

Resonant triad interactions on an extended equatorial β -plane

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1 Introduction and Motivation

Due to the low value of the vertical component of the Earth's rotation the tropics possess dynamical properties that are quite different in nature to other regions of higher latitude. One property is that the equator acts as a wave guide due to the changing in sign of the Coriolis parameter.

Linear equatorial wave theory (Matsuno, 1966) has shown that equatorially trapped waves (particularly Rossby and Kelvin waves) are important waves in determining large scale climatic processes such as the El Niño-Southern Oscillation (ENSO; e.g. Philander, 1990; Clarke, 2008). Other factors that determine properties like ENSO periodicity and strength are stochastic processes (Kleeman, 2008). There are, however, nonlinear equatorial processes that are well observed and are thought to impact on phenomena such as ENSO, but many of these processes remain relatively poorly understood from a theoretical point of view.

In the rest of this section, we shall examine previous studies on nonlinear equatorial waves. We shall also review a class of waves that are highly nonlinear in nature and have been observed for several decades now, however, recent insights by linear resonance theory have shown to give insights into their dynamics. The end of this section briefly reviews the properties of the equatorial β -plane, and the consequences for theoretical models of using such an approximation.

Section 2 examines the hydrostatic, inviscid, Boussinesq and rigid-lid momentum and continuity equations on a sphere and then transforms them to the Mercator projection. In section 3 we examine the linear shallow water equations in geopotential coordinates, but, we use some inspiration from the Mercator projection by using a tanh profile for the coriolis parameter (the “extended β -plane”), rather than the traditional linear β -plane approximation.

The modal equations and dispersion relations for the extended β -plane are then derived and investigated in section 4. Finally, section 5 examines the effects of weak nonlinear resonance using a multiscale expansion to derive the coupling coefficients for a system of three baroclinic Rossby waves interacting. Finally, a summary and a discussion of potential future works is presented in section 6.

1.1 Previous Studies of Equatorial Nonlinear Wave Dynamics

There is a large literature on linear equatorial wave dynamics. The nonlinear literature, on the other hand, is somewhat smaller. Furthermore, most of the studies have used either a single layer, or reduced gravity model with only a handful of studies including baroclinic waves. Herewith, we shall give a brief overview of the classes of nonlinearities and waves that have been studied.

Boyd (1980b) studied the nonlinear Kelvin wave. The linear Kelvin wave is a dispersionless wave, and as such, nonlinear effects are very small. Using the method of strained coordinates, he showed how nonlinearity can cause frontogenesis, an alteration in the phase speed and breaking. Boyd (1980a) examined how long equatorial Rossby waves can be described by the Korteweg-de Vries equation (Korteweg and de Vries, 1895) or the modified Korteweg-de Vries equation. Boyd (1983) examines how highly dispersive waves (short Rossby waves, the Yanai wave and inertia-gravity waves) can be described by the nonlinear Schrödinger equation and propagate as a solitary wave packets of permanent form.

More recently Le Sommer et al. (2004) have shown that there is a dynamical split between fast nonlinear waves (fast Yanai and inertia-gravity waves) and slow nonlinear waves (slow Rossby and Kelvin waves).

Ripa (1983a,b) examines nonlinear resonance in a one layer reduced gravity model in the equatorial β -plane using the method of Ripa (1981). In those studies, the various types of interactions are classified into 19 categories and the properties of waves in the various categories are investigated.

Recently, it has been shown in that nonlinear resonance between two equatorially trapped baroclinic Rossby waves and one free barotropic Rossby wave are possible (Reznik and Zeitlin, 2006, 2007a,b).

1.2 An example of observed Nonlinear Equatorial Waves

Tropical Instability Waves (TIWs; Düing et al., 1975; Legeckis, 1977) are an equatorial wave in which the dynamics are still relatively poorly understood, despite the fact that they are a dominant feature of the monthly variability of the equatorial Pacific and Atlantic. In addition, TIWs are climatologically important, as it is thought that TIWs are closely associated with dynamics of the El Niño-Southern Oscillation (Yu and Liu, 2003)

TIWs are seen as cusps on the front north equatorial front (the front that delineates the equatorial cold tongue; see figure 1). TIWs are seen on the southern front as well, but the amplitudes are typically much smaller than on the northern side. It has been shown in the Pacific that there are two distinct periods of 17 and 33 days (Lyman et al., 2007), although the shorter period wave does not appear to exist in the Atlantic (von Schuckmann et al., 2008).

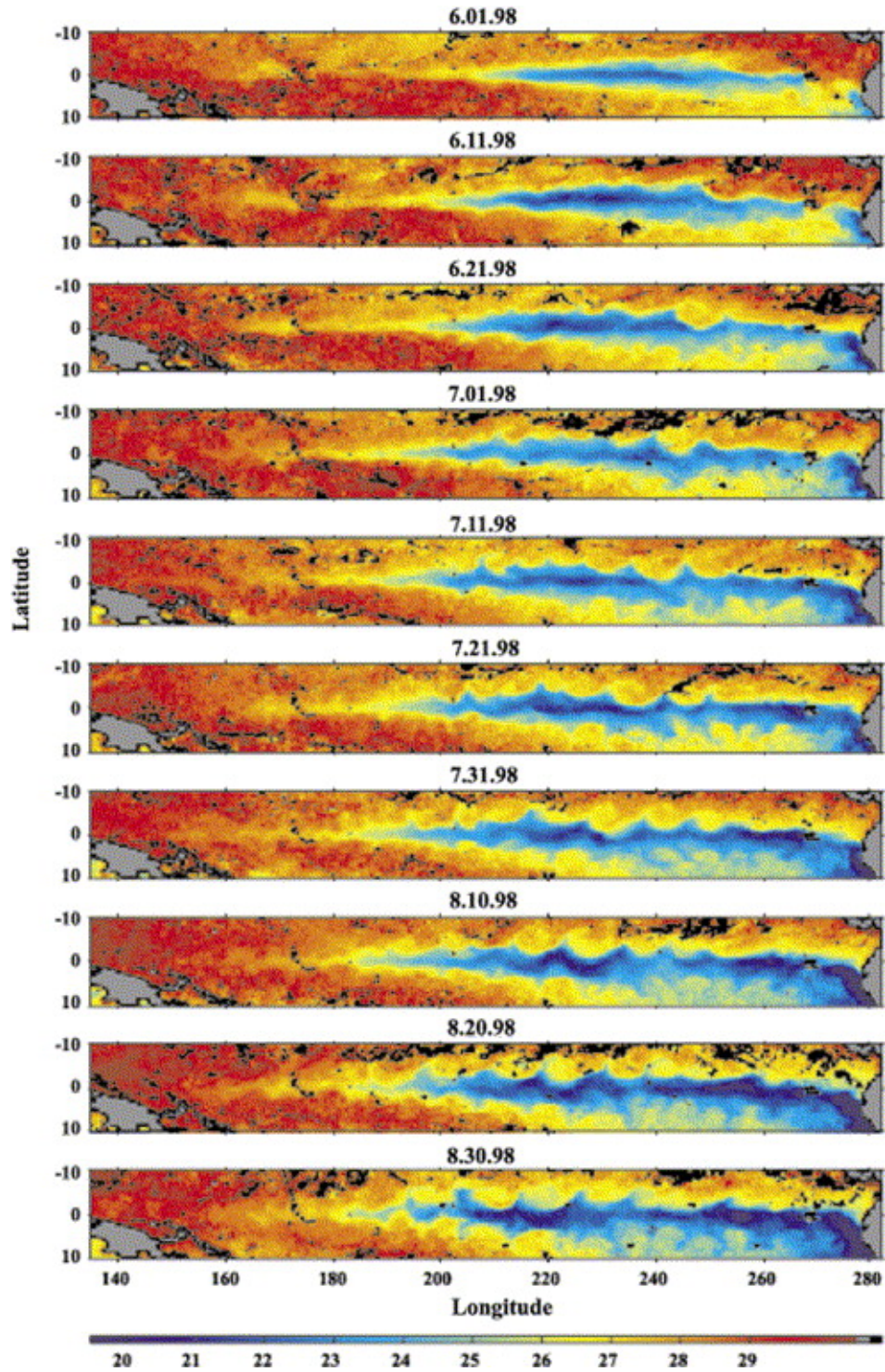


Figure 1: Sea Surface temperature signature of tropical instability waves, seen in imagery from the Tropical Rainfall Measuring Mission (TRMM) Microwave Imager, from Willett et al. (2006)

Various process studies have indicated that the waves are caused by instability in the intense zonal mean flow (for instance Philander, 1976, 1978; Proehl, 1996). Lawrence and Angell (2000) suggested that modification of the phase of TIWs by Rossby waves helps to explain some of the meridional asymmetry displayed by TIWs. More recently, it has been suggested that TIWs could be described as a linear unstable resonance between two baroclinic equatorial Rossby waves (Lyman et al., 2005). While linear analysis has offered some insights into the dynamics of TIWs, it is thought that these waves are nonlinear in nature (Philander, 1978; Kennan and Flament, 2000).

1.3 The Equatorial β -plane

The equatorial β -plane has been used extensively in the study of equatorial wave dynamics. The equatorial β -plane is not universally valid, particularly for barotropic waves (Boyd, 1985).

While the validity of using linear equations for the barotropic waves is somewhat limited, previous studies examining nonlinear resonance at the equator have avoided purely baroclinic systems. The reason stated by Reznik and Zeitlin (2007a) is that because of the rapid decay of the parabolic cylinder function (the function that describes the meridional structure of many equatorially trapped waves) for baroclinic waves, the nonlinear interaction of baroclinic waves is also small. On the other hand, the large meridional extent of barotropic waves allows for greater interaction.

2 Equations on a Sphere and the Mercator Projection

2.1 Equations on a Sphere

We begin by noting that in spherical coordinates, where θ is latitude, ϕ is longitude and r is the radial distance from the centre of the Earth, the divergence operator and gradient operators are (see chapter 4.12 of Gill, 1982),

$$\nabla \cdot \mathbf{F} \equiv \frac{1}{\cos \theta} \left\{ \frac{\partial}{\partial \phi} \left(\frac{F_\phi}{r} \right) + \frac{\partial}{\partial \theta} \left(\frac{F_\theta \cos \theta}{r} \right) + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 F_r \cos \theta) \right\} \quad (1)$$

$$\nabla \varrho \equiv \left(\frac{\partial_\phi \varrho}{r \cos \theta}, \frac{\partial_\theta \varrho}{r}, \partial_r \varrho \right), \quad (2)$$

where \mathbf{F} is some arbitrary vector and ϱ is an arbitrary scalar. As the radial length scales are much smaller than the lateral length scales, we may treat r as a constant, a_r , unless we are taking a derivative with respect to r . The divergence and gradient operators become (to a good approximation for the oceans),

$$\nabla \cdot \mathbf{F} \approx \frac{\partial_\phi F_\phi}{a_r \cos \theta} + \frac{\partial_\theta (F_\theta \cos \theta)}{a_r \cos \theta} + \partial_r F_r \quad (3)$$

$$\nabla \varrho \approx \left(\frac{\partial_\phi \varrho}{a_r \cos \theta}, \frac{\partial_\theta \varrho}{a_r}, \partial_r \varrho \right). \quad (4)$$

We can thus say that the inviscid, Boussinesq, hydrostatic lateral momentum equation in spherical coordinates is,

$$\mathbf{u}_{,t} + \mathbf{u} \cdot \nabla \mathbf{u} + \left[f_0 + \frac{u}{a_r \cos \theta} \right] \mathbf{e}_r \wedge \mathbf{u} \sin \theta = -\frac{\nabla p}{\rho}, \quad (5a)$$

where we have ignored the horizontal component of the Earth's rotation and the radial metric terms, both of which follow naturally from assuming that the radial length scale is much smaller than the lateral. Here, $\mathbf{u} = (u, v)$ is the zonal and meridional velocity and the gradient operator only operates in two dimensions in the momentum equation and \mathbf{e} indicates a unit vector. Furthermore, we note that f_0 is the value of the radial component of the Earth's rotation vector (the Coriolis parameter) at the North Pole. We also note that $\nabla \mathbf{u} \equiv (\nabla u, \nabla v)$, where ∇ is that defined in equation (2). The terms are, from left to right, the rate of change of velocity, the advective term, the Coriolis term, a metric term and the pressure gradient term. As we are considering an incompressible fluid, the continuity equation becomes,

$$\nabla \cdot (\mathbf{u}, w) = 0, \quad (5b)$$

where w is the radial velocity and the gradient operator is acting in all three directions. Written in their full form, the lateral momentum equations and the continuity equation is,

$$u_{,t} + \frac{uu_{,\phi}}{a_r \cos \theta} + \frac{vu_{,\theta}}{a_r} - vf_0 \sin \theta - \frac{uv \tan \theta}{a_r} = -\frac{p_{,\phi}}{\rho a_r \cos \theta} \quad (6a)$$

$$v_{,t} + \frac{uv_{,\phi}}{a_r \cos \theta} + \frac{vv_{,\theta}}{a_r} + uf_0 \sin \theta - \frac{u^2 \tan \theta}{a_r} = -\frac{p_{,\theta}}{\rho a_r} \quad (6b)$$

$$\frac{u_{,\phi}}{a_r \cos \theta} + \frac{(v \cos \theta)_{,\theta}}{a_r \cos \theta} + w_{,r} = 0, \quad (6c)$$

where we have used the comma notation to denote partial derivatives (see Aris, 1962).

2.2 The Mercator Projection

We now set about transforming the equations from spherical coordinates to Mercator coordinates. Firstly, we note that the unit vectors for each coordinate system are equivalent,

$$\mathbf{e}_x = \mathbf{e}_\phi \quad (7a)$$

$$\mathbf{e}_y = \mathbf{e}_\theta \quad (7b)$$

$$\mathbf{e}_z = \mathbf{e}_r, \quad (7c)$$

Here, x is the zonal distance from a reference $x = 0$ and y is the meridional distance from $y = 0$ (the equator), such that $-\infty < y < \infty$ while, $-\pi/2 < \theta < \pi/2$. In addition, z is the radial distance from the bottom of the ocean, a_r . Spherical coordinates and Mercator coordinates are thus related by,

$$x = a_r \phi \quad (8a)$$

$$y = a_r \tanh^{-1}(\sin \theta) \quad (8b)$$

$$z = r - a_r, \quad (8c)$$

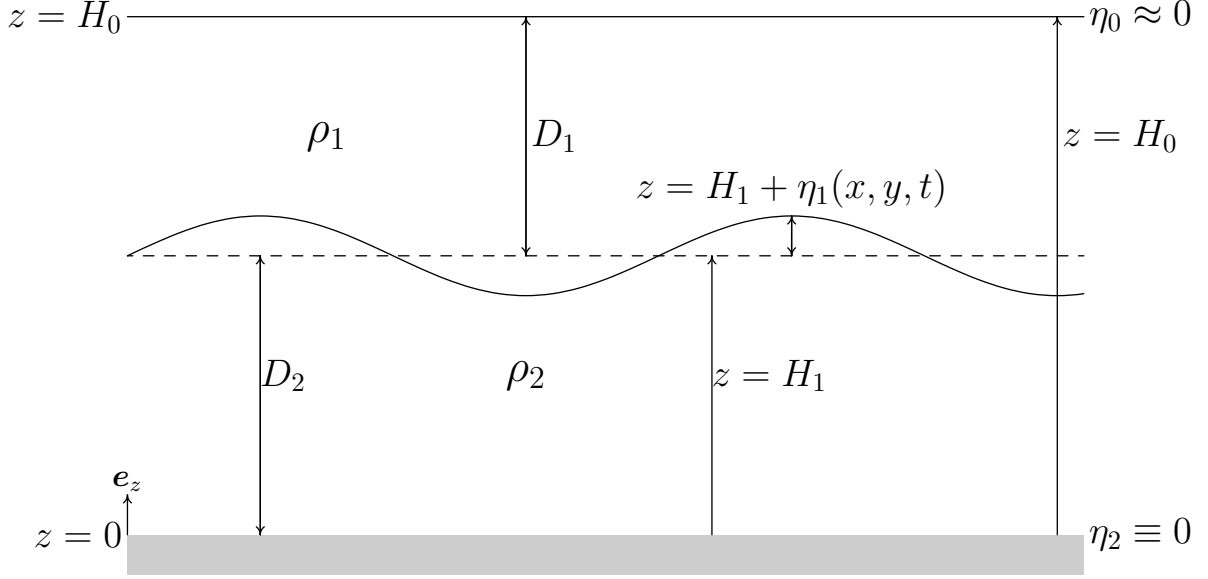


Figure 2: The shaded region is the (constant depth) ocean floor, the dashed line indicates mean sea level and the solid line represents the free surface. H is distance from mean sea level to the ocean floor, and $\eta(x, y, t)$ is the deviation of the free surface from mean sea level.

which essentially projects the sphere onto a cylinder of height $2a_r$. Applying this transform to the spherical equations (6) yields,

$$u_{,t} + \frac{uu_{,x}}{\operatorname{sech}(y/a_r)} + \frac{vu_{,y}}{\operatorname{sech}(y/a_r)} - v f_0 \tanh(y/a_r) - \frac{uv \tanh(y/a_r)}{a_r \operatorname{sech}(y/a_r)} = -\frac{p_{,x}}{\rho \operatorname{sech}(y/a_r)} \quad (9a)$$

$$v_{,t} + \frac{uv_{,x}}{\operatorname{sech}(y/a_r)} + \frac{vv_{,y}}{\operatorname{sech}(y/a_r)} + u f_0 \tanh(y/a_r) - \frac{u^2 \tanh(y/a_r)}{a_r \operatorname{sech}(y/a_r)} = -\frac{p_{,y}}{\rho \operatorname{sech}(y/a_r)} \quad (9b)$$

$$[u \operatorname{sech}(y/a_r)]_{,x} + [v \operatorname{sech}(y/a_r)]_{,y} + w_{,z} \operatorname{sech}^2(y/a_r) = 0 \quad (9c)$$

where we note from equations (8) that $x_{,\phi} = a_r$, $y_{,\theta} = \frac{a_r}{\cos \theta}$, $z_{,r} = 1 \sin \theta = \tanh(y/a_r)$ and $\cos \theta = \operatorname{sech}(y/a_r)$. If we now make the substitution $\mathcal{U} = u \operatorname{sech}(y/a_r)$ and $\mathcal{V} = v \operatorname{sech}(y/a_r)$ (which is equivalent to the $\cos \theta$ substitution made by Longuet-Higgins, 1965),

$$\operatorname{sech}^2(y/a_r) \mathcal{U}_{,t} + \mathcal{U} \mathcal{U}_{,x} + \mathcal{V} \mathcal{U}_{,y} - \operatorname{sech}^2(y/a_r) \mathcal{V} f_0 \tanh(y/a_r) = -\frac{p_{,x}}{\rho} \operatorname{sech}^2(y/a_r) \quad (10a)$$

$$\operatorname{sech}^2(y/a_r) \mathcal{V}_{,t} + \mathcal{U} \mathcal{V}_{,x} + \mathcal{V} \mathcal{V}_{,y} + \frac{(\mathcal{U}^2 + \mathcal{V}^2) \tanh(y/a_r)}{a_r} + \operatorname{sech}^2(y/a_r) \mathcal{U} f_0 \tanh(y/a_r) = -\frac{p_{,y}}{\rho} \operatorname{sech}^2(y/a_r) \quad (10b)$$

$$\mathcal{U}_{,x} + \mathcal{V}_{,y} + \operatorname{sech}^2(y/a_r) w_{,z} = 0 \quad (10c)$$

If we now examine these equations for a two layer fluid (which is illustrated in figure 2), where we label each layer with $i = 1, 2$, 1 being the top layer and 2 being the lower layer

and η_0 being the free surface. In both layers we use the hydrostatic approximation,

$$p_{,z} = -\rho g, \quad (11)$$

If we also employ the rigid lid approximation, we write the pressure for the top layer as $P = \rho_1 g \eta_0$ and we ignore the deviations of the surface elsewhere in the equations, $\eta_0 = 0$. Furthermore, linearising gives a more simple expression for the momentum equation of the top layer,

$$\mathcal{U}_{1,t} - \mathcal{V}_1 f_0 \tanh(y/a_r) = -P_{,x} \quad (12a)$$

$$\mathcal{V}_{1,t} + \mathcal{U}_1 f_0 \tanh(y/a_r) = -P_{,y}. \quad (12b)$$

In the bottom layer we make the Boussinesq approximation in addition to the same approximations to the top layer,

$$\mathcal{U}_{2,t} - \mathcal{V}_2 f_0 \tanh(y/a_r) = -(P_{,x} + g' \eta_{1,x}) \quad (12c)$$

$$\mathcal{V}_{2,t} + \mathcal{U}_2 f_0 \tanh(y/a_r) = -(P_{,y} + g' \eta_{1,y}). \quad (12d)$$

Here, the Boussinesq approximation means that we have assumed that $\rho_1/\rho_2 \approx 1$. Note that we have written $g' = g(\rho_2 - \rho_1)/\rho_2$ to represent the ‘‘reduced gravity’’. Integration of the continuity equation across each layer, assuming that the layers are immiscible, the bottom is impermeable, $\eta_2 = 0$ and that deviations of the interface between the fluids from its resting state are small $w(\eta_i) \approx \partial_t \eta_i$, we obtain the conservation equation for each layer,

$$-\text{sech}^2(y/a_r) \eta_{1,t} + (D_1 \mathcal{U}_1)_{,x} + (D_1 \mathcal{V}_1)_{,y} = 0 \quad (13a)$$

$$\text{sech}^2(y/a_r) \eta_{1,t} + (D_2 \mathcal{U}_2)_{,x} + (D_2 \mathcal{V}_2)_{,y} = 0. \quad (13b)$$

To arrive at these expressions, we note that the rigid lid approximation implies that, $\eta_{0,t} \ll \eta_{1,t}$ and $\eta_0 \ll D_1$; and the small amplitude assumption implies that $\eta_1 \ll D_1$ and $\eta_1 \ll D_2$.

It is important to note that one of the important factors in the validity of the linearization, is that the equatorial Rossby radius of deformation (Gill and Clarke, 1974) must be small when compared to the radius of the Earth, $Ro \ll a_r$. We note that this is true for baroclinic modes, whose wave speed is $0.5\text{--}3.0\text{ms}^{-1}$, however, it is only marginally true for the barotropic modes, whose wave speed is $\sim 200\text{ms}^{-1}$ (chapter 11.5 of Gill, 1982), and as such, the barotropic results must be interpreted very carefully.

If we now define a mean velocity \mathcal{U} and a shear velocity $\mathcal{R} = (\mathcal{R}, \mathcal{S})$,

$$H_0 \mathcal{U} = D_1(\mathcal{U}_1, \mathcal{V}_1) + D_2(\mathcal{U}_2, \mathcal{V}_2) \quad (14a)$$

$$\mathcal{R} = (\mathcal{U}_2 - \mathcal{U}_1, \mathcal{V}_2 - \mathcal{V}_1). \quad (14b)$$

We may combine equations (12) and (13),

$$\mathcal{U}_{,t} + f_0 \tanh(y/a_r) \mathbf{e}_z \wedge \mathcal{U} + \frac{\nabla(P + g' \eta_1)}{H_0} = 0 \quad (15a)$$

$$\mathcal{R}_{,t} + f_0 \tanh(y/a_r) \mathbf{e}_z \wedge \mathcal{R} + c^2 \nabla h = 0 \quad (15b)$$

$$\nabla \cdot \mathcal{U} = 0 \quad (15c)$$

$$\text{sech}^2(y/a_r) h_{,t} + \nabla \cdot \mathcal{R} = 0 \quad (15d)$$

where $c^2 = g'D_1D_2/H_0$, $h = H_0\eta_1/(D_1D_2)$. We note that c is the baroclinic phase speed. Employment of the rigid lid approximation implies that the mean velocity field is non-divergent, and as such, we may define a stream function, ψ ,

$$\mathcal{U} = \left(-\frac{\partial\psi}{\partial y}, \frac{\partial\psi}{\partial x} \right). \quad (16)$$

We now assume that the solutions to equations (15) are separable, having the form

$$\psi(x, y, t) = \Psi(y) [Ae^{i\varphi} + A^*e^{-i\varphi}] \quad (17a)$$

$$\mathcal{R}(x, y, t) = \Xi(y) [iAe^{i\varphi} - iA^*e^{-i\varphi}] \quad (17b)$$

$$\mathcal{S}(x, y, t) = \Phi(y) [Ae^{i\varphi} + A^*e^{-i\varphi}] \quad (17c)$$

$$h(x, y, t) = G(y) [iAe^{i\varphi} - iA^*e^{-i\varphi}], \quad (17d)$$

where $\varphi = kx - \omega t$, noting that k is the wavenumber and ω is the frequency. Here A is the amplitude of the wave and A^* is the complex conjugate of the amplitude.

Using the definition of the streamfunction, equation (16), and taking a combination of equations (15a) and (15c), we obtain an equation for the evolution of the streamfunction,

$$\nabla^2\psi_{,t} + f'(y)\psi_{,x} = 0, \quad (18)$$

where $f(y) = f_0 \tanh(y/a_r)$. We note that the barotropic equation is uncoupled from the baroclinic equations.

To find the baroclinic modal equation we may take various combinations of equations (15b) and (15d) to eliminate \mathcal{R} , and then applying the solutions in equation (17), we find two modal equations,

$$-\omega^2 \operatorname{sech}^2(y/a_r)G + kf(y)\Phi + c^2k^2G - \omega\Phi_{,y} = 0 \quad (19a)$$

$$-k\omega\Phi + f(y)\omega \operatorname{sech}^2(y/a_r)G - f(y)\Phi_{,y} - kc^2G_{,y} = 0 \quad (19b)$$

Rearranging equation (19a) in terms of $G(y)$, we get,

$$G = \frac{\omega\Phi_{,y} - kf\Phi}{c^2k^2 - \omega^2 \operatorname{sech}^2(y/a_r)}. \quad (20)$$

Differentiating with respect to y , we find

$$G_{,y} = \frac{k(f'\Phi + f\Phi_{,y}) - \omega\Phi_{,yy}}{\omega^2 \operatorname{sech}^2(y/a_r) - c^2k^2} + \frac{1}{a_r}(kf(y)\Phi - \omega\Phi_{,y}) \frac{2\omega^2 \tanh(y/a_r) \operatorname{sech}^2(y/a_r)}{(\omega^2 \operatorname{sech}^2(y/a_r) - c^2k^2)^2}. \quad (21)$$

We note that under the assumption that the wave length must be much smaller than the radius of the Earth, the second term in equation (21) will be small relative to other terms, and as such, we shall ignore it. Substitution of equations (20) and (21) into equation (19b) yields a modal equation in Φ ,

$$\Phi_{,yy} + \left\{ \operatorname{sech}^2(y/a_r) \left[\frac{\omega^2}{c^2} - \frac{f^2(y)}{c^2} \right] - k^2 - \frac{kf'(y)}{\omega} \right\} \Phi = 0. \quad (22)$$

We wish to nondimensionalize the barotropic and baroclinic modal equations by introducing the length and time scales,

$$L_* = \left(\frac{c}{\beta} \right)^{\frac{1}{2}} \quad (23a)$$

$$T_* = \frac{1}{\beta L_*}. \quad (23b)$$

where we have noted that the equatorial Rossby radius of deformation is an appropriate length scale. Associated with these length scales are the dimensionless variables,

$$\omega = \frac{\tilde{\omega}}{T_*} \quad (24a)$$

$$k = \frac{\tilde{k}}{L_*} \quad (24b)$$

$$f_0 = \beta L_* \tilde{f}_0 \quad (24c)$$

We also note that as we are considering the case close to the equator (see chapter 3.17 of Pedlosky, 1987),

$$a_r \approx \frac{f_0}{\beta}. \quad (25)$$

Substitution into equation (18) for the barotropic case and equation (22) for the baroclinic case yield, respectively,

$$\Psi_{,\tilde{y}\tilde{y}} - \left\{ \tilde{k}^2 + \frac{\tilde{k}\tilde{f}'(\tilde{y})}{\tilde{\omega}} \right\} \Psi = 0 \quad (26a)$$

$$\Phi_{,\tilde{y}\tilde{y}} + \left\{ \text{sech}^2(\tilde{y}/\tilde{f}_0) \left[\tilde{\omega}^2 - \tilde{f}(\tilde{y}) \right] - \tilde{k}^2 - \frac{\tilde{k}\tilde{f}'(\tilde{y})}{\tilde{\omega}} \right\} \Phi = 0. \quad (26b)$$

Using the Mercator projection as a motivation for a tanh profile for the Coriolis parameter, we examine the more familiar two layer shallow water equations in geopotential coordinates.

3 The β -Plane and “Extended” β -Plane

We begin by stating the inviscid, linear shallow water equations for a two layer fluid in geopotential coordinates,

$$\mathbf{u}_{i,t} + f(y)\mathbf{e}_z \wedge \mathbf{u}_i + \frac{\nabla p_i}{\rho_i} = 0 \quad (27)$$

$$\eta_{i-1,t} - \eta_{i,t} + \nabla \cdot ([D_i + \eta_{i-1} - \eta_i]\mathbf{u}_i) = 0, \quad (28)$$

for $i = 1, 2$. Here, the pressure, p_i , is given by,

$$p_1 = P \quad (29a)$$

$$p_2 = P + g'\eta_1. \quad (29b)$$

Similarly with section 2.2 we employ the rigid lid approximation, which implies that $P = g\rho_1\eta_0$ but that $\eta_0 = 0$ anywhere else it appears in an equation. We also utilise the Boussinesq approximation, assuming that $\rho_1/\rho_2 \approx 1$.

Again, as in section 2.2, we define a mean velocity, $\mathbf{U} = (U, V)$ and a shear velocity, $\mathbf{R} = (R, S)$,

$$H_0\mathbf{U} = D_1\mathbf{u}_1 + D_2\mathbf{u}_2 \quad (30a)$$

$$\mathbf{R} = \mathbf{u}_2 - \mathbf{u}_1. \quad (30b)$$

Applying these definitions to the momentum equation (27), and continuity equation (28) in conjunction with the rigid lid approximation and small amplitude assumption gives,

$$\mathbf{U}_{,t} + f(y)\mathbf{e}_z \wedge \mathbf{U} + \nabla Q = 0 \quad (31a)$$

$$\mathbf{R}_{,t} + f(y)\mathbf{e}_z \wedge \mathbf{R} + c^2\nabla h = 0 \quad (31b)$$

$$\nabla \cdot \mathbf{U} = 0 \quad (31c)$$

$$h_{,t} + \nabla \cdot \mathbf{R} = 0, \quad (31d)$$

where, $Q = P + g'\eta_1$, $h = H_0\eta_1/(D_1D_2)$ and $c^2 = g'D_1D_2/H_0$. We may take various combinations of equations (31) and assume solutions of the same form as in equations (17) to get a barotropic equation for Ψ ,

$$\Psi_{,yy} - \left\{ k^2 + \frac{kf'(y)}{\omega} \right\} \Psi = 0 \quad (32)$$

and a baroclinic equation for Φ

$$\Phi_{,yy} + \left\{ \frac{\omega^2}{c^2} - \frac{f^2(y)}{c^2} - k^2 - \frac{kf'(y)}{\omega} \right\} \Phi = 0. \quad (33)$$

Through the use of equations (31), we can find expressions for the modal functions for Ξ and G ,

$$G(y) = \frac{\omega\Phi_{,y} - kf(y)\Phi}{c^2k^2 - \omega^2} \quad (34a)$$

$$\Xi(y) = \frac{c^2k\Phi_{,y} - \omega f(y)\Phi}{c^2k^2 - \omega^2}. \quad (34b)$$

We may non-dimensionalize the modal equations (32) and (33) using the definitions in equations (23),

$$\Psi_{,\tilde{y}\tilde{y}} - \left\{ \tilde{k}^2 + \frac{\tilde{f}'(\tilde{y})\tilde{k}}{\tilde{\omega}} \right\} \Psi = 0 \quad (35)$$

$$\Phi_{,\tilde{y}\tilde{y}} + \left\{ \tilde{\omega}^2 - \tilde{f}(\tilde{y})^2 - \tilde{k}^2 - \frac{\tilde{k}\tilde{f}'(\tilde{y})}{\tilde{\omega}} \right\} \Phi = 0. \quad (36)$$

Furthermore, using the scaling

$$G = \frac{\tilde{G}}{c}, \quad (37)$$

we may non-dimensionalize the modal expressions for h and R ,

$$\tilde{G}(\tilde{y}) = \frac{\tilde{\omega}\Phi_{,\tilde{y}} - \tilde{k}\tilde{f}(\tilde{y})\Phi}{\tilde{k}^2 - \tilde{\omega}^2} \quad (38a)$$

$$\Xi(y) = \frac{\tilde{k}\Phi_{,\tilde{y}} - \tilde{\omega}\tilde{f}(\tilde{y})\Phi}{\tilde{k}^2 - \tilde{\omega}^2}. \quad (38b)$$

The traditional β -plane approximation locally approximates the effects of the curvature of the Earth by assuming a plane surface (thus ignoring the geometric effects), but allows the coriolis parameter to vary linearly with latitude. The traditional equatorial β -plane is generally written as $f(y) = \beta y$. In dimensionless form the coriolis parameter is $\tilde{f}(\tilde{y}) = \tilde{y}$, $\tilde{f}'(\tilde{y}) = 1$. We can see that in this case, we may assume a solution of the form $\tilde{\psi} = e^{i(\tilde{k}\tilde{x} + \tilde{l}\tilde{y} - \tilde{\omega}\tilde{t})}$, which gives a modal equation,

$$\tilde{\omega} = \frac{-\tilde{k}}{\tilde{k}^2 + \tilde{l}^2}. \quad (39)$$

Importantly, the modal equation for the β -plane has no trapped wave solutions, unlike the modal equation (26a) for the Mercator projection. We note also that with the β -plane approximation, the solution for the meridional modal function is a parabolic cylinder function (see Miller, 1965), yielding a dispersion relation of the form,

$$\tilde{\omega}_m^3 - (\tilde{k}^2 + 2m + 1)\tilde{\omega}_m - \tilde{k} = 0 \quad m = 0, 1, 2, \dots \quad (40)$$

Both equations are common in the literature and specifically, are in agreement with Reznik and Zeitlin (2006, 2007a,b), as is equations (38) when the β -plane approximation is substituted. We further note that there is an additional solution for the trapped waves, corresponding to $\Phi \equiv 0$, which yields a dispersion relation of $\tilde{\omega} = \tilde{k}$. This solution is the equatorial Kelvin wave (Matsuno, 1966).

Another special case of equation (40) is that when $m = 0$. In this instance, there are three branches. One is spurious, yielding the unphysical relation $\tilde{\omega} = -\tilde{k}$, which is unbounded. The other two branches are described by $\tilde{\omega}(\tilde{\omega} - \tilde{k}) = 1$. This dispersion relation corresponds to the Yanai wave (Matsuno, 1966).

In addition to the traditional β -plane approximation, there is also the δ -plane approximation (Yang, 1987) which is a second order Taylor expansion of the Coriolis parameter. Our aim in the present study is to use the full expression for the coriolis parameter, but, to ignore the metric terms (and thus, ignore geometric effects).

4 Meridional Structure and Dispersion Relations

We begin by deriving the properties of the system discussed in section 3.

4.1 Solving for the Meridional structure

Taking the barotropic modal equation (35), and using our chosen coriolis profile, $\tilde{f}(\tilde{y}) = \tanh(\tilde{Y})$, where $\tilde{Y} = \tilde{y}/\tilde{f}_0$, we find that the modal equation becomes,

$$\Psi_{,\tilde{Y}\tilde{Y}} + \{\alpha_{\text{T}} + \gamma_{\text{T}}(1 - \lambda^2)\} \Psi = 0 \quad (41)$$

where, $\lambda = \tanh(\tilde{Y})$ and

$$\alpha_{\text{T}} = -\tilde{k}^2 \tilde{f}_0^2 \quad (42a)$$

$$\gamma_{\text{T}} = -\tilde{f}_0^2 \tilde{k}/\tilde{\omega}. \quad (42b)$$

If we use the chain rule, $\Psi_{,\tilde{Y}} = \Psi_{,\lambda} \lambda_{,\tilde{Y}}$, noting that $\lambda_{,\tilde{Y}} = 1 - \lambda^2$, we transform equation (41) to,

$$\{(1 - \lambda^2)\Psi_{,\lambda}\}_{,\lambda} + \left(\gamma_{\text{T}} + \frac{\alpha_{\text{T}}}{1 - \lambda^2} \right) \Psi = 0. \quad (43)$$

Using the same procedure, the baroclinic modal equation (36) becomes,

$$\{(1 - \lambda^2)\Phi_{,\lambda}\}_{,\lambda} + \left(\gamma_{\text{C}} + \frac{\alpha_{\text{C}}}{1 - \lambda^2} \right) \Phi = 0, \quad (44)$$

where,

$$\alpha_{\text{C}} = \tilde{f}_0^2 (-\tilde{f}_0^2 - \tilde{k}^2 + \tilde{\omega}^2) \quad (45a)$$

$$\gamma_{\text{C}} = -\tilde{f}_0^2 (\tilde{f}_0^2 + \tilde{k}/\tilde{\omega}), \quad (45b)$$

noting that the subscript T indicates barotropic and subscript C indicates baroclinic. Note that subsequent statements regarding α and γ that do not explicitly indicate whether it is discussing the barotropic or baroclinic case are equally valid for the barotropic and baroclinic variables.

With the chosen form of the coriolis parameter, $\tilde{f}(\tilde{y})$, we now have the same differential equation for the barotropic and baroclinic equations, which are second order ordinary differential equations, namely the associated Legendre equation.

In the case where $\alpha < 0$ we have discrete eigenvalues given by (Drazin and Johnson, 1989)

$$\alpha_m = - \left\{ (\gamma + 1/4)^{\frac{1}{2}} - (m + 1/2) \right\}^2 \quad m = 0, 1, 2, \dots, N. \quad (46)$$

We note that from its definition, equation (42a), that α_{T} is always less than zero, and as such the barotropic modes are always trapped. On the other hand, α_{C} is not necessarily zero, and it has trapped modes only when $\tilde{\omega}^2 < \tilde{f}_0^2 + \tilde{k}^2$ (see section 4.3).

The solutions for the barotropic case are proportional to associated Legendre functions (see, for example Stegun, 1965),

$$\Psi(\lambda) = C_{\Psi} P_{\nu}^{\mu}(\lambda) + D_{\Psi} Q_{\nu}^{\mu}(\lambda), \quad (47)$$

where $P_\nu^\mu(\lambda)$ and $Q_\nu^\mu(\lambda)$ are the associated Legendre functions of the first and second kinds respectively. C_Ψ and D_Ψ are constants of integration. Noting that the associated Legendre function of the second kind, $Q_\nu^\mu(\lambda)$, goes unbounded as $\lambda \rightarrow 1$, we thus say $D_\Psi = 0$, thus leaving,

$$\Psi(\lambda) = C_\Psi P_\nu^\mu(\lambda). \quad (48)$$

Analogously for $\Phi(\tilde{Y})$,

$$\Phi(\lambda) = C_\Phi P_\nu^\mu(\lambda). \quad (49)$$

The integer N is determined by the criterion in the bracket $\{\cdot\}$, on the right hand side of equation (46), that is, $N = [(\gamma + 1/4)^{\frac{1}{2}} - 1/2] + 1$. In this instance the square brackets $[\cdot]$ denote the integral part, which take a real number greater than zero as an argument and rounds it down to the nearest integer. If the integral part is already an integer, then the $+1$ is omitted from the expression. The eigenfunctions for the associated Legendre functions, equations (48) and (49), are given by $\mu = (\gamma + 1/4)^{\frac{1}{2}} - (m + 1/2)$ and exist if there is a root, $\nu(\nu + 1) = \gamma$.

We note from equation (42b) that for $m > 0$, $\gamma_T > 0 \forall \tilde{k}$. On the other hand, we note from equation (45b) that γ_C is not necessarily greater than zero. This point shall be elaborated upon in section 4.3.

4.2 Barotropic dispersion relations

Substitution of α_T and γ_T into the eigenvalue relation, equation (46), gives the barotropic dispersion relation,

$$\tilde{\omega} = -\frac{\tilde{k}\tilde{f}_0^2}{\tilde{k}^2\tilde{f}_0^2 + |\tilde{k}|\tilde{f}_0(m + \frac{1}{2}) + m(m + 1)}. \quad (50)$$

The first four modes for the barotropic waves are plotted in figure 3. We can see that they attain a maximum value in frequency-wavenumber space. For reasons that will become clear later, we wish to find the maximum frequency and its relative size compared to \tilde{f}_0 .

We note that the maximum occurs at a turning point in wavenumber-frequency space. As such, in order to find the maximum frequency, we need to find where the gradient of the dispersion relation, equation (50), is zero (which, incidentally, corresponds to a zero group velocity). Thus, the maximum value of $\tilde{\omega}$ occurs when,

$$\tilde{k}\tilde{f}_0 = -\sqrt{m(m + 1)}, \quad (51)$$

where we have chosen the negative root for \tilde{k} based on the anti-symmetric nature of $\tilde{\omega}$.

Substitution of equation (51) into the dispersion relation gives the maximum frequency, $\tilde{\omega}_{\max}$, as a function of \tilde{f}_0 . A more useful measure, however, is the ratio of the maximum frequency to \tilde{f}_0 ,

$$\frac{\tilde{\omega}_{\max}}{\tilde{f}_0} = \frac{1}{2\sqrt{m(m + 1)} + (m + \frac{1}{2})} \quad (52)$$

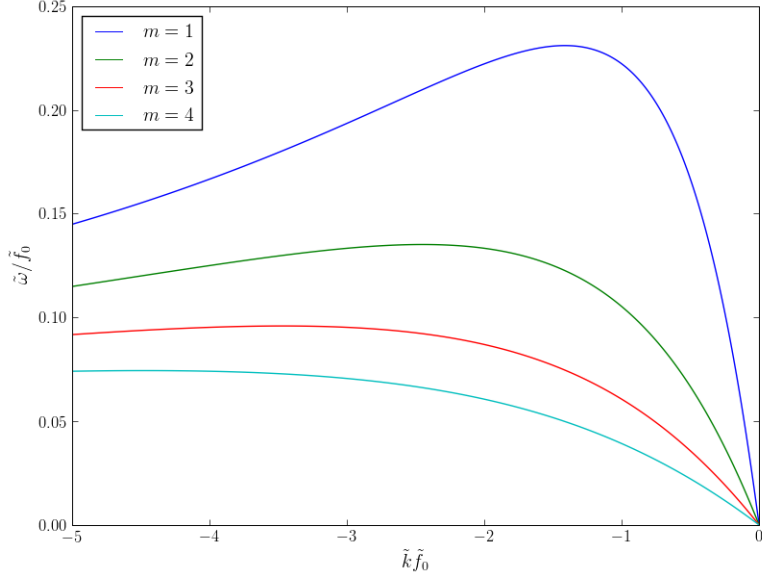


Figure 3: The first four meridional modes for the barotropic dispersion relation, equation (50). Note, that \tilde{f}_0 has been scaled out of the plot by making $\tilde{\omega}/\tilde{f}_0$ the dependent variable and $\tilde{k}\tilde{f}_0$ the independent variable.

We can thus see that ratio of the maximum frequency for barotropic modes to \tilde{f}_0 remains constant and that the ratio $\tilde{\omega}_{\max}/\tilde{f}_0$ decreases with increasing mode number.

Interestingly, we note through the use of the symmetry condition, $\tilde{k}(\omega) = -\tilde{k}(-\tilde{\omega})$, that the phase speed for the modes considered must be negative (i.e. westward).

4.3 Baroclinic dispersion relation

For the baroclinic case, we firstly consider the case for $\alpha_C < 0$, necessitating that $-k/\omega > f_0^2$, which, similarly to the barotropic case, indicates that the phase speed is negative and is also bounded in absolute value. Furthermore, by the definition of α_C , we also require that,

$$\omega^2 - f_0^2 < k^2, \quad (53)$$

The discrete dispersion relation – from equation (46) – is therefore given by,

$$\tilde{k}^2 + \tilde{f}_0^2 - \tilde{\omega}^2 = \left\{ \left(\tilde{f}_0^2 - \frac{\tilde{k}}{\tilde{\omega}} + \frac{1}{4\tilde{f}_0^2} \right)^{\frac{1}{2}} - \frac{m+1/2}{\tilde{f}_0} \right\}^2. \quad (54)$$

As $\tilde{f}_0 \rightarrow \infty$, we see that equation (54) reduces to the dispersion relation of Reznik and Zeitlin (2006, 2007a,b), equation (40) when we expand for large \tilde{f}_0 . When equation (53) is not satisfied, this implies that there is a continuous spectrum in wavenumber-frequency

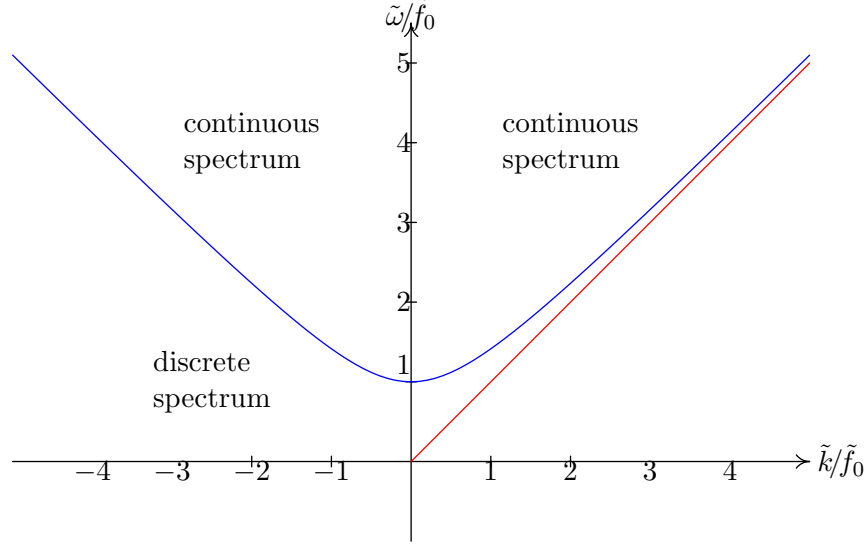


Figure 4: The blue line is $\tilde{\omega}/\tilde{f}_0 = \sqrt{\tilde{k}^2/\tilde{f}_0^2} + 1$, which deliniates the continuous spectrum of frequency-wavenumber solutions from the discrete spectrum and the red line is $\tilde{\omega} = \tilde{k}$ which is the dispersion relation for equatorial Kelvin waves.

space.

Solutions for the discrete eigenvalues of the barotropic and baroclinic modes may all be found using associated Legendre functions. More general solutions to the associated Legendre equation can be found with the use of hypergeometric functions (see, for instance Oberhettinger, 1965). Again, following Drazin and Johnson (1989), we find that the solution for equation (43) is given by,

$$\Phi(\tilde{Y}) = a2^{ip}[\text{sech}(\tilde{Y})]^{-ip}F(\check{a}, \check{b}; \check{c}; [\tanh(\tilde{Y})]/2), \quad (55)$$

where $a = a(\alpha_c)$ is a constant of integration, $p^2 = \alpha_c$, $\check{a} = \frac{1}{2} - ip + (\gamma_c + \frac{1}{4})^{\frac{1}{2}}$, $\check{b} = \frac{1}{2} - ik - (\gamma_c + \frac{1}{4})^{\frac{1}{2}}$, $\check{c} = 1 - ip$ and F is the hypergeometric function. Specifically, we note that for this type of function, the solutions for $\alpha_c > 0$ yields a continuous spectrum for the baroclinic modes.

Similarly to the equatorial β -plane, we get a solution for $\Phi \equiv 0$, which again corresponds to an equatorial Kelvin wave, of the form $G(y) = \text{sech}^{f_0^2}(Y)$. Again similarly to the β -plane, $m = 0$ has a spurious branch for $\tilde{\omega} = -\tilde{k}$ and the remaining solution is the Yanai wave. Finally, the modes that correspond to $m = 1, 2, \dots$ are the trapped low-frequency Rossby waves. We can see that the high frequency branch of the dispersion relation for $m = 1, 2, \dots$ actually lie within the continuous spectrum. Figure 4 shows the regions in frequency-wavenumber space where discrete and continuous solutions exist.

4.4 Resonance Conditions

We wish to investigate two different classes of interaction. The first is where we have three waves that lie within the discrete spectrum, and the second is where we have two waves in the discrete spectrum interacting with a third wave that lies in the continuous spectrum (recall, baroclinic waves that lie in the continuous spectrum satisfy the relation $\tilde{\omega}^2 > \tilde{k}^2 + \tilde{f}_0^2$). Most of the discussion in the remainder of this section pertains to the latter scenario, as there are some additional constraints that the waves from the discrete spectrum must satisfy in order to form part of a resonant triad.

Resonant triads (that is, the non-linear interaction between three waves) have resonance conditions of the form,

$$k_1 + k_2 + k_3 = 0 \quad (56a)$$

$$\omega_1 + \omega_2 + \omega_3 = 0. \quad (56b)$$

If we consider the case where one of the waves in the triad lies in the continuous spectrum, we note that the frequency of the continuous spectrum has a minimum value of \tilde{f}_0 , we need to investigate the circumstances under which the resonance conditions will be satisfied with two waves from the discrete spectrum (which we label waves 1 and 2), and one from the continuous spectrum (which we label wave 3). A necessary (but not sufficient), condition for resonance to occur is for $\tilde{\omega}_{1\max} + \tilde{\omega}_{2\max} > \tilde{f}_0$, where $\tilde{\omega}_{\max}$ is the maximum frequency of a particular mode. As can be seen from equation (52), a combination of any two barotropic Rossby waves ($m \geq 1$) will not be able to satisfy this condition. However, if a Rossby wave is interacting with a Yanai wave ($m = 0$) or Kelvin wave ($m = -1$), then there may be circumstances where this condition may be satisfied.

The operations required to obtain an equivalent expression relating the maximum frequency to \tilde{f}_0 for the baroclinic dispersion relation, equation (54) requires more subtle analysis. We begin by taking the long wave limit, $\tilde{k} \rightarrow 0$. We define the dimensionless phase speed as $\tilde{c} = \tilde{\omega}/\tilde{k}$. We will examine two different limits,

- $\tilde{\omega} \rightarrow 0$ as $\tilde{c} \rightarrow \tilde{c}_{(0)}$ (the long Rossby limit), and
- $\tilde{c} \rightarrow 0$ as $\tilde{\omega} \rightarrow \tilde{\omega}_{(0)}$ (the long inertia-gravity limit);

The long Rossby limiting case yields

$$\tilde{f}_0 = \left(\tilde{f}_0^2 - \frac{1}{\tilde{c}_{(0)}} + \frac{1}{4\tilde{f}_0^2} \right)^{\frac{1}{2}} - \frac{m + \frac{1}{2}}{\tilde{f}_0} \quad (57a)$$

$$\Rightarrow \lim_{\tilde{\omega} \rightarrow 0} \tilde{c}_{(0)} = \frac{-\tilde{f}_0^2}{\tilde{f}_0^2(2m + 1) + m(m + 1)}. \quad (57b)$$

The equation for $\tilde{c}_{(0)}$ tells us that the long wave speed is negative for all modes $m \geq 0$ and positive for $m = -1$. Furthermore, the phase speed decreases in absolute value with increasing mode number, m .

The second case to consider yields the equation,

$$\sqrt{\tilde{f}_0^2 - (\tilde{\omega}_{(0)})^2} = \left(\tilde{f}_0^2 + \frac{1}{4\tilde{f}_0^2} \right)^{\frac{1}{2}} - \frac{m + \frac{1}{2}}{\tilde{f}_0} \quad (58a)$$

$$\Rightarrow \lim_{\tilde{c} \rightarrow 0} (\tilde{\omega}_{(0)})^2 = (2m + 1) \left(1 + \frac{1}{4\tilde{f}_0^4} \right)^{\frac{1}{2}} - \frac{(2m + 1)^2 + 1}{4\tilde{f}_0^2}. \quad (58b)$$

The left hand side of equation (58a) tells us that the frequency must always be such that $0 < \tilde{\omega}_{(0)} < \tilde{f}_0$. To satisfy this requirement, the right hand side of equation (58a) must be

$$\begin{aligned} (m + \frac{1}{2})\tilde{f}_0^{-2} &< \sqrt{\tilde{f}_0^2 + (4\tilde{f}_0^2)^{-1}} \\ \Rightarrow m(m + 1) &< \tilde{f}_0^4, \end{aligned} \quad (59)$$

in the long wave limit. We can see from equation (59) that the Yanai wave has a solution for all $\tilde{f}_0 > 0$. On the other hand, the inertia-gravity waves (with $m > 0$ and $\tilde{\omega}_{(0)} > 0$) are only trapped for $\tilde{k} = 0$ if \tilde{f}_0^4 is large enough. A corollary of this is that for some finite \tilde{f}_0 only low mode waves are trapped.

We also wish to examine whether any trapped modes can intercept the continuous spectrum. We determine this by recognising that the left hand side of the dispersion relation, equation (54), is also the boundary between the continuous and discrete spectra (the blue curve in figure 4). Equating each side to zero yields intersection conditions,

$$\tilde{\omega}^2 = \tilde{k}^2 + \tilde{f}_0^2 \quad (60)$$

$$\frac{\tilde{k}}{\tilde{\omega}} = -\frac{m(m + 1)}{\tilde{f}_0^2} + \tilde{f}_0^2. \quad (61)$$

If we set $m = 0$ to search for a Yanai wave, we note that the two curves intercept at $\tilde{k}_0^2 = \tilde{f}_0^6(1 - \tilde{f}_0^4)^{-1}$, where \tilde{k}_0 is the wavenumber of intersection for the $m = 0$ wave. Thus, the dispersion relation for a Yanai wave intercepts the boundary between the continuous and discrete spectra at \tilde{k}_0 when $0 < \tilde{f}_0 < 1$, having a discrete dispersion relation for $-\infty < \tilde{k} < \tilde{k}_0$ and a continuous spectrum for $\tilde{k} > \tilde{k}_0$. Conversely for $\tilde{f}_0 > 1$ they do not intercept the Yanai wave has a discrete dispersion relation for $-\infty < \tilde{k} < \infty$.

Using similar reasoning, we find that inertia-gravity waves cross from discrete relations to continuous if $\{m(m + 1)\}^{\frac{1}{2}} < \tilde{f}_0^2 < m + 1$, there is a solution $\tilde{k} = \tilde{k}_m > 0$ so that the trapped (discrete) baroclinic modes exist only for $-\infty < \tilde{k} < \tilde{k}_m$ and that $\tilde{k}_0 > \tilde{k}_1 > \tilde{k}_2 > \dots$. Otherwise, if $\tilde{f}_0^2 > m + 1$, the trapped waves exist for all \tilde{k} . If, however, $m < \tilde{f}_0^2 < \{m(m + 1)\}^{\frac{1}{2}}$, there is a solution such that $k = \tilde{k}_m < 0$ where the inertia gravity wave crosses from the discrete spectrum to the continuous spectrum. Otherwise if $\tilde{f}_0^2 < m$, there are no trapped inertia-gravity modes.

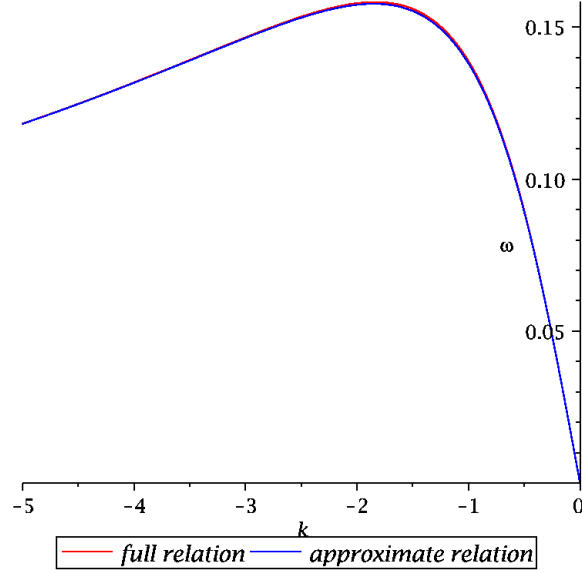


Figure 5: The red line is a plot of the full baroclinic dispersion relation, equation (54) and the blue line is the low frequency approximate dispersion relation, equation (62). The values used were $m = 1$ and $\tilde{f}_0 = 1$, and was made using the implicit plotting capabilities of Maple.

On the other hand, baroclinic Rossby waves remain trapped for all \tilde{k} , and as such, we may justifiably use a low frequency approximation, i.e. that the $\tilde{\omega}^2$ terms are small compared with the other terms in equation (54). This yields a dispersion relation,

$$\tilde{\omega} \approx \frac{-\tilde{k}\tilde{f}_0^2}{\tilde{k}^2\tilde{f}_0^2 + \tilde{f}_0(2m+1)\sqrt{\tilde{k}^2 + \tilde{f}_0^2} + m(m+1)}. \quad (62)$$

Figure 5 shows a plot of the full discrete dispersion relation, equation (54), and the approximate low frequency dispersion relation for $m = 1$ and $f_0 = 1$. The difference between the two curves is quite small, and is of similar magnitude for higher modes. This confirms that the low frequency relation is an appropriate approximation to use for this study. A similarly small error is present for the first five trapped Rossby modes are plotted in figure 6 using the low frequency approximation.

As in section 4.2 we wish to ascertain under what circumstances we may obtain triad interactions between the discrete and continuous spectra. As we have a good approximate dispersion relation for Rossby waves that is explicit in $\tilde{\omega}$, we again begin by trying to find the maximum frequency as a function of \tilde{f}_0 , we differentiate equation (62),

$$\frac{d\omega}{d\tilde{k}} = -\frac{\tilde{f}_0}{\tilde{k}^2\tilde{f}_0^2 + \tilde{f}_0(2m+1)\sqrt{\tilde{k}^2 + \tilde{f}_0^2} + m(m+1)} + \tilde{k}\tilde{f}_0 \frac{2\tilde{k}\tilde{f}_0^2 + (\tilde{f}_0\tilde{k}(2m+1))(\tilde{k}^2 + \tilde{f}_0^2)^{-1}}{[\tilde{k}^2\tilde{f}_0^2 + \tilde{f}_0(2m+1)\sqrt{\tilde{k}^2 + \tilde{f}_0^2} + m(m+1)]^2}. \quad (63)$$

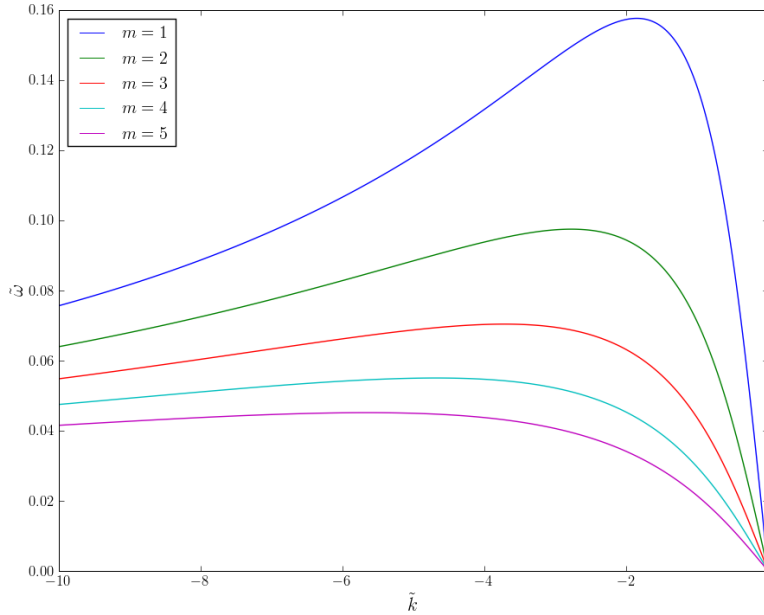


Figure 6: A dispersion relation for the first five low baroclinic frequency, trapped modes with $\tilde{f}_0 = 1$.

Setting the equation to zero and isolating $\tilde{k} = \tilde{k}_{max}$, and substituting back into the dispersion relation allows us to plot the ratio of the maximum frequency, $\tilde{\omega}_{max}$, to \tilde{f}_0 as well as the value of \tilde{k} where this occurs, \tilde{k}_{max} , as a function of \tilde{f}_0 . Such a plot is shown in figure 7 for $m = 1$.

We note that for both the barotropic case (see section 4.2) and the baroclinic case, we are unable to find resonant triads consisting of two discrete Rossby waves and one wave from the continuous spectrum. Again, it is necessary to use at least one Yanai or Kelvin wave in order to satisfy the resonance conditions, equations (56). The remainder of this report shall focus on weakly nonlinear resonant interactions between three discrete baroclinic Rossby waves. The properties of interactions involving Yanai or Kelvin waves, as well as the continuous spectrum are potential avenues of further research.

5 Resonant Triads

We now wish to examine “near-linear theory,” for waves of moderately small amplitude. Small amplitude expansions to linear wave theory allow us to examine the interactions, and energy sharing between different Fourier components (see chapter 14 of Whitham, 1974).

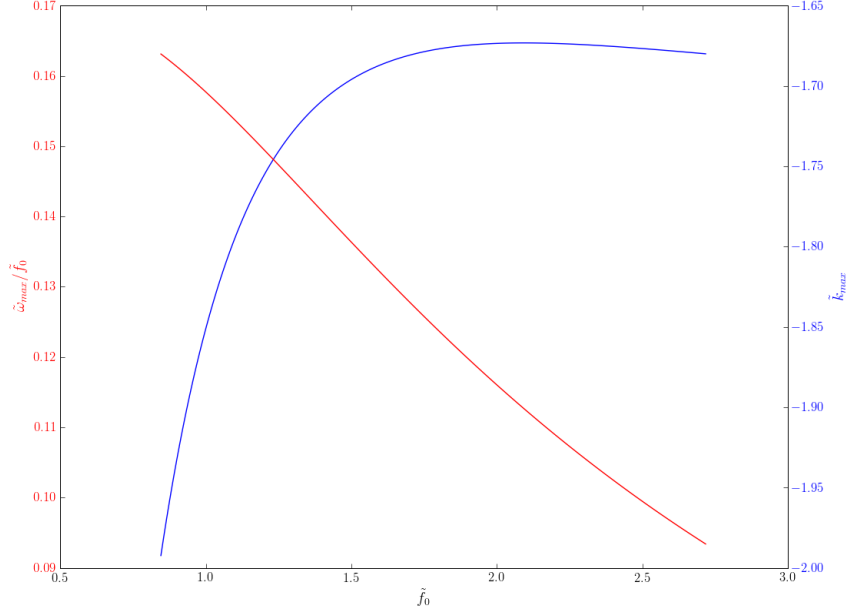


Figure 7: The ratio of the value of \tilde{k} where the maximum frequency occurs (blue line and right axis) and the value of the ratio of the maximum value of $\tilde{\omega}$ to \tilde{f}_0 (red line and left axis), as a function of \tilde{f}_0 , using the low frequency approximation for the dispersion relation, equation (62).

5.1 Nonlinear Equations

We need to examine a non-linear model to identify the important terms for the wave interactions. We use the baroclinic momentum and continuity equations (3.6a-d) of Benilov and Reznik (1996), which are modified here for an extended $f(y)$ profile,

$$\begin{aligned} \nabla^2 \psi_{,t} + J(\psi, \nabla^2 \psi) + \frac{D_1 D_2}{H_0^2} (\partial_{xx} - \partial_{yy}) \left[\left(1 + \frac{D_1 - D_2}{H_0} h - \frac{D_1 D_2}{H_0^2} h^2 \right) RS \right] \\ - \frac{D_1 D_2}{H_0^2} \partial_{xy} \left[\left(1 + \frac{D_1 - D_2}{H_0} h - \frac{D_1 - D_2}{H_0^2} h^2 \right) (R^2 - S^2) \right] + f'(y) \psi_{,x} = 0 \end{aligned} \quad (64a)$$

$$\begin{aligned} \mathbf{R}_{,t} + J(\psi, \mathbf{R}) - \mathbf{R} \cdot \nabla (\mathbf{e}_z \wedge \nabla \psi) \\ - \frac{2D_1 D_2}{H_0^2} h \mathbf{R} \cdot \nabla \mathbf{R} - \frac{D_1 D_2}{H_0^2} \mathbf{R} \mathbf{R} \cdot \nabla h + c^2 \nabla h = -f(y) \mathbf{e}_z \wedge \mathbf{R} \end{aligned} \quad (64b)$$

$$h_{,t} + J(\psi, h) + \nabla \cdot \mathbf{R} + ((D_1 - D_2) H_0^{-1} - D_1 D_2 H_0^{-2}) \nabla \cdot (h \mathbf{R}) - \nabla \cdot (h^2 \mathbf{R}) = 0 \quad (64c)$$

where $J(A, B) = A_{,x} B_{,y} - A_{,y} B_{,x}$ is the Jacobi operator. We also recall that $h = H_0 \eta_1 / D_1 D_2$ and $c^2 = g' D_1 D_2 H_0^{-1}$. From these equations, we obtain the non-dimensional equations using

the length and time scales defined in equations (23),

$$\begin{aligned} \tilde{\nabla}^2 \tilde{\psi}_{,\tilde{t}} + \tilde{f}'(\tilde{y}) \tilde{\psi}_{,\tilde{x}} = & -J(\tilde{\psi}, \tilde{\nabla}^2 \tilde{\psi}) - \varpi(\partial_{\tilde{x}\tilde{x}} - \partial_{\tilde{y}\tilde{y}})[(1 + q\tilde{h} - \varpi\tilde{h}^2)\tilde{R}\tilde{S}] \\ & - \partial_{\tilde{x}\tilde{y}}[(1 + q\tilde{h} - \varpi\tilde{h}^2)(\tilde{R}^2 - \tilde{S}^2)] \end{aligned} \quad (65a)$$

$$\begin{aligned} \tilde{\mathbf{R}}_{,\tilde{t}} + \tilde{\nabla}\tilde{h} + \tilde{f}(\tilde{y})\mathbf{e}_z \wedge \tilde{\mathbf{R}} = & -J(\tilde{\psi}, \tilde{\mathbf{R}}) + \tilde{\mathbf{R}} \cdot \tilde{\nabla}(\mathbf{e}_z \wedge \nabla\tilde{\psi}) - \varpi q \tilde{\mathbf{R}} \cdot \tilde{\nabla}\tilde{\mathbf{R}} \\ & + \varpi [2\tilde{h}\tilde{\mathbf{R}} \cdot \tilde{\nabla}\tilde{\mathbf{R}} + \tilde{\mathbf{R}}\tilde{\mathbf{R}} \cdot \tilde{\nabla}\tilde{h}] \end{aligned} \quad (65b)$$

$$\tilde{h}_{,\tilde{t}} + \nabla \cdot \tilde{\mathbf{R}} = -J(\tilde{\psi}, \tilde{h}) - \tilde{\nabla} \cdot (\tilde{h}\tilde{\mathbf{R}}) + \varpi \tilde{\nabla} \cdot (\tilde{h}^2 \tilde{\mathbf{R}}), \quad (65c)$$

where, $\varpi = \tilde{D}_1 \tilde{D}_2 / \tilde{H}_0^2$ and $q = (\tilde{D}_1 - \tilde{D}_2) / \tilde{H}_0$.

5.2 Solutions

Let us now consider an asymptotic, multitimescale solution (Bretherton, 1964) to the above equations, such that,

$$\tilde{\psi} = \epsilon^1 \tilde{\psi}^{(1)}(\tilde{x}, \tilde{y}, \tilde{t}, \tilde{\tau}) + \epsilon^2 \tilde{\psi}^{(2)}(\tilde{x}, \tilde{y}, \tilde{t}, \tilde{\tau}) + \dots \quad (66a)$$

$$\tilde{\mathbf{R}} = \epsilon^1 \tilde{\mathbf{R}}^{(1)}(\tilde{x}, \tilde{y}, \tilde{t}, \tilde{\tau}) + \epsilon^2 \tilde{\mathbf{R}}^{(2)}(\tilde{x}, \tilde{y}, \tilde{t}, \tilde{\tau}) + \dots \quad (66b)$$

$$\tilde{S} = \epsilon^1 \tilde{S}^{(1)}(\tilde{x}, \tilde{y}, \tilde{t}, \tilde{\tau}) + \epsilon^2 \tilde{S}^{(2)}(\tilde{x}, \tilde{y}, \tilde{t}, \tilde{\tau}) + \dots \quad (66c)$$

$$\tilde{h} = \epsilon^1 \tilde{h}^{(1)}(\tilde{x}, \tilde{y}, \tilde{t}, \tilde{\tau}) + \epsilon^2 \tilde{h}^{(2)}(\tilde{x}, \tilde{y}, \tilde{t}, \tilde{\tau}) + \dots, \quad (66d)$$

where, ϵ is a small parameter and $\tilde{\tau} = \epsilon \tilde{t}$ is a slow time variable. Recalling equations (17), we assume that the solutions take the form,

$$\tilde{\psi}^{(1)}(\tilde{x}, \tilde{y}, \tilde{t}, \tilde{\tau}) = \sum_{j=1}^3 \Psi_j^{(1)}(\tilde{y}) [A_j(\tilde{\tau})e^{i\tilde{\varphi}_j} + A_j^*(\tilde{\tau})e^{-i\tilde{\varphi}_j}] \quad (67a)$$

$$\tilde{\mathbf{R}}^{(1)}(\tilde{x}, \tilde{y}, \tilde{t}, \tilde{\tau}) = \sum_{j=1}^3 \Xi_j^{(1)}(\tilde{y}) [iA_j(\tilde{\tau})e^{i\tilde{\varphi}_j} - iA_j^*(\tilde{\tau})e^{-i\tilde{\varphi}_j}] \quad (67b)$$

$$\tilde{S}^{(1)}(\tilde{x}, \tilde{y}, \tilde{t}, \tilde{\tau}) = \sum_{j=1}^3 \Phi_j^{(1)}(\tilde{y}) [A_j(\tilde{\tau})e^{i\tilde{\varphi}_j} + A_j^*(\tilde{\tau})e^{-i\tilde{\varphi}_j}] \quad (67c)$$

$$\tilde{h}^{(1)}(\tilde{x}, \tilde{y}, \tilde{t}, \tilde{\tau}) = \sum_{j=1}^3 \tilde{G}_j^{(1)}(\tilde{y}) [iA_j(\tilde{\tau})e^{i\tilde{\varphi}_j} - iA_j^*(\tilde{\tau})e^{-i\tilde{\varphi}_j}], \quad (67d)$$

where a superscripted asterisk, *, indicates a complex conjugate, $\tilde{\varphi} \equiv \tilde{k}\tilde{x} - \tilde{\omega}\tilde{t}$ and $j = 1, 2, 3$ identifies the wave in the triad. Here, we have assumed that the amplitude of the waves is slowly varying. We substitute the first two orders of our expansion into equations (65a) to (65c) and equate powers of ϵ (Luke, 1966). For the barotropic equations, we equate terms that are multiplied by ϵ ,

$$\begin{aligned} \epsilon \tilde{\nabla}^2 \tilde{\psi}_{,\tilde{t}}^{(1)} + \epsilon \tilde{f}'(\tilde{y}) \tilde{\psi}_{,\tilde{x}}^{(1)} &= 0 \\ \tilde{\nabla}^2 \tilde{\psi}_{,\tilde{t}}^{(1)} + \tilde{f}'(\tilde{y}) \tilde{\psi}_{,\tilde{x}}^{(1)} &= 0, \end{aligned} \quad (68)$$

which may be recognised as the linear equation (18). Doing the same for \tilde{h} and $\tilde{\mathbf{R}}$ yields,

$$\tilde{h}_{,\tilde{t}}^{(1)} + \nabla \cdot \tilde{\mathbf{R}}^{(1)} = 0 \quad (69a)$$

$$\tilde{\mathbf{R}}_{,\tilde{t}}^{(1)} + \tilde{\nabla} \tilde{h}^{(1)} + \tilde{f}(\tilde{y}) \mathbf{e}_z \wedge \tilde{\mathbf{R}}^{(1)} = 0 \quad (69b)$$

which can be recognised as the linear equations (31b) and (31d) respectively.

Similarly, we may equate terms that are multiplied by ϵ^2 ,

$$\tilde{\nabla}^2 \tilde{\psi}_{,\tilde{t}}^{(2)} + \tilde{f}'(\tilde{y}) \tilde{\psi}_{,\tilde{x}}^{(2)} = -\tilde{\nabla}^2 \tilde{\psi}_{,\tilde{\tau}}^{(1)} + N^\psi \quad (70a)$$

$$\tilde{\mathbf{R}}_{,\tilde{t}}^{(2)} + \tilde{h}_{,\tilde{x}}^{(2)} - \tilde{f}(\tilde{y}) \tilde{\mathcal{S}}^{(2)} = -\tilde{\mathbf{R}}_{,\tilde{\tau}}^{(1)} + N^R \quad (70b)$$

$$\tilde{\mathcal{S}}_{,\tilde{t}}^{(2)} + \tilde{h}_{,\tilde{y}}^{(2)} + \tilde{f}(\tilde{y}) \tilde{R}^{(2)} = -\tilde{\mathcal{S}}_{,\tilde{\tau}}^{(1)} + N^S \quad (70c)$$

$$\tilde{h}_{,\tilde{t}}^{(2)} + \nabla \cdot \tilde{\mathbf{R}}^{(2)} = -\tilde{h}_{,\tilde{\tau}}^{(1)} + N^h \quad (70d)$$

where we note that via use of the chain rule $\tilde{\nabla}^2 \tilde{\psi}_{,\tilde{\tau}}^{(1)} \tau_{,\tilde{t}} = \tilde{\nabla}^2 \tilde{\psi}_{,\tilde{\tau}}^{(1)} \epsilon$, and similarly for $\tilde{\mathbf{R}}_{,\tilde{\tau}}^{(1)}$, $\tilde{\mathcal{S}}_{,\tilde{\tau}}^{(1)}$ and $\tilde{h}_{,\tilde{\tau}}^{(1)}$. Here, the nonlinear terms N^ψ , N^R and N^h are given by,

$$N^\psi = -J(\tilde{\psi}^{(1)}, \tilde{\nabla}^2 \tilde{\psi}^{(1)}) - \varpi(\partial_{\tilde{x}\tilde{x}} - \partial_{\tilde{y}\tilde{y}}) \tilde{\mathbf{R}}^{(1)} \tilde{\mathcal{S}}^{(1)} - \partial_{\tilde{x}\tilde{y}} (\tilde{\mathbf{R}}^{(1)2} - \tilde{\mathcal{S}}^{(1)2}) \quad (71a)$$

$$N^R = -J(\tilde{\psi}^{(1)}, \tilde{\mathbf{R}}^{(1)}) + \tilde{\mathbf{R}}^{(1)} \cdot \tilde{\nabla} \tilde{\psi}_{,\tilde{x}}^{(1)} - \varpi q \tilde{\mathbf{R}}^{(1)} \cdot \tilde{\nabla} \tilde{\mathbf{R}}^{(1)} \quad (71b)$$

$$N^S = -J(\tilde{\psi}^{(1)}, \tilde{\mathcal{S}}^{(1)}) + \tilde{\mathbf{R}}^{(1)} \cdot \tilde{\nabla} \tilde{\psi}_{,\tilde{y}}^{(1)} - \varpi q \tilde{\mathbf{R}}^{(1)} \cdot \tilde{\nabla} \tilde{\mathcal{S}}^{(1)} \quad (71c)$$

$$N^h = -J(\tilde{\psi}^{(1)}, \tilde{h}^{(1)}) - \tilde{\nabla} \cdot (\tilde{h}^{(1)} \tilde{\mathbf{R}}^{(1)}) \quad (71d)$$

If we apply the resonance conditions, equations (56), to these nonlinear equations, we note that both the left hand side and the right hand sides have terms that are proportional $\exp(i\tilde{\varphi}_j)$. For example, if we apply the resonance condition for the $j = 1$ wave, then terms that are not discarded are those that are proportional to $\exp(i\tilde{\varphi}_1)$ and $\exp(-i[\tilde{\varphi}_2 + \tilde{\varphi}_3])$. In order to find a compatibility condition we substitute our assumed barotropic streamfunction solution, equation (67a) into equation (70a) to yield,

$$i\tilde{\omega}_j \Psi_{j,\tilde{y}\tilde{y}}^{(2)} + (\tilde{k}_j \tilde{f}' - \tilde{\omega}_j \tilde{k}_j^2) i\Psi_j^{(2)} = \mathcal{F}_j^\Psi, \quad (72)$$

where \mathcal{F}_j^Ψ may thought of as a forcing function arising from the nonlinear wave interactions and is given by,

$$\mathcal{F}_j^\Psi = (\tilde{k}_j^2 \Psi_j^{(1)} - \Psi_{j,\tilde{y}\tilde{y}}^{(1)}) A_{j,\tilde{\tau}} + \mathcal{N}_j^\psi, \quad (73)$$

in which \mathcal{N}_j^ψ are the non-linear terms from equation (71a) that are proportional to $\exp(i\tilde{\varphi}_j)$.

Similarly, combining equations (70b) to (70d), we find a forced modal equation for the second order effects,

$$i\tilde{\omega}_j \Phi_{j,\tilde{y}\tilde{y}}^{(2)} + \left\{ -\tilde{\omega}_j \tilde{f}^2 - \tilde{\omega}_j \tilde{k}_j^2 + \tilde{\omega}_j^3 - \tilde{k}_j \tilde{f}' \right\} i\Phi_j^{(2)} = \mathcal{F}_j^\Phi, \quad (74)$$

where \mathcal{F}_j^Φ is the baroclinic forcing term due to interaction with other waves and is given by,

$$\begin{aligned}\mathcal{F}_j^\Phi = & \Phi_j^{(1)}(\tilde{\omega}_j^2 - \tilde{k}_j^2)A_{j,\tilde{\tau}} + (\partial_{\tilde{t}\tilde{t}} - \partial_{\tilde{x}\tilde{x}})\mathcal{N}_j^S A_2^* A_3^* \\ & + (\tilde{k}_j \Xi_{j,\tilde{y}}^{(1)} + \tilde{f} \tilde{\omega}_j \Xi_j^{(1)})A_{j,\tilde{\tau}} + (\partial_{\tilde{x}\tilde{y}} - \tilde{f} \partial_{\tilde{t}})\mathcal{N}_j^R A_2^* A_3^* \\ & + (\tilde{f} \tilde{k}_j \tilde{G}_j^{(1)} + \tilde{\omega}_j \tilde{G}_{j,\tilde{y}}^{(1)})A_{j,\tilde{\tau}} + (\tilde{f} \partial_{\tilde{x}} - \partial_{\tilde{t}\tilde{y}})\mathcal{N}_j^h A_2^* A_3^*,\end{aligned}\quad (75)$$

where \mathcal{N}^R , \mathcal{N}^S and \mathcal{N}^h are the nonlinear terms that are proportional to $\exp(i\tilde{\varphi}_j)$.

Equations (72) and (74) have compatibility conditions,

$$\int_{-\infty}^{\infty} \mathcal{F}_j^\Psi \Psi_j d\tilde{y} = 0 \quad (76a)$$

$$\int_{-\infty}^{\infty} \mathcal{F}_j^\Phi \Phi_j d\tilde{y} = 0, \quad (76b)$$

where Ψ_j and Φ_j are functions that satisfy the modal equations (43) and (44), respectively. Although there appear to be two compatibility conditions, there is indeed only one case required, as for a barotropic mode $\Phi_j(\tilde{y}) \equiv 0$ and for the baroclinic case, $\Psi_j(\tilde{y}) \equiv 0$.

In the next section, we examine form of \mathcal{N} for three trapped Rossby waves, which will eventually allow us to find the equations for the evolution of the wave amplitude.

5.3 Triad Interactions of Three Baroclinic Rossby Waves

We wish to examine the interaction of three baroclinic waves. Firstly, we use the graphical technique (for example, Simmons, 1969) to illustrate the possibility of resonant interactions. The approximate baroclinic Rossby wave dispersion relation, equation (62), is plotted in figure 8. The origin of the $m = 1, 2, 4, 5$ waves (blue curves) is translated to an arbitrary point along the $m = 3$ baroclinic wave (the red line) while $m = 1, 2, 4, 5$ are also plotted in black. As can be seen, there are many possible intersections for just the first five modes, and indeed further possibilities may be found by translating the origin of the blue lines to a different place on the red line, or, considering a different mode to $m = 3$, or indeed, considering more modes than just the first five.

We note that in the purely baroclinic case, there will be no barotropic terms, as $\tilde{\psi}^{(1)} \equiv 0$. This allows us to expand the nonlinear terms in the forcing function, equation (75). To illustrate, we begin by considering the wave labelled 1. We examine the non-linear terms, equation (71), paying particular attention to those terms that are proportional to $\exp(-i[\tilde{\varphi}_2 + \tilde{\varphi}_3])$,

$$N_1^R = i\varpi q \left(\tilde{k}_1 \Xi_2^{(1)} \Xi_3^{(1)} + \Phi_2^{(1)} \Xi_{3,\tilde{y}}^{(1)} + \Xi_{2,\tilde{y}}^{(1)} \Phi_3^{(1)} \right) e^{-i(\tilde{\varphi}_2 + \tilde{\varphi}_3)} + \dots \quad (77a)$$

$$N_1^S = \varpi q \left(\tilde{k}_2 \Phi_2^{(1)} \Xi_3^{(1)} + \tilde{k}_3 \Xi_2^{(1)} \Phi_3^{(1)} - \partial_{\tilde{y}}[\Phi_2^{(1)} \Phi_3^{(1)}] \right) e^{-i(\tilde{\varphi}_2 + \tilde{\varphi}_3)} + \dots \quad (77b)$$

$$N_1^h = i \left(\partial_{\tilde{y}}[\Phi_2 \tilde{G}_3 - \tilde{G}_2 \Phi_3] + \tilde{k}_1 [\Xi_2 \tilde{G}_3 - \tilde{G}_2 \Xi_3] \right) e^{-i(\tilde{\varphi}_2 + \tilde{\varphi}_3)} + \dots, \quad (77c)$$

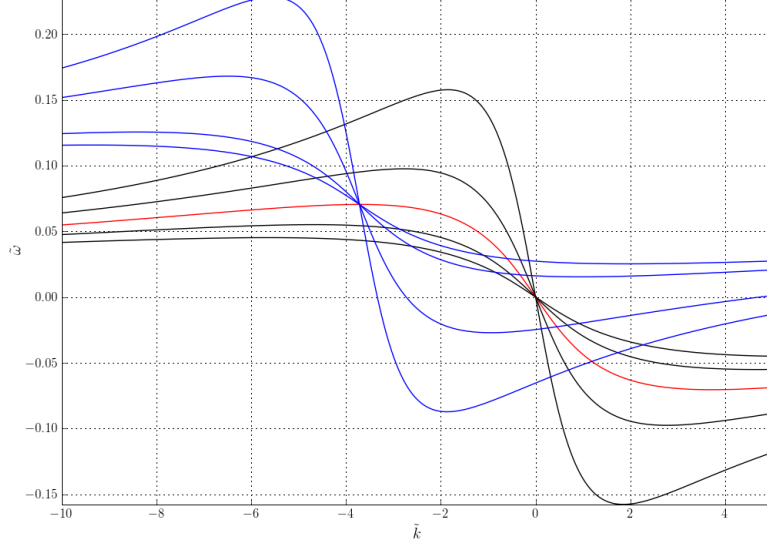


Figure 8: A rich array of three wave resonant interactions can be found in the system under consideration. The approximate low frequency dispersion relation for baroclinic Rossby waves with $\tilde{f}_0 = 1$ is plotted for $m = 1, 2, 4, 5$ (black lines) and $m = 3$ (red line). The origin for the $m = 1, 2, 4, 5$ waves is translated along the $m = 3$ relation to an arbitrary point. Intersection of the blue lines with black lines indicate resonant triads.

where ... represents terms that are not proportional to $e^{-i\tilde{\varphi}_1}$, and are thus ignored. The terms shown are \mathcal{N}_1^R , \mathcal{N}_1^S and \mathcal{N}_1^h in equation (75) for the forcing term. In order to find the forcing term, we need to operate on the nonlinear terms,

$$\mathcal{N}_1^R = \varpi q (k_1 \partial_{\tilde{y}} + \tilde{f} \tilde{\omega}_1) \left(\tilde{k}_1 \Xi_2^{(1)} \Xi_3^{(1)} + \Phi_2^{(1)} \Xi_{3,\tilde{y}}^{(1)} + \Xi_{2,\tilde{y}}^{(1)} \Phi^{(1)} \right) \quad (78a)$$

$$\mathcal{N}_1^S = \varpi q (\tilde{\omega}_1^2 - \tilde{k}_1^2) \left(\tilde{k}_2 \Phi_2^{(1)} \Xi_3^{(1)} + \tilde{k}_3 \Xi_2^{(1)} \Phi_3^{(1)} - \partial_{\tilde{y}} [\Phi_2^{(1)} \Phi_3^{(1)}] \right) \quad (78b)$$

$$\mathcal{N}_1^h = (\tilde{\omega}_1 \partial_{\tilde{y}} + \tilde{f} \tilde{k}_1) \left(\partial_{\tilde{y}} [\Phi_2 \tilde{G}_3 - \tilde{G}_2 \Phi_3] + \tilde{k}_1 [\Xi_2 \tilde{G}_3 - \tilde{G}_2 \Xi_3] \right) \quad (78c)$$

where

$$\mathcal{N}_1^R = -(\partial_{\tilde{x}\tilde{y}} - \tilde{f} \partial_{\tilde{t}}) \mathcal{N}_1^R, \quad \mathcal{N}_1^S = -(\partial_{\tilde{t}\tilde{t}} - \partial_{\tilde{x}\tilde{x}}) \mathcal{N}_1^S, \quad \mathcal{N}_1^h = -(\tilde{f} \partial_{\tilde{x}} - \partial_{\tilde{t}\tilde{y}}) \mathcal{N}_1^h.$$

We can see that the equivalent expressions for \mathcal{N}_2 and \mathcal{N}_3 can be found by swapping indicies.

As the governing equation for the nonlinear interactions, equation (74), is a forced modal equation, we use the compatibility condition – noting that $\Phi_1^{(1)}$ satisfies the modal equation (44) – to yield an equation for the time evolution of the amplitude,

$$\begin{aligned} A_{1,\tilde{\tau}} \int_{-\infty}^{\infty} (\tilde{\omega}_1^2 - \tilde{k}_1^2) \left(\Phi_1^{(1)} \right)^2 + \Phi_1^{(1)} (\tilde{f} \tilde{\omega}_1 + \tilde{k}_1 \partial_{\tilde{y}}) \Xi_1^{(1)} + \Phi_1^{(1)} (\tilde{f} \tilde{k}_1 + \tilde{\omega}_1 \partial_{\tilde{y}}) \tilde{G}_1^{(1)} d\tilde{y} \\ = A_2^* A_3^* \int_{-\infty}^{\infty} \Phi_1^{(1)} \mathcal{N}_1^R + \Phi_1^{(1)} \mathcal{N}_1^S + \Phi_1^{(1)} \mathcal{N}_1^h d\tilde{y}. \end{aligned} \quad (79)$$

Before being able to solve such an equation, we use the relations \tilde{G} and Ξ in terms of Φ , equations (38). It is known that Φ is an associated Legendre function, equation (49). Substituting, we find that,

$$\begin{aligned} & (\tilde{\omega}_1^2 - \tilde{k}_1^2) (\Phi_1)^2 + \Phi_1(\tilde{f}\tilde{\omega}_1 + \tilde{k}_1\partial_{\tilde{y}})\Xi_1 + \Phi_1(\tilde{f}\tilde{k}_1 + \tilde{\omega}_1\partial_{\tilde{y}})\tilde{G}_1 \\ &= \Phi_1\Phi_{1,\tilde{y}\tilde{y}}\frac{\tilde{k}_1^2 + \tilde{\omega}_1^2}{\tilde{k}_1^2 - \tilde{\omega}_1^2} - \Phi_1(\Phi_1\tilde{f})_{,\tilde{y}}\frac{2\tilde{k}_1\tilde{\omega}_1}{\tilde{k}_1^2 - \tilde{\omega}_1^2} - \Phi_1^2\tilde{f}^2\frac{\tilde{k}_1^2 + \tilde{\omega}_1^2}{\tilde{k}_1^2 - \tilde{\omega}_1^2} + \tilde{f}\Phi_{1,\tilde{y}}^2\frac{\tilde{k}_1\tilde{\omega}_1}{\tilde{k}_1^2 - \tilde{\omega}_1^2} - \Phi_1^2(\tilde{k}_1^2 - \tilde{\omega}_1^2), \end{aligned} \quad (80)$$

where we have dropped the superscript, (1), for brevity, as all modal functions are first order. In order to garner information about the time evolution of the amplitude of the waves, we need to find out information about the integrals in equation (79). We begin with the left hand side. Firstly we note that from the approximate baroclinic dispersion relation, equation (62), $|\tilde{k}| > |\tilde{\omega}|$ for $|\tilde{k}| > 0$ and as such, $\tilde{k}^2 - \tilde{\omega}^2 > 0$. We also note from the dispersion relation that $\tilde{\omega}\tilde{k} < 0$ for $|\tilde{k}| > 0$.

We use integration by parts and the property that $\Phi(\tilde{y}) \rightarrow 0$ as $\tilde{y} \rightarrow \pm\infty$, to note the following,

$$\begin{aligned} \int_{-\infty}^{\infty} \Phi\Phi_{,\tilde{y}\tilde{y}}d\tilde{y} &= \Phi\Phi_{,\tilde{y}}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \Phi^2_{,\tilde{y}}d\tilde{y} \\ &= - \int_{-\infty}^{\infty} \Phi^2_{,\tilde{y}}d\tilde{y} \end{aligned} \quad (81a)$$

$$\begin{aligned} \frac{1}{2} \int_{-\infty}^{\infty} \tilde{f}\Phi^2_{,\tilde{y}}d\tilde{y} &= \frac{1}{2} \left(\tilde{f}\Phi^2\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \Phi^2\tilde{f}'d\tilde{y} \right) \\ &= -\frac{1}{2} \int_{-\infty}^{\infty} \Phi^2\tilde{f}'d\tilde{y} \end{aligned} \quad (81b)$$

$$\begin{aligned} \int_{-\infty}^{\infty} \Phi(\Phi\tilde{f})_{,\tilde{y}}d\tilde{y} &= \Phi^2\tilde{f}\Big|_{-\infty}^{\infty} - \frac{1}{2} \int_{-\infty}^{\infty} \tilde{f}\Phi^2_{,\tilde{y}}d\tilde{y} \\ &= \frac{1}{2} \int_{-\infty}^{\infty} \Phi^2\tilde{f}'d\tilde{y}, \end{aligned} \quad (81c)$$

We say that Υ_1 is equivalent to equation (80) and use the results of equations (81) to obtain,

$$\int_{-\infty}^{\infty} \Upsilon_1d\tilde{y} = -\frac{1}{\tilde{k}_1^2 - \tilde{\omega}_1^2} \int_{-\infty}^{\infty} (\tilde{k}_1^2 + \tilde{\omega}_1^2)\Phi^2_{,\tilde{y}} + \frac{3\tilde{k}_1\tilde{\omega}_1}{2}\tilde{f}'\Phi_1^2 + \Phi_1^2\tilde{f}^2(\tilde{k}_1^2 + \tilde{\omega}_1^2) + \Phi_1^2(\tilde{k}_1^2 - \tilde{\omega}_1^2)^2d\tilde{y}. \quad (82)$$

We furthermore note that one property of the associated Legendre function is that it is either symmetric or anti-symmetric. In the present case it is the meridional mode number, m , that determines whether the modal function is symmetric or anti-symmetric – see equation (46). We also note that $\tilde{f}(\tilde{y})$ is an anti-symmetric function and that $\tilde{f}'(\tilde{y}) > 0$ is a positive, symmetric function. Using these properties we find the following for the various terms in

equations (80) and (81),

$$\int_{-\infty}^{\infty} \Phi_{,\tilde{y}}^2 d\tilde{y} > 0 \quad (83a)$$

$$\int_{-\infty}^{\infty} \Phi^2 \tilde{f}' d\tilde{y} > 0 \quad (83b)$$

$$\int_{-\infty}^{\infty} \tilde{f}^2 \Phi^2 d\tilde{y} > 0 \quad (83c)$$

$$\int_{-\infty}^{\infty} \Phi^2 d\tilde{y} > 0, \quad (83d)$$

Use of these results tells us that the integral of Υ_1 , equation (82) is negative.

With respect to the right hand side of equation (79) we find, after much algebra, that,

$$\begin{aligned} \Phi_1 \mathcal{N}_1^R = & -\varpi q \left\{ \frac{\tilde{k}_1 \Phi_1 (\tilde{\omega}_1 \tilde{f} + \tilde{k}_1 \partial_{\tilde{y}}) \left[\tilde{k}_2 \tilde{k}_3 \Phi_{2,\tilde{y}} \Phi_{3,\tilde{y}} - \tilde{k}_2 \tilde{\omega}_3 \tilde{f} \Phi_{2,\tilde{y}} \Phi_3 - \tilde{\omega}_2 \tilde{k}_3 \tilde{f} \Phi_2 \Phi_{3,\tilde{y}} + \tilde{\omega}_2 \tilde{\omega}_3 \tilde{f}^2 \Phi_2 \Phi_3 \right]}{(\tilde{k}_2^2 - \tilde{\omega}_2^2)(\tilde{k}_3^2 - \tilde{\omega}_3^2)} \right. \\ & + \Phi_1 \frac{\tilde{k}_1 \tilde{k}_2 (\Phi_{2,\tilde{y}\tilde{y}} \Phi_3)_{,\tilde{y}} - \tilde{k}_1 \tilde{\omega}_2 [(\Phi_2 \tilde{f}' \Phi_3)_{,\tilde{y}} + (\tilde{f} \Phi_{2,\tilde{y}} \Phi_3)_{,\tilde{y}}] + \tilde{\omega}_1 \tilde{f} [\tilde{k}_2 \Phi_{2,\tilde{y}\tilde{y}} \Phi_3 - \tilde{\omega}_2 \Phi_3 (\tilde{f} \Phi_2)_{,\tilde{y}}]}{\tilde{k}_2^2 - \tilde{\omega}_2^2} \\ & \left. + \Phi_1 \frac{\tilde{k}_1 \tilde{k}_3 (\Phi_2 \Phi_{3,\tilde{y}\tilde{y}})_{,\tilde{y}} - \tilde{k}_1 \tilde{\omega}_3 [(\Phi_2 \tilde{f}' \Phi_3)_{,\tilde{y}} + (\tilde{f} \Phi_2 \Phi_{3,\tilde{y}})_{,\tilde{y}}] + \tilde{\omega}_1 \tilde{f} [\tilde{k}_3 \Phi_2 \Phi_{3,\tilde{y}\tilde{y}} - \tilde{\omega}_3 \Phi_2 (\tilde{f} \Phi_3)_{,\tilde{y}}]}{\tilde{k}_3^2 - \tilde{\omega}_3^2} \right\} \quad (84a) \end{aligned}$$

$$\begin{aligned} \Phi_1 \mathcal{N}_1^S = & \varpi q (\tilde{k}_1^2 - \tilde{\omega}_1^2) \left\{ \Phi_1 (\Phi_2 \Phi_3)_{,\tilde{y}} - \tilde{k}_2 \tilde{k}_3 \left(\frac{\Phi_1 \Phi_{2,\tilde{y}} \Phi_3}{\tilde{k}_2^2 - \tilde{\omega}_2^2} + \frac{\Phi_1 \Phi_2 \Phi_{3,\tilde{y}}}{\tilde{k}_3^2 - \tilde{\omega}_3^2} \right) \right. \\ & \left. + \tilde{f} \Phi_1 \Phi_2 \Phi_3 \left(\frac{\tilde{k}_3 \tilde{\omega}_2}{\tilde{k}_2^2 - \tilde{\omega}_2^2} + \frac{\tilde{k}_2 \tilde{\omega}_3}{\tilde{k}_3^2 - \tilde{\omega}_3^2} \right) \right\} \quad (84b) \end{aligned}$$

$$\begin{aligned} \Phi_1 \mathcal{N}_1^h = & \frac{\Phi_1 (\tilde{f} \tilde{k}_1 + \tilde{\omega}_1 \partial_{\tilde{y}}) [\tilde{k}_3 (\tilde{f} \Phi_2 \Phi_3)_{,\tilde{y}} - \tilde{\omega}_3 (\Phi_2 \Phi_{3,\tilde{y}})_{,\tilde{y}}]}{\tilde{k}_3^2 - \tilde{\omega}_3^2} \\ & - \frac{\Phi_1 (\tilde{f} \tilde{k}_1 + \tilde{\omega}_1 \partial_{\tilde{y}}) [\tilde{k}_2 (\tilde{f} \Phi_2 \Phi_3)_{,\tilde{y}} - \tilde{\omega}_2 (\Phi_{2,\tilde{y}} \Phi_3)_{,\tilde{y}}]}{\tilde{k}_2^2 - \tilde{\omega}_2^2} \\ & + \frac{\Phi_1 \tilde{k}_1 (\tilde{f} \tilde{k}_1 + \tilde{\omega}_1 \partial_{\tilde{y}}) [(\tilde{k}_2 \tilde{k}_3 - \tilde{\omega}_2 \tilde{\omega}_3) (\tilde{f} \Phi_{2,\tilde{y}} \Phi_3 - \tilde{f} \Phi_2 \Phi_{3,\tilde{y}}) + (\tilde{k}_2 \tilde{\omega}_3 - \tilde{k}_3 \tilde{\omega}_2) (\tilde{f}^2 \Phi_2 \Phi_3 - \Phi_{2,\tilde{y}} \Phi_{3,\tilde{y}})]}{(\tilde{k}_2^2 - \tilde{\omega}_2^2)(\tilde{k}_3^2 - \tilde{\omega}_3^2)}. \quad (84c) \end{aligned}$$

The sign of the integrals of equation (84) are not as easy to determine as that for equation (82). We can simplify these equations, again through the use of integration by parts and the property of the modal functions that $\Phi \rightarrow 0$ as $\tilde{y} \rightarrow \pm\infty$,

$$\int_{-\infty}^{\infty} \Phi_1 \partial_{\tilde{y}} F d\tilde{y} = - \int_{-\infty}^{\infty} \Phi_{1,\tilde{y}} F d\tilde{y}, \quad (85)$$

where $F = F(\tilde{y})$. When integrating equations (84), we can thus replace all terms of the form $\Phi_1 \partial_{\tilde{y}} F$ with $-\Phi_{1,\tilde{y}} F$, making the equations somewhat more transparent. Unfortunately, there was insufficient time to fully examine what conditions determine the sign of the integrals of equations (84).

We can thus write down the coupling coefficient for time evolution equation (79) of the amplitude of wave 1,

$$\delta_1 = \frac{\int_{-\infty}^{\infty} \Phi_1 \mathcal{N}_1^R + \Phi_1 \mathcal{N}_1^S + \Phi_1 \mathcal{N}_1^h d\tilde{y}}{\int_{-\infty}^{\infty} \Upsilon_1 d\tilde{y}}. \quad (86)$$

The coupling coefficients δ_2 and δ_3 can now be trivially gained by changing the labels appropriately in equation (86), which gives the standard, ordinary coupled differential equations for the time evolution of amplitude of a weakly non-linear triad interaction,

$$A_{1,\tilde{\tau}} = \delta_1 A_2^* A_3^* \quad (87a)$$

$$A_{2,\tilde{\tau}} = \delta_2 A_1^* A_3^* \quad (87b)$$

$$A_{3,\tilde{\tau}} = \delta_3 A_1^* A_2^*. \quad (87c)$$

The sign of the coupling coefficients is important, as, if they are single signed, then we shall encounter explosive instability (Coppi et al., 1969), which is an unphysical circumstance in the situation we are considering. We have already seen that the sign the integral of Υ_1 , equation (82), is fixed. Thus, the sign of the coupling coefficients is determined by the sign of the sum of the integrals of equations (84), which is the numerator in equation (86).

In practise, it is not particularly practical to attempt to evaluate the coupling coefficient, equation (86), analytically and a future possibility would be to numerically integrate these equations.

6 Conclusion and Discussion

The governing equations for the equations on a Mercator projection centred about the equator were derived. We took inspiration from these equations to examine an “extended β -plane” in which we use the standard Cartesian, geopotential coordinates except employing the use of a tanh profile for the coriolis parameter, rather than the usual linear profile of the standard equatorial β -plane.

The properties of the solutions of such a system were investigated, with the finding that both a discrete and continuous spectrum exist for baroclinic waves, while only the discrete spectrum exists for barotropic waves. There exist many avenues for potential future study of these systems, a few of which shall now be discussed.

6.1 Potential Future Work

As with any interesting line of enquiry, we are left with more questions than answers. A brief list of potential future works, based upon this project is presented.

- Examine the properties of the Mercator projection including the metric terms, and compare with the solutions on a sphere, as described by Longuet-Higgins (1964, 1965).
- Examine the properties of interactions between a Rossby wave and a Yanai or Kelvin and the continuous spectrum.
- The derivation and numerical evaluation of the Manly-Rowe relations for the system of 3 baroclinic Rossby waves considered in section 5.3. Such an evaluation would help to ascertain whether any observed phenomena may be explained by such interactions (viz-a-viz the motivation given in section 1).
- It may be beneficial, and easier to conduct manipulations and further studies into these phenomena if the system were described using a Lagrangian or Hamiltonian method (for example, Ripa, 1981).
- The re-derivation of the system with a background shear (which is a more realistic set up for near equatorial oceanic dynamics).

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