle {B_1'}^2\rangle \tag{20}$$

To get rid of the remaining linear terms we will use the transformation $v=w-\mathbf{i}V(y)$ and $b=\beta-\mathbf{i}H(y)$ where

$$\nu V''=B_2B_1' \quad \text{and} \quad \eta H''=B_2U' \tag{21}$$

then

$$\begin{aligned}\partial_t\mathcal{E}+\frac{1}{2}\mathcal{D} &= -\langle(w_1w_2-\beta_1\beta_2)U'\rangle-\langle(w_1\beta_2-\beta_1w_2)B_1'\rangle+\frac{1}{2}\nu\langle|\nabla w|^2\rangle+\frac{1}{2}\eta\langle|\nabla\beta|^2\rangle\\ &\quad+\frac{1}{2}\nu\langle U'^2+V'^2\rangle+\frac{1}{2}\eta\langle {B_1'}^2+H'^2\rangle\end{aligned} \tag{22}$$

where $\mathcal{E}=\frac{1}{2}\langle v^2+b^2\rangle$. We can write the above equation (22) in the form

$$2\partial_t\mathcal{E}+\mathcal{D}=-\mathcal{Q}_{UB_1}+\mathcal{D}_{bg} \tag{23}$$

where $\mathcal{Q}_{UB_1}$ is a quadratic functional on v and b that depends on our choice of the background fields U and B_1, and $\mathcal{D}_{bg}$ is the dissipation due to the background field. Our aim now is to choose an appropriate background field so that the quadratic term $\mathcal{Q}_{UB_1}$ is positive definite. If we succeed the we can prove by integrating over time that the the total energy is bounded in time. More by taking the time average of (23) that the total time averaged dissipation is $\mathcal{D}\le\mathcal{D}_{bg}$.

From the form of $\mathcal{Q}_{UB_1}$ a natural choice for the background magnetic field is going to be $B_1=0$. For U we are going to use the piece-wise linear profile

$$U(y)=\begin{cases}(U^*/2\delta)y & \text{if} \quad -d/2 \le y \le -d/2+\delta\\ U^*/2 & \text{if} \quad -d/2+\delta \le y \le d/2-\delta\\ (U^*/2\delta)(d/2-y) & \text{if} \quad d/2-\delta \le y \le d/2.\end{cases} \tag{24}$$

From (21) and the boundary conditions for b we also have that

$$H(y)=\frac{B_2}{\eta}\int_0^y U(y')-\langle U\rangle dy'. \tag{25}$$

The background fields U and H are shown in figure (4). We can easily now evaluate the dissipation of the background field and it is found to be

$$\frac{1}{2}\mathcal{D}_{bg}=\frac{\nu U^{*2}}{4d}\frac{1}{\delta}+\frac{B_2^2U^{*2}}{12\eta d}\delta=\frac{\nu U^{*2}}{4d}\left(\frac{1}{\delta}+\frac{B_2^2\delta}{3\nu\eta}\right) \tag{26}$$

$\mathcal{D}_{bg}$ obtains its minimum value for $\delta_{min}=\sqrt{3\eta\nu/B_2^2}$, giving $\min\{\mathcal{D}_{bg}\}=\frac{1}{\sqrt{3}}\frac{U^*B_2}{d}$

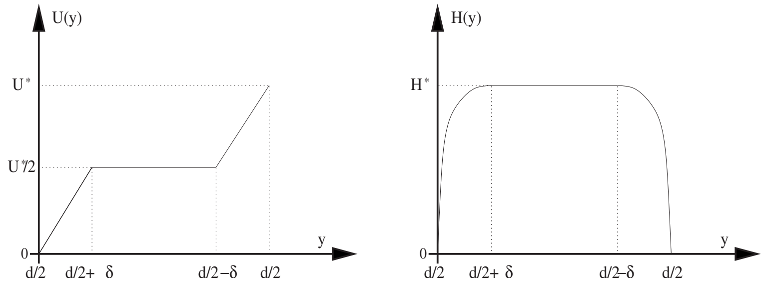

Figure 4: The two background fields $U(y)$ and $H(y)$. H^* is equal to $B_2\delta/4\eta$.

Now we focus on the quadratic term $\mathcal{Q}_{UB_1}$ and try to determine the values of the free parameter δ that make it definite positive. Formally we would need to solve for the minimum of $\mathcal{Q}_{UB_1}$ that would lead to an eigenvalue problem that we would have to solve numerically. We are not going to follow this procedure here though but instead we are going to give rigorous estimates for the values of δ that guarantee the positivity of $\mathcal{Q}_{UB_1}$. Using the fundamental theorem of calculus and the Cauchy-Schwartz inequality we can show that

$$|w_i| = \left|\int_{-d/2}^{y} \frac{\partial w_i}{\partial y}(y')dy'\right| = \left|\int_{-d/2}^{y} 1\cdot\frac{\partial w_i}{\partial y}(y')dy'\right| \leq \sqrt{y+d/2}\left|\int_{-d/2}^{y}\left(\frac{\partial w_i}{\partial y}(y')\right)^2 dy'\right|^{1/2}. \tag{27}$$

This implies

$$\left|\int_{-d/2}^{d/2} U'(y)w_1w_2dx^3\right| \leq \frac{U_*}{2\delta}\int dxdz\int_0^{\delta} y\left(\int_{-d/2}^{0}\left(\frac{\partial w_1}{\partial y}\right)^2 dy'\right)^{1/2}\left(\int_{-d/2}^{0}\left(\frac{\partial w_2}{\partial y}\right)^2 dy'\right)^{1/2} dy+$$

$$\frac{U_*}{2\delta}\int dxdz\int_{d/2-\delta}^{d/2}(d/2-y)\left(\int_0^{d/2}\left(\frac{\partial w_1}{\partial y}\right)^2 dy'\right)^{1/2}\left(\int_0^{d/2}\left(\frac{\partial w_2}{\partial y}\right)^2 dy'\right)^{1/2} dy. \tag{28}$$

Including all the other terms in $|\nabla w|^2$ we obtain

$$\left|\int U'(y)w_1w_2dx^3\right| \leq \frac{U_*}{2\delta}\frac{\delta^2}{2}\frac{1}{2}\int|\nabla w|^2dx^3 = \frac{U_*\delta}{8}\int|\nabla w|^2dx^3. \tag{29}$$

and similarly for β

$$\left|\int U'(y)\beta_1\beta_2dx^3\right| \leq \frac{U_*\delta}{8}\int|\nabla\beta|^2dx^3. \tag{30}$$

This implies for $\mathcal{Q}_{UB_1}$:

$$\begin{aligned}
\mathcal{Q}_{UB_1}\{w,\beta\} &= \nu\left\langle|\nabla w|^2\right\rangle + \eta\left\langle|\nabla\beta|^2\right\rangle + 2\left\langle U'(w_1w_2-\beta_1\beta_2)\right\rangle \\
&\geq \nu\left\langle|\nabla w|^2\right\rangle + \eta\left\langle|\nabla\beta|^2\right\rangle - 2\left|\left\langle U'w_1w_2\right\rangle\right| - 2\left|\left\langle U'\beta_1\beta_2\right\rangle\right| \\
&\geq \nu\left\langle|\nabla w|^2\right\rangle + \eta\left\langle|\nabla\beta|^2\right\rangle - \frac{U_*\delta}{4}\left\langle|\nabla w|^2\right\rangle - \frac{U_*\delta}{4}\left\langle|\nabla\beta|^2\right\rangle \\
&\geq \left(\nu-\frac{U_*\delta}{4}\right)\left\langle|\nabla w|^2\right\rangle + \left(\eta-\frac{U_*\delta}{4}\right)\left\langle|\nabla\beta|^2\right\rangle \\
&\geq \frac{\pi^2}{d^2}\left(\min\{\nu,\eta\}-\frac{U_*\delta}{4}\right)\left\langle w^2+\beta^2\right\rangle. \qquad (31)
\end{aligned}$$

So $\mathcal{Q}_{UB_1}\{w,\beta\}\geq 0$ if we choose $\delta\leq 4\nu/U_* = 4d/Re$. This is the maximum value of δ that our estimates allow us to use.

$$\delta \leq \frac{4\min\{\eta,\nu\}}{U^*} = \delta_Q \qquad (32)$$

The smallest value of $\mathcal{D}_{bg}$ (keeping $\mathcal{Q}_{UB_1}$ positive) is obtained for $\delta=\min\{\delta_{min},\delta_Q,d/2\}$. So we end up with our final result on the Couette flow that if $\delta_{min}<\delta_Q$ we are going to use δ_{min} to evaluate the background dissipation, which means that if

$$\boxed{\frac{4\min\{\nu,\eta\}B_2}{U^*\sqrt{3\nu\eta}} > 1 \quad \text{then} \quad \mathcal{D}\leq\frac{U^{*2}B_2}{\sqrt{3}d}\sqrt{\frac{\nu}{\eta}},} \qquad (33)$$

or in the non-dimensional form

$$\boxed{\text{if} \qquad 8\mathbf{Q} > \sqrt{3}\max\{\mathbf{Re},\mathbf{R_M}\} \quad \text{then} \quad \mathbf{D}\leq\frac{2\mathbf{Q}}{\sqrt{3}\mathbf{Re}}.} \qquad (34)$$

If on the other hand $\delta_{min}>\delta_Q$ we are forced to use δ_Q in the evaluation of the background dissipation. So if

$$\boxed{\frac{4\min\{\nu,\eta\}B_2}{U^*\sqrt{3\nu\eta}} < 1 \quad \text{then} \quad \mathcal{D}\leq\frac{1}{8}\frac{\nu}{\min\{\nu,\eta\}}\frac{U^{*3}}{d}+\frac{2}{3}\frac{\min\{\nu,\eta\}}{\eta}\frac{B_2^2U^*}{d}} \qquad (35)$$

or in the non-dimensional form

$$\boxed{\text{if} \quad 8\mathbf{Q} < \sqrt{3}\max\{\mathbf{Re},\mathbf{R_M}\} \quad \text{then} \quad \mathbf{D}\leq\frac{1}{8}\max\{\mathbf{Pr},1\}+\frac{8}{3}\frac{\mathbf{Q^2}}{\mathbf{Re}\max\{\mathbf{R_M},\mathbf{Re}\}}.} \qquad (36)$$

The first inequality (34) we have shows that for large enough magnetic field the dissipation is bounded by a function with the same dependence on **Re** and **Q** as the laminar. The prefactor has only a 15% difference. This gives an indication that the flow should be close to the laminar solution. If the magnetic field on the other hand is not strong enough then the dissipation becomes independent of the Reynolds number **Re** and has only a dependence on the Prandtl number **Pr**. The increase of the bound on the dissipation with Prandtl number is an interesting result that we cannot yet determine if it is the outcome of a bad estimate or it corresponds to a physical mechanism for increase of the dissipation.

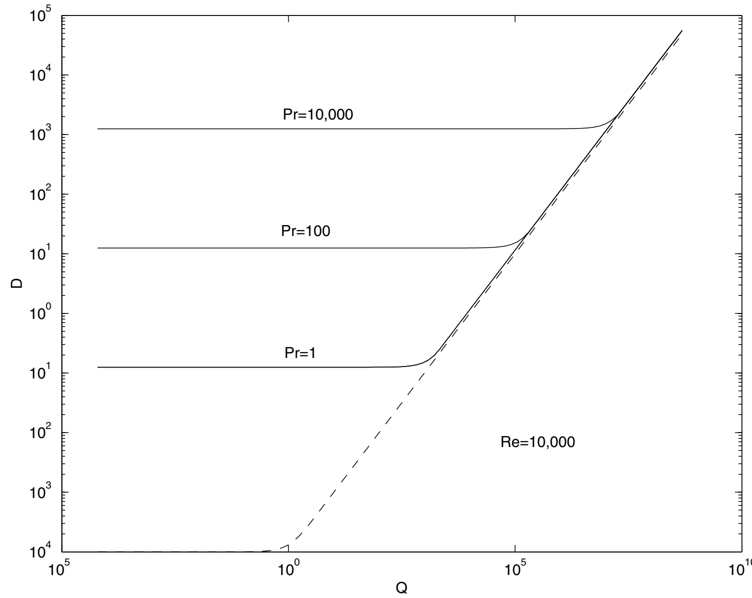

Figure 5: The dissipation as a function of **Q** for different Prandtl numbers. The dashed line shows the laminar solutions dissipation.

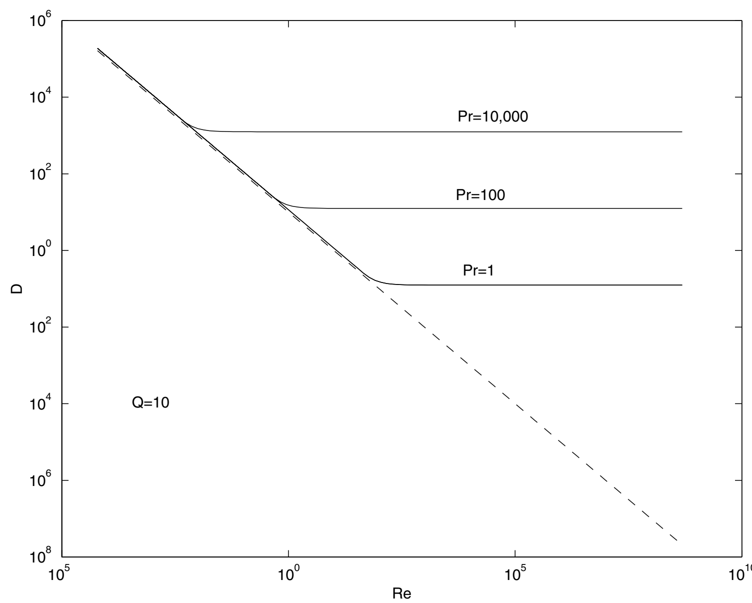

Figure 6: The dissipation as a function of **Re** for different Prandtl numbers. The dashed line shows the laminar solutions dissipation.

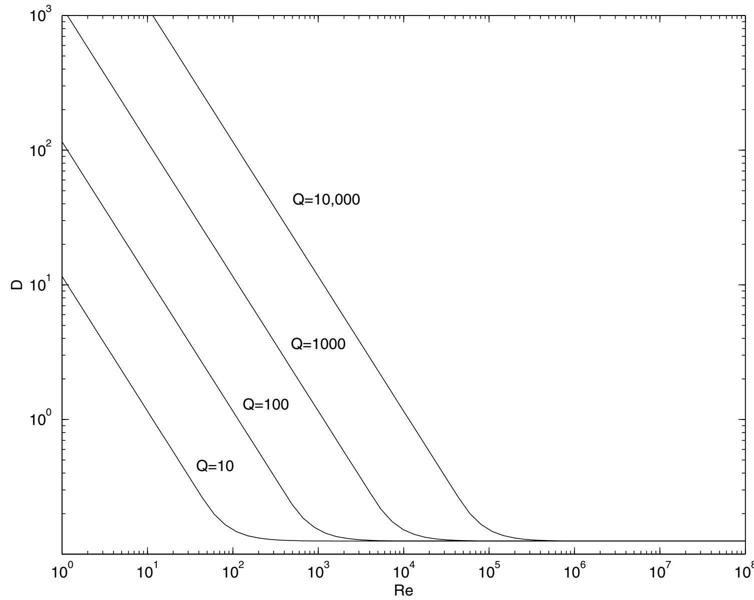

Figure 7: The dissipation as a function of $\mathbf{Re}$ for different values of $\mathbf{Q}$.

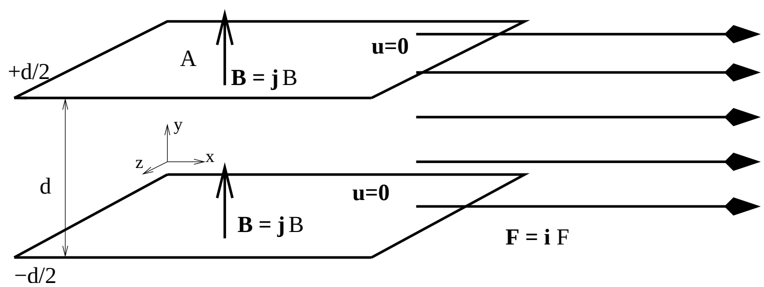

Figure 8: The setup for the Hartmann flow

3 Magnetic Poiseuille (Hartmann) Flow

3.1 Preliminaries

Next we turn to examine the magnetic Poiseuille or Hartmann flow named after Hartmann who first examined this kind of flow [4]. We consider the same set up as in §2, only this time both the top and bottom plate are held fixed and there is a constant pressure gradient or a uniform force field F in the $\mathbf{i}$ direction. The same equations govern the current setup as in §2 with the addition of the force field in the momentum equation:

$$\partial_t u + u \cdot \nabla u = -\nabla P + B \cdot \nabla B + \nu \nabla^2 u + F \,. \tag{37}$$

The non-dimensional numbers that parametrize our system are the Hartmann number defined as before, and the Grashoff number $\mathbf{G}$ and magnetic Grashoff $\mathbf{G_M}$ number defined

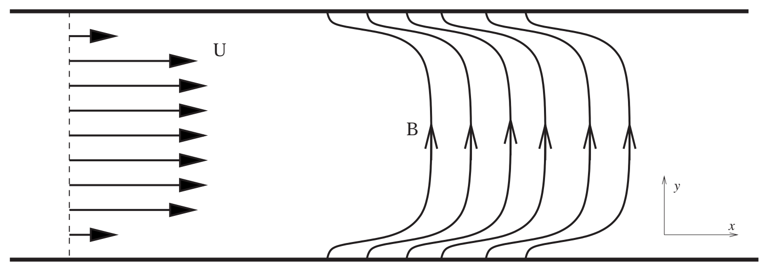

Figure 9: The laminar velocity profile and the magnetic field lines.

as

$$\mathbf{G} = \frac{Fd^3}{2\nu^2}, \quad \mathbf{G_M} = \mathbf{GPr}$$

The energy dissipation is given by

$$\mathcal{D} = \nu\langle|\nabla u|^2\rangle + \eta\langle|\nabla B|^2\rangle = F^{3/2}d^{1/2}\mathbf{D} \tag{38}$$

where $\mathbf{D}$ is again the non-dimensional form of the dissipation we are going to use.

3.2 Laminar Solution

Assuming time and $x - z$ independence again we end up with the system of equations

$$\begin{aligned} 0 &= B_2 \cdot \partial_y B_1 + \nu\partial_y^2 U + F & (39)\\ 0 &= B_2 \cdot \partial_y U + \eta\partial_y^2 B_1 & (40)\\ B_2 &= \text{constant.} & (41) \end{aligned}$$

They can be solved easily and the solution is given by:

$$U = \frac{Fd}{2B_2}\sqrt{\frac{\eta}{\nu}}\left[\frac{\cosh\left(\frac{B_2 d}{2\sqrt{\nu\eta}}\right) - \cosh\left(\frac{B_2 y}{\sqrt{\nu\eta}}\right)}{\sinh\left(\frac{B_2 d}{2\sqrt{\nu\eta}}\right)}\right], \qquad B_1 = \frac{Fd}{2B_2}\left[\frac{\sinh\left(\frac{B_2 y}{\sqrt{\nu\eta}}\right)}{\sinh\left(\frac{B_2 d}{2\sqrt{\nu\eta}}\right)} - \frac{2y}{d}\right]. \tag{42}$$

The laminar velocity and the magnetic field lines are shown in figure (9). Again the limit $\mathbf{Q} \to 0$ brings us back to Poiseuille flow.

We evaluate the dissipation again and find it to be

$$\mathcal{D} = \frac{F^2 d}{2B_2}\sqrt{\frac{\eta}{\nu}}\left[\coth\left(\frac{B_2 d}{2\sqrt{\nu\eta}}\right) - \frac{2\sqrt{\nu\eta}}{B_2 d}\right] \tag{43}$$

or in the non-dimensional form

$$\mathbf{D} = \frac{\mathbf{G^{1/2}}}{2\sqrt{2}\mathbf{Q}}\left[\coth(\mathbf{Q}) - \frac{1}{\mathbf{Q}}\right]. \tag{44}$$

$\mathbf{D}$ goes to $\frac{\mathbf{G^{1/2}}}{6\sqrt{2}}$ for $\mathbf{Q}$ going to zero, and $\mathbf{D}$ goes to $\frac{\mathbf{G^{1/2}}}{2\sqrt{2}\mathbf{Q}}$ for $\mathbf{Q}$ going to infinity. Also as in Couette flow the dissipation goes to a finite limit as ν and η go to zero, keeping their ratio (Prandtl number) fixed.

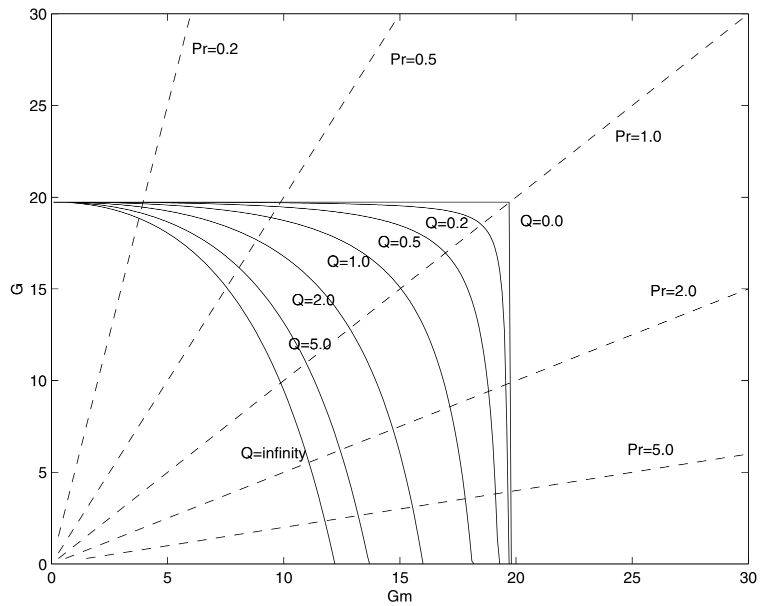

Figure 10: Energy stability reagions for Hartmann flow. The solid lines indicate the estimated stability boundaries for different values of **Q**. The dashed lines indicate constant Prandtl number

3.3 Stability

Next we examine the energy stability of the Hartmann flow. The evolution of the energy is given by:

$$\frac{1}{2}\partial_t\langle v^2 + b^2\rangle = -\langle(v_1v_2 - b_1b_2)U'\rangle - \langle(v_1b_2 - b_1v_2)B_1'\rangle - \eta\langle|\nabla b|^2\rangle - \nu\langle|\nabla v|^2\rangle. \tag{45}$$

Using the same inequalities as in the Couette flow we obtain

$$B_{1max}'^{\,2} \leq \left(\nu\frac{2\pi^2}{d^2} - U_{max}'\right)\left(\eta\frac{2\pi^2}{d^2} - U_{max}'\right) \tag{46}$$

or

$$\frac{F^2}{B_2^2}\left[\frac{B_2 d}{2\sqrt{\nu\eta}}\coth\left(\frac{B_2 d}{2\sqrt{\nu\eta}}\right) - 1\right]^2 \leq \left(\nu\frac{2\pi^2}{d^2} - \frac{Fd}{2\nu}\right)\left(\eta\frac{2\pi^2}{d^2} - \frac{Fd}{2\nu}\right) \tag{47}$$

that gives in the non-dimensional form

$$\mathbf{G_M^2}\Big[\mathbf{Q}\coth(\mathbf{Q}) - 1\Big]^2 \leq \mathbf{Q^2}(2\pi^2 - \mathbf{G})(2\pi^2 - \mathbf{G_M}). \tag{48}$$

As before we find that the energy stability is decreased as we increase **Q**. Unlike the Couette flow though in the limit of large Q the stability curve goes to the finite limit given by $\mathbf{G_M^2} < 3(2\pi^2 - \mathbf{G})(2\pi^2 - \mathbf{G_M})$ Our stability results are summarized in figure (10).

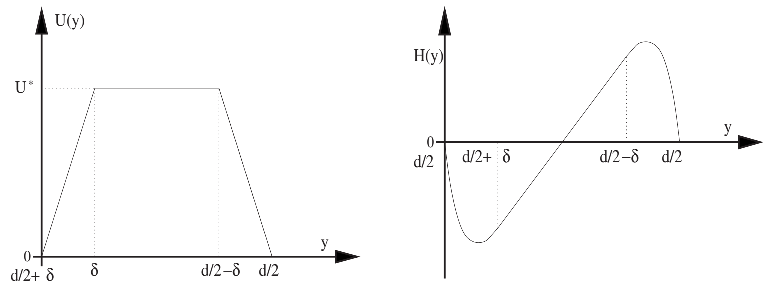

Figure 11: The two background fields $U(y)$ and $H^*(y)$

3.4 Background Method

Next examine the dissipation in the turbulent regime. Separating the flow to a background $U, B_1, B2$ and a fluctuating component b, v multiplying with b, v and taking the spacial average as before we obtain

$$\frac{1}{2}\partial_t\langle v^2+b^2\rangle = \langle v_1 B_2 B_1'\rangle + \langle b_1 B_2 U'\rangle - \langle (v_1v_2 - b_1b_2)U'\rangle - \langle (b_1v_2 - v_1b_2)B_1'\rangle$$

$$-\nu\langle|\nabla v|^2\rangle - \eta\langle|\nabla b|^2\rangle + \nu\langle v_1 U''\rangle + \eta\langle b_1 B_1''\rangle + \langle F\cdot v\rangle. \tag{49}$$

Adding half the dissipation we get

$$\partial_t\mathcal{E} + \frac{1}{2}\mathcal{D} = \langle v_1 B_2 B_1'\rangle + \langle b_1 B_2 U'\rangle + \langle F v_1\rangle - \langle (v_1v_2 - b_1b_2)U'\rangle - \langle (v_1b_2 - b_1v_2)B_1'\rangle$$

$$-\frac{1}{2}\nu\langle|\nabla v|^2\rangle - \frac{1}{2}\eta\langle|\nabla b|^2\rangle + \frac{1}{2}\nu\langle U'^2\rangle + \frac{1}{2}\eta\langle {B_1'}^2\rangle. \tag{50}$$

Using $\mathcal{D} = \langle F\cdot u\rangle = \langle FU\rangle + \langle F v_1\rangle$ and $v = w - \mathbf{i}V(y)$ and $b = \beta - \mathbf{i}H(y)$ where $\nu V'' = B_2 B_1'$ and $\eta B_1'' = B_2 U'$ we can write (50) as

$$2\partial_t\mathcal{E} - \mathcal{D} = 2F\langle U\rangle - \mathcal{D}_{bg} + \mathcal{Q}_{UB_1} \tag{51}$$

with $\mathcal{D}_{bg} = \nu\langle U'^2\rangle + \eta\langle H'^2\rangle$ and $H = -\frac{B_2}{\eta}(U - \langle U\rangle)$ and

$$\mathcal{Q}_{UB_1} = \nu\langle|\nabla w|^2\rangle + \eta\langle|\nabla\beta|^2\rangle + 2\langle (w_1w_2 - \beta_1\beta_2)U'\rangle + 2\langle (w_1\beta_2 - w_2\beta_1)U'\rangle$$

where we already picked $B_1 = 0$ for a background profile.

Contrary to the Couette flow case that the positivity of $Q_U B_1$ was leading to an upper bound on the dissipation, if $\mathcal{Q}_{UB_1} \geq 0$ then we have that $\mathcal{D} \geq F\langle U\rangle - \frac{1}{2}\mathcal{D}_{bg}$ which gives a lower bound on the dissipation.

The velocity back ground field we are going to choose is going to be

$$U(y) = \begin{cases} (U^*/\delta)y & \text{if} \quad -d/2 \quad \leq y \quad \leq -d/2+\delta \\ U^*) & \text{if} \quad -d/2+\delta \quad \leq y \quad \leq d/2-\delta \\ (U^*/\delta)(d/2-y) & \text{if} \quad d/2-\delta \quad \leq y \quad \leq d/2. \end{cases} \tag{52}$$

with U^* and δ undetermined parameters. $U(y)$ and $H(y)$ are shown in figure (11). Evaluating the background dissipation and $F\langle U\rangle$ we get:

$$F\langle U\rangle - \frac{1}{2}\mathcal{D}_{bg} = FU^* - FU^*\left(\frac{\delta}{d}\right) - \frac{\nu U^{*2}}{d^2}\left(\frac{d}{\delta}\right) - \frac{B_2^2 U^{*2}}{2\eta}\left[\frac{2}{3}\left(\frac{\delta}{d}\right) - \left(\frac{\delta}{d}\right)^2\right]$$

$$\simeq FU^* - \frac{\nu U^{*2}}{d^2}\left(\frac{d}{\delta}\right) - \frac{B_2^2 U^{*2}}{2\eta}\frac{2}{3}\left(\frac{\delta}{d}\right) \tag{53}$$

where we dropped out terms of order $(\delta/d)^2$.
The above expression takes its minimum value when $\delta = \delta_{min} = \sqrt{3\nu\eta}/B_2$ and $U^* = U^*_{min} = (\sqrt{3}/4)(Fd/B_2)\sqrt{\eta/\nu}$

Now we turn to the quadratic term $\mathcal{Q}_{UB_1}$ and try to determine the constraint on on δ and U^*. The calculation is identical with the Couette flow and gives that for $\mathcal{Q}_{UB_1} \geq 0$ we have to have $U^*\delta \leq 2\min\{\nu,\eta\} = (U^*\delta)_Q$. All we have to do now is to find the values of U^* and δ that give the maximum possible of $2F\langle U\rangle - \mathcal{D}_{bg}$ with out violating the constraint $\mathcal{Q}_{UB_1} \geq 0$. If $U^*_{min}\delta_{min} \leq (U^*\delta)_Q$ then the obvious choice for U^* and δ is $U^*_{\min}$ and $\delta_{\min}$ that gives

$$\boxed{\text{If } U^*_{min}\delta_{min} = \frac{3Fd\eta}{4B_2^2} \leq 2\min\{\nu,\eta\} \text{ then} \mathcal{D}_{bg} \geq \frac{\sqrt{3}F^2 d}{4B_2}\sqrt{\frac{\eta}{\nu}}} \tag{54}$$

or in dimensionless form

$$\boxed{\text{If } 3\max\{\mathbf{G},\mathbf{G_M}\} \leq 16\mathbf{Q^2} \text{ then} \mathbf{D} \geq \frac{\sqrt{3}}{2}\left(\frac{\mathbf{G^{1/2}}}{2\sqrt{2}\mathbf{Q}}\right)} \tag{55}$$

If the condition $U^*_{min}\delta_{min} \leq (U^*\delta)_Q$ is violated then we have to evaluate the maximum of $2F\langle U\rangle - \mathcal{D}_{bg}$ over U^* and δ under the constraint that $U^*\delta = (U^*_\delta)_Q$ after some algebra we end up with

$$\boxed{\text{if } \frac{3F\eta}{4B_2^2} \geq 2\min\{\nu,\eta\} \text{ then } \mathcal{D}_{bg} \geq \frac{4\sqrt{2d}}{3\sqrt{3}}\left(F - \frac{2B^2\min\{\nu,\eta\}}{3\eta}\right)^{3/2}\sqrt{\frac{\min\{\nu,\eta\}}{\nu}}} \tag{56}$$

or in dimensionless form

$$\boxed{\text{if } 3\max\{\mathbf{G},\mathbf{G_M}\} \geq 16\mathbf{Q^2} \text{ then } \mathbf{D} \geq \frac{4\sqrt{2}}{3\sqrt{3}}\left(1 - \frac{4\mathbf{Q^2}}{3\max\{\mathbf{G},\mathbf{G_M}\}}\right)^{3/2}\min\{1,\mathbf{Pr^{-1/2}}\}} \tag{57}$$

As in the Couette case for strong enough magnetic field the first inequality (55) indicates that the bound is very close (15% difference) to the laminar dissipation. On the other hand for small enough magnetic fields the bound on the dissipation becomes independent of $\mathbf{Q}$ and $\mathbf{Re}$ and decreases as the inverse square root of the Prandtl number for $\mathbf{Pr} > 1$. This

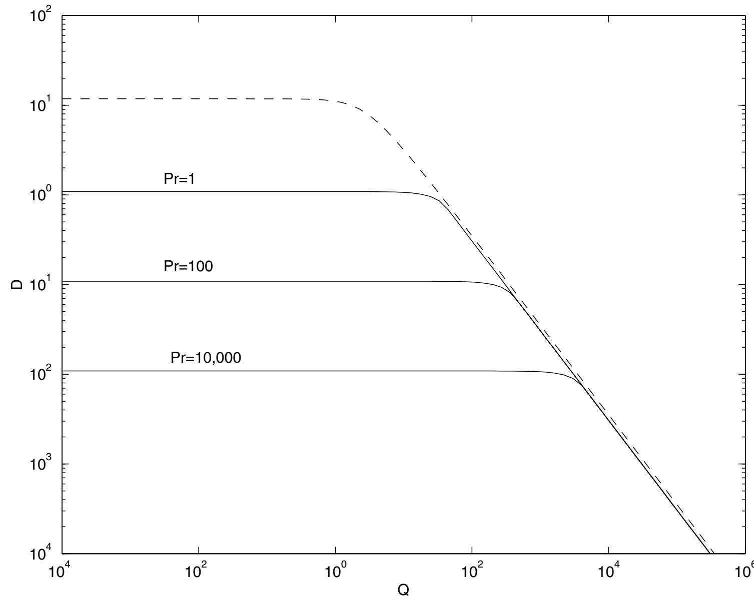

Figure 12: The dissipation as a function of $\mathbf{Q}$ for different Prandtl numbers. The dashed line shows the laminar solutions dissipation.

result is not in contradiction with the related result of the Couette flow that was giving a linear increase with the Prandtl number. The reason for the difference is the definition for the non-dimensional dissipation. If we had chosen $\tilde{D} = \mathcal{D}d/\langle u\rangle^3$ as our non dimensional dissipation we would have

$$\tilde{D} = \frac{\mathcal{D}d}{\langle u\rangle^3} = \frac{\mathcal{D}d}{(\mathcal{D}/F)^3} = \frac{F^3 d}{\mathcal{D}^2} = \frac{1}{\mathbf{D^2}}$$

that gives the same scaling with Couette flow. The figures below (12,13,14) summarize our results.

4 Discussion

We have examined the dissipation for two different kinds of flows in conducting fluids with an imposed vertical (to the flow) magnetic field, namely magnetic Couette flow and Hartmann flow. We have derived bounds on the dissipation and determined the bounds behavior at high Reynolds and magnetic Reynolds number. One of our basic results is that the dissipation is tending to the laminar value if the magnetic field is strong enough. If the magnetic field is not very strong and the Reynolds number is large the dissipation is independent of $\mathbf{Re}$ and $\mathbf{Q}$ and scales as the first power of the Prandtl number if $\mathbf{Pr} > 1$ and is independent of it otherwise. The next figure (15) shows a quantitative comparison of experimental data [5] with our bound. The data show measurments of the drag coefficient $\mathbf{C_f}$ as a function of $\mathbf{Q}$. The coefficient $\mathbf{C_f}$ is defined as:

$$\mathbf{C_f} = \frac{Fd}{\langle U\rangle^2} = \frac{d\mathcal{D}}{\langle U\rangle^3} = \tilde{\mathbf{D}} = \frac{1}{\mathbf{D}^2}$$

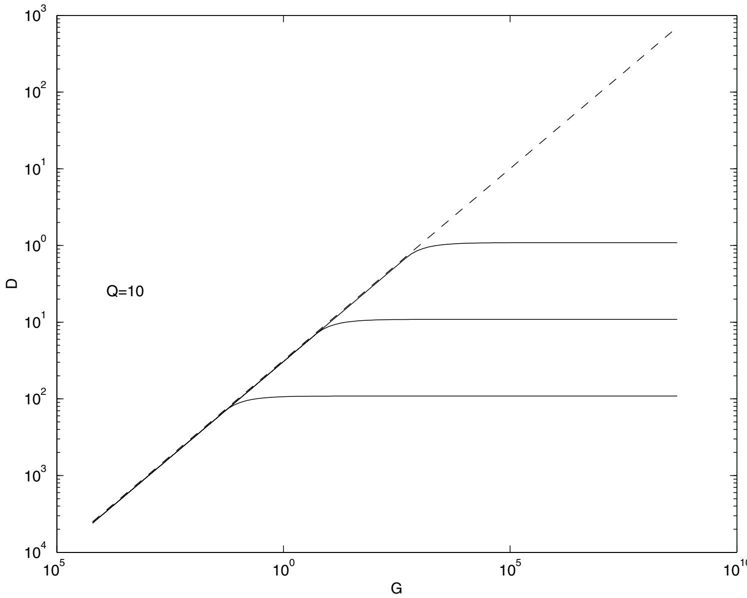

Figure 13: The dissipation as a function of **Re** for different Prandtl numbers. The dashed line shows the laminar solutions dissipation.

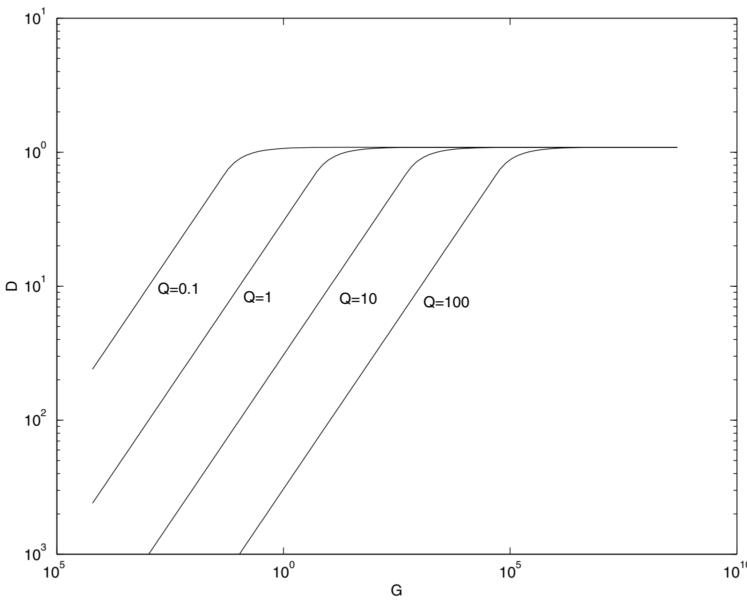

Figure 14: The dissipation as a function of **Re** for different values of **Q**.

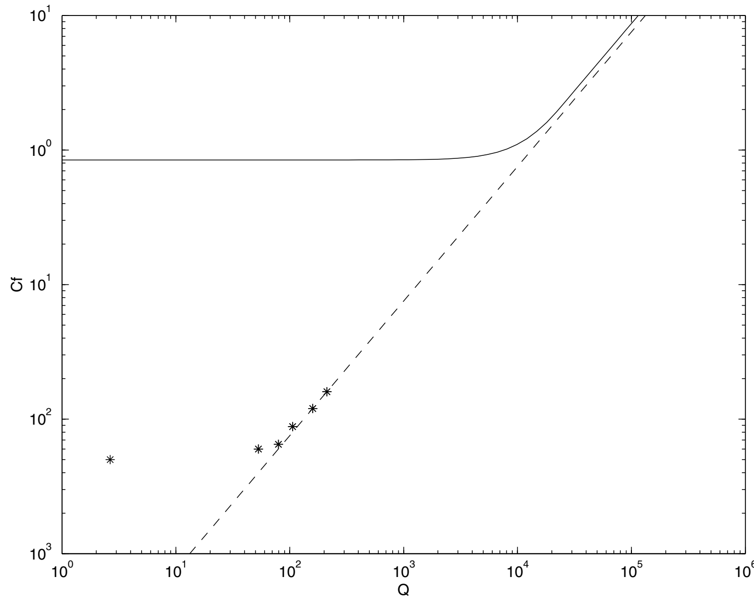

Figure 15: The drag coefficient $\mathbf{C_f}$ as a function of $\mathbf{Q}$

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Although there is a two orders of magnitude difference from our bound which is not suprizing for the rough estimates we used, the bound seems to capture the behavior of the dissipation up to a prefactor.

5 References

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