A Correlated Stochastic Model for the Large-scale Advection, Condensation and Diffusion of a Passive Scalar

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1 Introduction

1.1 Distribution of moisture in the atmosphere

At first glance, the distribution of atmospheric water vapor is highly inhomogeneous and varies significantly on very short distances. Lagrangian trajectories of each individual moist parcels in the Tropics strongly depend on the wind field and the intensity of tropical convection, and cloud microphysical processes will have a big influence on the parcel's water content [7]. However understanding the distribution of water vapor in the atmosphere is crucial for several reasons:

- **Its climatic importance**: Water vapor is the main greenhouse gas in the atmosphere, because of its molecular characteristics (eg a permanent dipole moment) and its abundance. Furthermore, the condensation of water produces clouds, which reflect, scatter and absorb solar and terrestrial radiation. Finally, the condensation of water produces latent heat which affects the global atmospheric circulation. Thus, it is impossible to quantitatively study the distribution of clouds [3, 11] or even climate change [19, 21] without understanding the atmospheric water content.
- **Its central meteorological role**: It is impossible to understand weather without knowing the distribution of water vapor and liquid water in the atmosphere. Important weather systems, such as cloud clusters, cyclones, or the Madden-Julian oscillation, are all examples of aggregates of water in the atmosphere. Understanding the self-aggregation of water vapor is a significant research challenge [25, 28], and knowledge of the large-scale advection/distribution of water vapor is required to understand the evolution of moist clusters in time.

The comparison of observations with simple numerical models gives us insights in how atmospheric water vapor is distributed:

- Understanding the mixing occurring on dry/moist isentropic surfaces between the moist Tropics, the dry subtropics, and the sub-saturated mid-latitudes is central to understanding the distribution of water vapor [4, 2, 1, 10, 16].
- The Probability Density Function (PDF) of water vapor is bimodal with a dry and a moist spike [15].

- There is a consensus that microphysical processes do not need to be known precisely to understand the distribution of water vapor, at least outside of the Tropics, and that focusing on computing the advecting wind with precision is more relevant in most cases [5, 20].

The last point justifies the advection-condensation paradigm, which assumes that the distribution of water vapor can be deduced from the velocities in the atmosphere and the saturation specific humidity profile, without a precise knowledge of the microphysical and convective processes producing this profile. It is an interesting alternative to models that need a precise knowledge of the atmospheric properties at each location [17], and was initially used to compute the short term evolution of a given water vapor field in the atmosphere using the observed wind field data. Pierrehumbert et al. [15] started using a stochastic advection-condensation model to understand the fundamentals of water vapor distribution. Assuming a one-dimensional Brownian motion for the air parcels and an exponentially decreasing saturation specific humidity profile, they were able to solve initial value problems and obtain the PDF of moisture numerically. O'Gorman and Schneider [12, 13] added correlation to the stochastic velocity field by assuming that it was governed by an Ornstein-Uhlenbeck process (Gaussian colored noise). They were able to analytically compute the moisture flux and the condensation rate in the Ballistic and Brownian limits; they computed the same quantities numerically in the 1D/2D case. Sukhatme and Young [23] were able to solve for the PDF, the moisture flux and the condensation rate in the general case (giving significant insight in the physics of the process), but they had to come back to a white noise process for the velocity field.

Thus, our first motivating questions are: **What are the physics of the advection-condensation model in the case where the velocity process has a finite time-correlation?** Is it possible to analytically approximate:

- The bimodal PDF of specific humidity?
- The moisture flux?

Understanding how the flux of a condensing scalar behaves in the case of a stochastic process correlated in time has broader applications and leads to new motivations, as we show in paragraph 1.2.

1.2 Diffusivity of a passive scalar

A passive scalar is a diffusive substance with no dynamical effect (such as a change in buoyancy) on the fluid flow that contains it [27]. Under certain conditions, anthropogenically introduced dyes, biological nutrients, and of course water vapor can be approximated as passive scalars. Here, we show how the Lagrangian evolution equation of a passive scalar q influences its diffusion, ie its movement down a given concentration gradient G (in m^{-1}). If the scalar has no sources nor sinks, it can be shown [24] that in a statistically steady state, its flux $\mathcal{F}_q$ verifies Fick's law:

$$\mathcal{F}_q \overset{\text{def}}{=} \overline{v'q'} = -DG \tag{1}$$

where $(\overline{X})$ is an appropriate ensemble average on the fluid's parcels, and the deviations from this average are denoted with a prime. v is the parcel's velocity, defined as the Lagrangian time-derivative of its displacement, and assumed to have a mean value equal to zero without loss of generality. In Fick's law, we define the coefficient D (in $\mathrm{m}^2.\mathrm{s}^{-1}$) to be the diffusivity of any passive scalar q. In the absence of sources and sinks:

$$D = \int_0^{+\infty} C(\tau) d\tau \tag{2}$$

where C is the steady covariance of the velocity process v:

$$C(\tau) \stackrel{\text{def}}{=} \overline{v(t=\tau)v(t=0)} \tag{3}$$

and t (in s) is a measure of the time. This simple result shows that the sole understanding of C can fully describe the diffusion of q. Now, let us consider the case where q is linearly damped at a frequency λ. The evolution of the perturbation of q from its gradient G is given by:

$$\frac{dq'}{dt} + \lambda q' + Gv = 0 \tag{4}$$

Integrating this equation in time, applying a proper ensemble average, and assuming that the statistically steady state has been reached, we find that $\mathcal{F}_q$ follows an adapted Fick's law, with a special diffusivity that depends on the damping rate λ:

$$D_\lambda = \int_0^{+\infty} \exp(-\lambda\tau) C(\tau) d\tau \tag{5}$$

In the limit of a null damping ($\lambda = 0$), or a white noise velocity process ($C(\tau) = D_q \delta^+(\tau)$), we recover the case without sources or sinks. In general, linear damping decreases $\mathcal{F}_q$ while keeping it proportional to G (ie Fickian), since the rate of damping λ is the same everywhere in the fluid. The previous generalization of Taylor's diffusivity has applications in biology, where biological properties can be advected by the the flow as well as consumed at a linear rate, and allows to compute the eddy fluxes of the biological properties at all time [9]. If we try to apply the same reasoning to a condensing scalar q, the evolution of the perturbation q' is given by:

$$\frac{dq'}{dt} + Gv + \lambda[q - q^*(y)]\mathcal{H}[q - q^*(y)] = 0 \tag{6}$$

where condensation brings back linearly the scalar q to its saturation value q^* when $q > q^*$. We have defined the Heaviside function $\mathcal{H}$ and the condensation time λ^{-1}. **A key motivation of this work is to find how $\mathcal{F}_q$ deviates from its Fickian upper bound $-DG$ in the case of a condensing scalar.** We immediately see that we need a positive damping ($\lambda > 0$) and a correlated process ($C(\tau > 0) \neq 0$), to avoid the case $D_\lambda = D$ which would not provide any insight on $\mathcal{F}_q$.

2 The Advection-Condensation Model for Specific Humidity

2.1 Specific humidity

We begin by defining the specific humidity q of a parcel as:

$$\boxed{q \overset{\text{def}}{=} \frac{m_v}{m_t}} \tag{7}$$

where m_v is the mass of water vapor in the parcel and m_t is the total mass of the parcel. We assume that q is governed by the following advection-condensation equation:

$$\boxed{\frac{Dq}{Dt} = \mathcal{S} - \mathcal{C}} \tag{8}$$

which involves:

- The material derivative on an isentropic surface:

$$\frac{D}{Dt} \overset{\text{def}}{=} \frac{\partial}{\partial t} + \overrightarrow{u} \cdot \overrightarrow{\nabla} \tag{9}$$

  where if we define the position vector on the isentropic surface $\overrightarrow{x}$, the velocity on the isentropic surface is defined to be:

$$\overrightarrow{u} \overset{\text{def}}{=} \frac{\partial \overrightarrow{x}}{\partial t} \tag{10}$$

  and the gradient is defined to be:

$$\overrightarrow{\nabla} \overset{\text{def}}{=} \frac{\partial}{\partial \overrightarrow{x}} \tag{11}$$

- A condensation sink $\mathcal{C}$ that instantly brings back the specific humidity q to its saturation value q^*:

$$\mathcal{C} = \lim_{\lambda \to +\infty} \lambda(q - q^*)\mathcal{H}(q - q^*) \;\Rightarrow\; \boxed{\forall t,\; q = \min(q, q^*)} \tag{12}$$

- A source term $\mathcal{S}$, which we will model in different ways throughout this report.

2.2 An example of saturation specific humidity profile

2.2.1 Saturation specific humidity

The saturation specific humidity q^* can be expressed as a function of the saturation water vapor partial pressure e^* and the environmental pressure p using the ideal gas law (eg [6]):

$$\boxed{q^* = \frac{\epsilon e^*}{p - (1 - \epsilon)e^*}} \tag{13}$$

where ϵ is the ratio of the specific gas constant of dry air r_d and water vapor r_v:

$$\epsilon \stackrel{\text{def}}{=} \frac{r_d}{r_v} \approx \frac{287 J.K^{-1}.kg^{-1}}{461 J.K^{-1}.kg^{-1}} \approx 0.621 \tag{14}$$

We can then use an integral version of the Clausius-Clapeyron equation if we want to express the saturation water vapor partial pressure e^* as a function of the environmental temperature T. Clausius-Clapeyron relation, valid for an ideal gas undergoing a phase change in thermodynamic equilibrium, can be written:

$$\frac{de^*}{dT} = \frac{L_v e^*}{r_v T^2} \tag{15}$$

where $L_v(T)$ is the latent heat of vaporization of water vapor. An approximate integral of the previous relation is given by Bolton's formula:

$$e^*(p, T) \approx 6.112 \cdot \exp(\frac{17.67 T[°C]}{T[°C] + 243.5}) \tag{16}$$

2.2.2 Exponentially decreasing profile

Let's assume that the temperature only varies with latitude, and that its meridional gradient is constant:

$$T \approx T_0 - \frac{\Delta T}{\Delta y} \cdot y \tag{17}$$

where $(T_0, \Delta T, \Delta y)$ are known parameters. Integrating Clausius-Clapeyron's 15 relation keeping e^*'s prefactor constant, we obtain:

$$e^* = e^*(T_0) \exp[\frac{L_v(T_0)}{r_v T_0^2}(T - T_0)] = e^*(T_0) \exp[-\frac{L_v(T_0)}{r_v T_0^2} \frac{\Delta T}{\Delta y} \cdot y] \tag{18}$$

We then neglect the reduced saturation water vapor pressure $(1 - \epsilon)e^*$, compared to the total pressure p that we assume to be approximately constant $p \approx p_0$, so that the saturation specific humidity 13 can be written:

$$q^* = \frac{\epsilon e^*}{p - (1 - \epsilon)e^*} \approx \frac{\epsilon}{p_0} e^* \approx \frac{\epsilon e^*(T_0)}{p_0} \exp[-\frac{L_v(T_0)}{r_v T_0^2} \frac{\Delta T}{\Delta y} \cdot y] \tag{19}$$

The previous relation 19 gives us a saturation specific humidity profile that exponentially decreases with latitude:

$$\boxed{q^*(y) \approx q_{max} \exp(-\alpha y) \;\Leftrightarrow\; y^*(q) \approx -\frac{1}{\alpha} \ln(\frac{q}{q_{max}})} \tag{20}$$

where we have introduced the following physical parameters:

$$q_{max} \stackrel{\text{def}}{=} \frac{\epsilon e^*(T_0)}{p_0} \approx 2.5\% \tag{21}$$

$$\alpha \stackrel{\text{def}}{=} \frac{L_v(T_0)}{r_v T_0^2} \frac{\Delta T}{\Delta y} \approx 4.10^{-7} m^{-1} \approx \frac{1}{2500\text{km}} \tag{22}$$

where the following orders of magnitude have been used:

- For the planet Earth, the Equator's temperature annually averages at $T_0 \approx 30°C$ and the North Pole's temperature has an average around $T_0 - \frac{\Delta T}{\Delta y}(a\frac{\pi}{2}) \approx -20°C$, leading to a global temperature gradient of $\frac{\Delta T}{\Delta y} \approx 5.10^{-3}$°C.km^{-1}.
- The latent heat of vaporization of water: $L_v \approx 2.10^6$J.kg^{-1}.
- We can estimate the saturation water vapor pressure at T_0 using Bolton's formula: $e^*(T_0) \approx 42$hPa $\approx (4\%)p_0$.

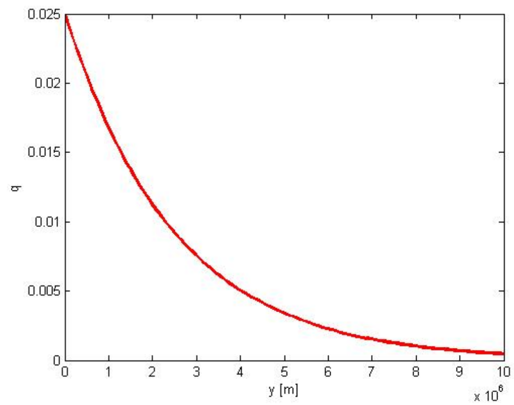

Figure 1: Approximate profile of q^* vs y for the Earth's atmosphere

2.3 Global distributions in the advection-condensation model

Under certain conditions, the global distributions of displacement and specific humidity have universal forms. Let's consider the following set of assumptions for an advection-condensation model defined by equation 8:

- We work on a latitudinal domain, extending from $y = 0$ to $y = L$.
- We assume that the saturation specific humidity is strictly decreasing from from $y = 0$ where $q^*(0) = q_{max}$ to $y = L$ where $q^*(L) = q_{min}$. We represent the domain in (q, y) space on figure 2 to make its boundaries more concrete to the reader.
- We assume that the system has reached a steady state so that its Probability Density function (PDF) $\rho(q, y, v)$ satisfies:
$$\frac{\partial \rho}{\partial t} = 0 \tag{23}$$
- We assume that the velocity v of the system is governed by an isotropic and homogeneous stochastic process.

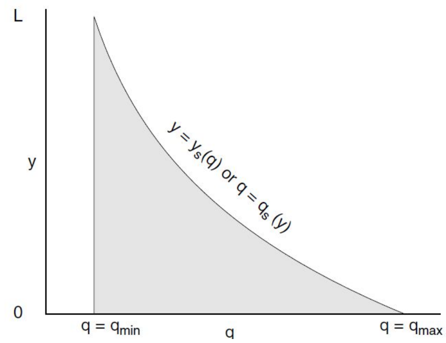

Figure 2: Domain in (q, y) space

Under the previous assumptions, we expect the global distribution of displacement:

$$P_Y(y) \stackrel{\text{def}}{=} \int_{\mathbb{R}} dv \int_{q_{min}}^{q^*(y)} dq\rho(q, y, v) \tag{24}$$

to have no gradients in y, ie:

$$\frac{dP_Y}{dy} = 0 \tag{25}$$

It is then possible to compute the constant P_Y from the normalization condition on the steady PDF ρ:

$$1 = \int_0^L dy \int_{\mathbb{R}} dv \int_{q_{min}}^{q^*(y)} dq\rho(q, y, v) = \int_0^L dy P_Y = LP_Y \tag{26}$$

From the knowledge of P_Y, we can rewrite the normalization condition on ρ in a way that does not involve any integral in y:

$$\boxed{\int_{\mathbb{R}} dv \int_{q_{min}}^{q^*(y)} dq\rho(q, y, v) = \frac{1}{L}} \tag{27}$$

This very powerful constraint will allow us to compute exact analytical solutions for the PDF in special cases. Furthermore, it determines the form of the global distribution of specific humidity, as we will show now. To simplify the proof, the marginalization over the variable v is always implied. From the fact that P_Y is constant (equation 26):

$$\forall y,\ P_Y(y) = \frac{P_Y(0) + P_Y(L)}{2} \tag{28}$$

Integrating 28 from $y' = 0$ to $y' = y$ yields:

$$\int_{q_{min}}^{q^*(y)} dq \int_0^y dy' \rho(q, y') dq = \int_0^y dy' P_Y(y') = \frac{y}{2}[P_Y(0) + P_Y(L)] \tag{29}$$

where we have used Fubini's theorem to exchange integrals. As $\rho_\infty[q > q^*(y)] = 0$, we can prove from 29 that:

$$\int_{q^*(y)}^{q_{max}} dq \int_0^y dy' \rho(q, y') dq = \frac{y}{2}[P_Y(0) + P_Y(L)] \tag{30}$$

We change the independent variable from y to q in equation 30, and carefully adapt the bounds of the integrals using figure 2:

$$\int_{q}^{q_{max}} \int_{0}^{y^*(q')} \rho(q',y')dq'dy' = \frac{y^*(q)}{2} \int_{q}^{q_{max}} [\rho(q',0) + \rho(q',L)]dq' \tag{31}$$

Differentiating equation 31 with respect to q gives:

$$P_Q(q) = \frac{1}{2}\frac{d}{dq}\{y^*(q) \int_{q}^{q_{max}} [\rho(q',0) + \rho(q',L)]dq'\} \tag{32}$$

where we have defined the global distribution of specific humidity:

$$P_Q(q) \overset{\text{def}}{=} \int_{\mathbb{R}} dv \int_{0}^{y^*(q)} \rho_\infty(q,y',v)dy' \tag{33}$$

Because of the condensation rule 12, there can only be parcels of specific humidity $q = q_{min}$ at $y = L$. Adding the normalization condition 27, it means that:

$$\rho(q',L) = \frac{\delta^+(q - q_{min})}{L} \tag{34}$$

Using 34 in equation 32 gives:

$$\frac{d}{dq}\{y^*(q) \int_{q}^{q_{max}} \rho_\infty(q',L)dq'\} = \frac{d}{dq}\{\frac{y^*(q)}{L} \int_{q}^{q_{max}} \delta^+(q'-q_{min})dq'\} = \frac{1}{L}\frac{dy^*}{dq}\delta_{qq_{min}} + \delta^+(q-q_{min})$$

where $\delta_{qq_{min}}$ is the Kronecker symbol for $q = q_{min}$. The first term is only nonzero when $q = q_{min}$, and is finite, which means it will be negligible compared to the second term in practice. As a consequence, we obtain the following general expression for the global distribution of specific humidity:

$$\boxed{P_Q(q) = \frac{\delta^+(q - q_{min})}{2} + \frac{1}{2}\frac{d}{dq}[y^*(q) \int_{q}^{q_{max}} \rho(q',0)dq']} \tag{35}$$

Not only does this distribution show the presence of a dry peak in the PDF, but it also states something very fundamental about the parcel's history. Indeed, since the velocity process is isotropic and homogeneous:

- Half of the parcels have last hit the dry Northern boundary, and have been dried to $q = q_{min}$. They constitute the dry peak of $P_Q(q)$ which has a $\frac{1}{2}$ amplitude.
- Half of the parcels have last hit the moist Southern boundary, and have been remoistened by the Southern boundary condition coming through $\rho(q',0)dq'$ as well as condensed because of the saturation specific humidity profile, explaining the presence of $y^*(q)$.

We will see in section 4 where the advection-condensation model is solved for many different cases, that the form 35 of the global distribution of specific humidity universally applies as long as the assumptions stated at the beginning of this paragraph 2.3 apply.

3 Brownian Linearly Damped Motion on an Isentropic Surface

3.1 Preliminary assumptions

Before dealing with the complexity of condensation, we begin with a simple linear damping model for moisture, similarly to what has been tackled in the introduction (cf equation 4). We make the following simplifying assumptions:

1. We reduce our problem to a 1D Cartesian process on a line, parametrized by its ordinate y.

2. The domain we work on extends from the Equator ($y = 0$) to a given latitude $L > 0$.

3. We model the source term $\mathcal{S}$ following [23]. The model "resets" the specific humidity q to a random value chosen from a specified distribution $\Phi(q)$ when it encounters the Southern boundary of the domain ($y = 0$). Physically, we could think of the fictive domain $y \leq 0$ as the Tropics, that remoisten the dry parcels that are advected southward by the mid-latitude eddies. The normalization condition on Φ yields:

$$\int_{q_{min}}^{q_{max}} \Phi = 1 \tag{36}$$

4. The sink term is a linear damping with a typical decay frequency λ:

$$\mathcal{C}(q) = \lambda q \tag{37}$$

3.2 Stochastic differential equation and Fokker-Planck equation for the PDF

For simplicity sake, we assume that the moist parcels have a Brownian motion of diffusivity κ, leading to the following stochastic differential equation (SDE) for the position $Y(t)$ and the moisture $Q(t)$ of the moist parcels:

$$\boxed{\begin{cases} dY(t) = \sqrt{2\kappa} dW(t) \\ dQ(t) = \{\mathcal{S}[Y(t)] - \mathcal{C}[Q(t)]\} dt \end{cases}} \tag{38}$$

where $W(t)$ is a Wiener process. The corresponding steady Fokker-Planck equation (FPE) for the PDF ρ can be written by identifying the drift and the diffusion terms in the previous SDE 38:

$$\frac{\partial}{\partial q}[(\mathcal{S} - \mathcal{C})\rho] = \kappa \frac{\partial^2 \rho}{\partial y^2} \tag{39}$$

Note that integrating this equation from $q' = 0$ to $q' = q_{max}$ and using the normalization 27 gives:

$$-\lambda q_{max} \rho(q_{max}, y) = \kappa \int_0^{q_{max}} \frac{\partial^2 \rho}{\partial y^2} dq' = \kappa \frac{\partial^2}{\partial y^2} \int_0^{q_{max}} \rho(q', y) dq' = \kappa \frac{\partial^2}{\partial y^2} L^{-1} = 0$$

$$\boxed{\forall y,\ \rho(q_{max}, y) = 0} \tag{40}$$

In order to solve the FPE, we change variables:

$$\begin{cases} y \mapsto \sqrt{\frac{\lambda}{\kappa}} y \\ q \mapsto \ln(\frac{q_{max}}{q}) \end{cases} \tag{41}$$

We are then left with a diffusion-amplified equation in our new (q, y) space:

$$\boxed{(\frac{\partial}{\partial q} - 1 - \frac{\partial^2}{\partial y^2})\rho = 0} \tag{42}$$

Note that the normalization condition 27 in the new(q, y) space yields:

$$1 = Lq_{max} \int_0^{+\infty} \exp(-q) \cdot \rho(q, y) dq \tag{43}$$

3.3 Solution to the steady FPE 42

First, we need to define 3 boundary conditions:

1. The resetting at the Southern boundary along with the normalization condition impose:
$$\rho(y = 0) = \frac{\Phi(q_{\text{dim}})}{L} = \frac{\Phi[q_{max} \exp(-q)]}{L} = \tilde{\Phi}(q) \tag{44}$$
2. We impose a no-flux boundary conditions at the Northern boundary:
$$\frac{\partial \rho}{\partial y}(y = L) = 0 \tag{45}$$
where we have introduced the dimensionless extent of the domain:
$$L \overset{\text{def}}{=} L_{\text{dim}} \sqrt{\frac{\lambda}{\kappa}} \tag{46}$$
3. With the new variables 41, equation 40 can be written:
$$\rho(q = 0) = 0 \tag{47}$$

The mixed boundary value problem on a finite domain has an exact analytic solution, given by:

$$\rho(q, y) = \frac{\pi}{L^2} \int_0^q dq' \tilde{\Phi}(q') \exp(q - q') \sum_{n \in \mathbb{N}} (2n + 1) \sin[\frac{\pi(2n + 1)y}{2L}] \exp[-\frac{\pi^2(2n + 1)^2(q - q')}{4L^2}] \tag{48}$$

We now focus on given remoistening profiles $\Phi(q)$ in two specific cases:

1. In case I, we consider a complete remoistening at the Equator:

$$\Phi_I(q_{\text{dim}}) \overset{\text{def}}{=} \delta^-(q_{\text{dim}} - q_{max}) \tag{49}$$

corresponding to $\tilde{\Phi}_I(q) = \frac{\delta^-(q)}{L_{\text{dim}}}$. Coming back to the dimensional variables (ρ, q, y), the PDF in case I can be written:

$$\boxed{\rho_I(q,y) = \frac{\pi\kappa}{\lambda L^3} \sum_{n\in\mathbb{N}} (2n+1)\sin[\frac{\pi(2n+1)y}{2L}](\frac{q}{q_{max}})^{\frac{\pi^2\kappa(2n+1)^2}{4\lambda L^2}-1}} \tag{50}$$

2. In case II, we consider a uniform remoistening at the Equator:

$$\Phi_{II}(q_{\text{dim}}) = \frac{1}{q_{max} - q_{min}} \tag{51}$$

corresponding to $\tilde{\Phi}_{II}(q) = \frac{1}{L_{\text{dim}}(q_{max}-q_{min})}$. Coming back to the original variables (q, y), the PDF in case II can be written:

$$\rho_{II}(q,y) = \frac{\pi\kappa}{\lambda L^3(q_{max} - q_{min})} \sum_{n\in\mathbb{N}} \frac{(2n+1)4\frac{\lambda}{\kappa}L^2}{\pi^2(2n+1)^2 - 4\frac{\lambda}{\kappa}L^2} \sin[\frac{\pi(2n+1)y}{2L}][1-(\frac{q}{q_{max}})^{1-\frac{\pi^2\kappa(2n+1)^2}{4\lambda L^2}}] \tag{52}$$

3.4 Global steady PDF

From now on, we come back to dimensional variables. By definition, the global steady PDF is:

$$P_Q(q) = \int_0^L dy \rho(q,y)$$

1. In case I:

$$\boxed{P_Q(q) = \frac{2\kappa}{\lambda L^2} \sum_{n\in\mathbb{N}} (\frac{q}{q_{max}})^{\frac{\pi^2\kappa(2n+1)^2}{4\lambda L^2}-1}} \tag{53}$$

Note that P_Q can be expressed using an elliptic function:

$$P_Q(q) = \frac{2\kappa}{\lambda L^2}\frac{q_{max}}{q} \cdot \theta_2[0, (\frac{q}{q_{max}})^{\frac{\kappa}{\lambda}(\frac{\pi}{L})^2}]$$

where θ_2 is the Jacobi theta function of second type, defined by:

$$\theta_2[z,q] = \sum_{n\in\mathbb{Z}} q^{(n+\frac{1}{2})^2} \cdot \exp[(2n+1)iz]$$

2. In case II:

$$P_Q(q) = \frac{2\kappa}{\lambda L^2(q_{max} - q_{min})} \sum_{n\in\mathbb{N}} \frac{(2n+1)4\frac{\lambda}{\kappa}L^2}{\pi^2(2n+1)^2 - 4\frac{\lambda}{\kappa}L^2}[1-(\frac{q}{q_{max}})^{1-\frac{\pi^2\kappa(2n+1)^2}{4\lambda L^2}}] \tag{54}$$

3.5 Averages and moments of the distribution

By definition, the average of a function $f(q)$ is:

$$\overline{f}(y) \stackrel{\text{def}}{=} L \int_0^{q_{max}} f(q)\rho(q,y)dq \tag{55}$$

Using the expression computed for ρ, it is possible (but tedious) to estimate $< f >$ for any well-behaved function. For our problem, the moments of the distribution $< q^n >$ are particularly interesting as they correspond to the average moisture ($n = 1$), the average square moisture ($n = 2$)... To compute the average moisture, we multiply the steady FPE 39 by q and integrate the result from $q = 0$ to $q = q_{max}$:

$$\int_0^{q_{max}} qdq \frac{\partial}{\partial q}(\lambda q \rho_\infty) + \kappa \int_0^{q_{max}} q \frac{\partial^2 \rho_\infty}{\partial y^2} = 0 \tag{56}$$

Integrating 56 by parts, noting that the boundary term is zero, and using the definition of $< q >$, we find a very simple equation for $< q >$:

$$\lambda \overline{q} = \kappa \frac{\partial^2 \overline{q}}{\partial y^2} \tag{57}$$

More generally, for $< q_n >$, multiplying the FPE 39 by q^n and integrating by parts gives:

$$\boxed{n\lambda \overline{q^n} = \kappa \frac{\partial^2 \overline{q^n}}{\partial y^2} \;\Leftrightarrow\; n\overline{q^n} = \frac{\partial^2 \overline{q^n}}{\partial y^2}} \tag{58}$$

We are thus left with a simple second order differential equation with constant coefficients to solve, where the two boundary conditions are given by:

1. The resetting condition at the Southern boundary:

$$\overline{q^n}(y=0) = \int_0^{q_{max}} q^n \Phi(q) dq = \Phi_n \leq q_{max}^n$$

2. The no-flux condition at the Northern boundary:

$$\frac{\partial}{\partial y} \overline{q^n}(y = L) = 0$$

leading to the following solution:

$$\boxed{\overline{q^n}(y) = \Phi_n[\cosh(\sqrt{n}y) - \tanh(\sqrt{n}L)\sinh(\sqrt{n}y)]} \tag{59}$$

Note that in the specific case of a complete remoistening at the Equator 49, we obtain $\Phi_n = q_{max}^n$.

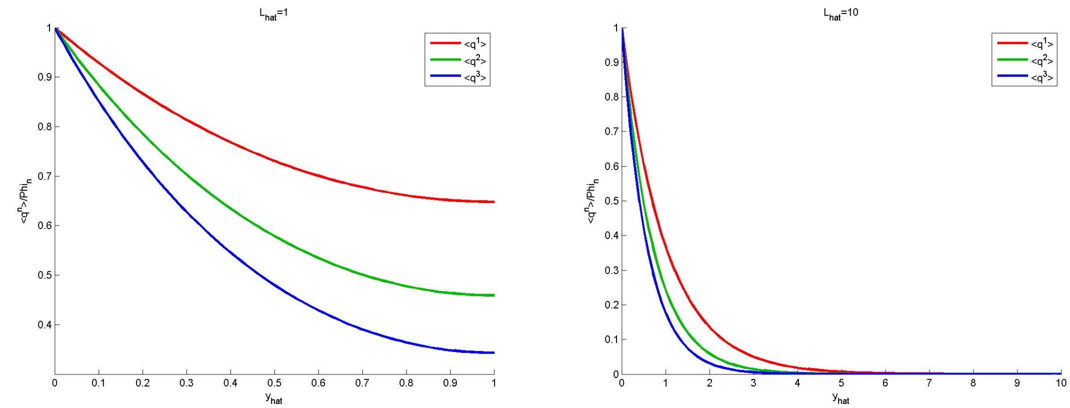

Figure 3: $\frac{<q^n>}{\Phi_n}$ for $n = 1$ (red) $n = 2$ (green) and $n = 3$ (blue) vs y for $L = 1$ (left) and $L = 10$ (right)

We can see that:

- The moments of q are damped faster throughout the domain as the dimensionless extent of the domain 46 (which varies proportionally to the damping frequency λ) increases.
- The higher moments of q are damped faster than the lower moments of q, which was to be expected with a linear damping, which effect is amplified when multiplied by q^n.

3.6 Diffusion in the linearly damped case

Let $y_0 \in]0, L[$: if we integrate the dimensional equation for $\overline{q}$ 57 from $y = y_0$ to $y = L$, we obtain:

$$\boxed{\lambda \int_{y_1}^{L} \overline{q} dy = -\kappa(\frac{\partial \overline{q}}{\partial y})_{y_0} = \mathcal{F}(y_0)} \tag{60}$$

This indicates that in steady state, the linear damping of moisture in the Northern part of the domain $y > y_0$is compensated by the Fickian diffusive flux of moisture $\mathcal{F}(y_0)$ from the Southern part of the domain $y < y_0$ to the Northern part. As shown in the more general case, we obtain that the diffusivity of the linearly damped moisture q is equal to κ in the case of a white noise velocity field.

3.7 Numerical experiments in the complete remoistening case $\Phi_I(q) = \delta^+(q - q_{min})$

To check the validity of our general solution in this specific case, we numerically simulate the SDE with the following dimensional values:

$$\begin{cases} N_{tot} = 3.10^4 \\ L = 1 \\ q_{max} = 1 \\ \kappa = 1 \end{cases}$$

and compare the numerical PDF to the marginal distributions $P_Y(y)$and $P_Q(q)$ for different values of the damping frequency λ:

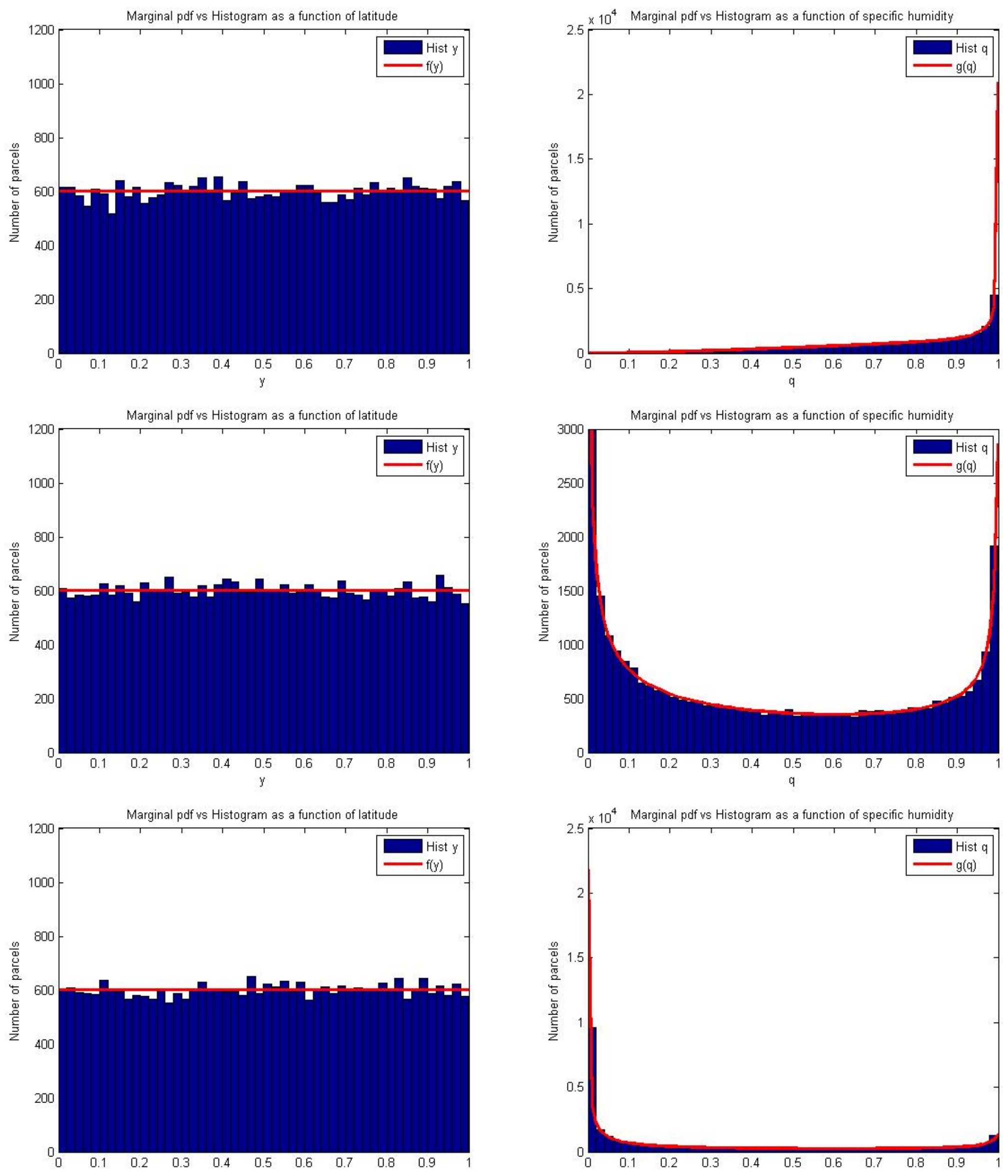

Figure 4: $P_Y(y)$ (left) and $P_Q(q)$ (right) for a square dimensionless length $L^2 \in \{1, 5, 10\}$ (top, middle, bottom)

To interpret the previous results, it is useful to remember that our system is governed by the dimensionless extent of the domain defined in 46:

- For small damping $L^2 \leq 1$:

$$P_Q(q) \propto \sum_{n\in\mathbb{N}^*} q^{(2n+1)^2 L^{-2}} \underset{(L\to 0)}{\to} \delta^-(q - q_{max})$$

  and the moist peak of the distribution near $q = q_{max}$ dominates. Physically, the moisture of the parcels do not have time to damp before the parcels encounters the Southern boundary and is completely remoistened again.

- For large damping $L^2 \geq 10$:

$$P_Q(q) \propto q^{-1} \underset{(L\to +\infty)}{\to} \delta^+(q)$$

  and the dry peak of the distribution near $q = 0$ dominates. Physically, the moisture of the parcels damps out very fast after the parcels have been remoistened, and it takes them a long time before they re-encounter the moistening Southern boundary of the domain.

- For intermediate $L^2 \in [1, 10]$, the distribution of the parcels is bimodal, with a moist and a dry peak.

4 Time-correlated Velocities

White noise does not exist in nature, and atmospheric data indeed show that velocities are correlated in time, the correlation time usually being the typical eddy turnover time.

4.1 The Ornstein-Uhlenbeck model

4.1.1 Gaussian colored noise

We replace the white noise in the previous part by Gaussian colored noise for the Lagrangian velocities. According to Doob's theorem, this noise can be obtained through an Ornstein-Uhlenbeck process for the velocities. If we consider 1D meridional motion, our new system of SDEs can be written:

$$\boxed{\begin{cases} dY(t) = V(t)dt \\ \tau dV(t) = -V(t)dt + \sqrt{2\kappa}dW(t) \\ dQ(t) = \{\mathcal{S}[Y(t)] - \mathcal{C}[Y(t), Q(t)]\}dt \end{cases}} \tag{61}$$

where $V(t) = \frac{dY}{dt}$ is the meridional Lagrangian velocity of the process, and κ its associated diffusivity. The Ornstein-Uhlenbeck process is a standard model of turbulent dispersion [26, 14, 18], and the novelty of the stochastic advection-condensation model is to introduce the condensing variable Q. The corresponding FPE for the PDF $\rho(t, q, y, v)$ can be written:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial q}[(\mathcal{S} - \mathcal{C})\rho] + \frac{\partial}{\partial y}(\rho v) = \frac{1}{\tau}\frac{\partial}{\partial v}(\rho v) + \frac{\kappa}{\tau^2}\frac{\partial^2 \rho}{\partial v^2} \tag{62}$$

It can be proven that the auto-correlation function for the Ornstein-Uhlenbeck process can be written:

$$\mathbb{E}[V(t_1)V(t_2)] = \frac{\kappa}{\tau}\exp(-\frac{|t_2 - t_1|}{\tau}) \neq 0 \tag{63}$$

which is what makes this noise "colored" in contrast to white.

4.1.2 Marginalized distribution

We make the Fokker-Planck equation non-dimensional by rescaling:

$$y \mapsto \frac{y}{\sqrt{\tau\kappa}} \tag{64}$$

$$v \mapsto \sqrt{\frac{\tau}{\kappa}}v \tag{65}$$

If we marginalize the distribution over q:

$$p(y,v) \overset{\text{def}}{=} \int_{q_{\min}}^{q^*(y)} \rho(t,q,y,v)dq \tag{66}$$

we recover the well-known FPE for the OU process from the FPE 62:

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial y}(vp) = \frac{\partial}{\partial v}(vp + \frac{\partial p}{\partial v}) \tag{67}$$

The solution of equation 67 along with the full normalization condition for the PDF ρ 26 is the well-known Maxwellian distribution:

$$p(y,v) = \frac{1}{\sqrt{2\pi}L}\exp(-\frac{v^2}{2}) \tag{68}$$

We have introduced the sole dimensionless parameter of our model, ie the dimensionless domain's extension:

$$L \overset{\text{def}}{=} \frac{L_{\text{dim}}}{\sqrt{\kappa\tau}} \tag{69}$$

In the ballistic limit $L \ll 1$, the domain is so small that the velocity of a given parcel is constant in time. In the Brownian limit $L \gg 1$, the domain is so large that the displacement of a given parcel is a white noise process. Both cases are studied in A.

4.1.3 Steady PDF in the general case

We transform the steady version of the FPE 67 by changing variable for the density ρ, using the solution 68:

$$\rho \mapsto \frac{p(y,v)}{\kappa}\rho \tag{70}$$

giving:

$$v\frac{\partial \rho}{\partial y} = \mathcal{D}\rho \overset{\text{def}}{=} \frac{\partial^2 \rho}{\partial v^2} - v\frac{\partial \rho}{\partial v} \tag{71}$$

Note that the normalization condition for ρ is now simply:

$$\forall (y,v) \ \int_{q_{min}}^{q^*(y)} \rho(q,y,v)dq = 1 \tag{72}$$

In order to solve 72, we first look for exponential solutions in y (or more rigorously we Laplace-transform the previous differential equation in y):

$$\rho(q,y,v) \propto \exp(\alpha y)\varphi_\alpha(v)$$

where the functions φ_α are the eigenfunctions of the following eigenvalue problem:

$$\boxed{\mathcal{D}\varphi_\alpha = \alpha v \varphi_\alpha} \tag{73}$$

It is shown in the appendix B that the functions φ_α for $\alpha^2 \in \mathbb{N}$, along with the generalized eigenfunction $v \mapsto v$, constitute an orthogonal basis in which we can expand any well-behaved function which verifies the initial differential equation. Solving a half-range problem is a hard mathematical problem, and adding the specific boundaries of our problem in (q,y) space makes the analytical solution too complicated to be tractable in a useful way. We will thus describe the four steps to solve this problem in the general case without doing the algebra, based on the fact that the PDF comprises a dry spike, a saturated spike, and a smooth part (cf appendix C):

1. First, we write ρ as a combination of the solutions to the differential equation:

   $$\rho(q,y,v) = B(q)[y-v] + \sum_{\alpha^2 \in \mathbb{N}} A_\alpha(q)\varphi_\alpha(v)\exp(\alpha y)$$

   where $\varphi_0(v) = 1$. The remoistening condition at the Equator can be written:

   $$\rho(q, y=0, v>0) = \Phi(q)$$

   giving a first relation between all the coefficients:

   $$\forall v > 0, \ -B(q)v + \sum_{\alpha^2 \in \mathbb{N}} A_\alpha(q)\varphi_\alpha(v) = \Phi(q) \tag{74}$$

   From this relation 74, it is possible to express the function B and half of the functions A_α using the function Φ and the other half of the functions A_α (we will call them C_α to avoid confusion). If the projection is done well, $\Phi(q)$ should only appear in $A_0(q)$ as it appears as a constant in y in the final solution.

2. Then, we decompose each function (C_α) in a dry and a smooth part:

   $$C_\alpha(q) = C_{\alpha,\text{dry}}\delta^+(q - q_{min}) + C_{\alpha,\text{smooth}}(q)$$

   which implies that all the functions of q : (A_α, B) have a dry and a smooth part. To find the coefficients $C_{\alpha,\text{dry}}$, we apply the normalization condition 72 for the southwards moving parcels at $y = L$ where $q = q^*(y) = q_{min}$. We can only apply it to the southwards moving parcels because the northwards moving parcels have a saturated peak, as we prove in appendix C:

$$\forall v < 0,\ 1 = \int_{q_{min}}^{q_{min}} \rho(q, y = L, v < 0) dq = B_{\text{dry}}[L - v] + \sum_{\alpha^2 \in \mathbb{N}} C_{\alpha,\text{dry}} \varphi_\alpha(v) \exp(\alpha L) \tag{75}$$

3. We then need to compute the smooth part of the functions: $C_{\alpha,\text{smooth}}(q)$. It is also a half-range problem in v, as we can only apply the normalization condition 72 for all y to the southwards parcels which do not have a saturated peak:

$$\forall y, \forall v < 0,\ 1 = \int_{q_{min}}^{q^*(y)} \rho(q, y, v) dq$$

$$\forall y, \forall v < 0,\ 1 = [B_{\text{dry}} + \int_{q_{min}}^{q^*(y)} B_{\text{smooth}}(q) dq][L - y] + \sum_{\alpha^2 \in \mathbb{N}} [C_{\alpha,\text{dry}} + \int_{q_{min}}^{q^*(y)} C_{\alpha,\text{smooth}}(q) dq] \varphi_\alpha(v) \exp(\alpha y)$$

Changing the independent variable from y to q and adapt the bounds of the integrals using 2:

$$\forall q, \forall v < 0,\ 1 = [B_{\text{dry}} + \int_{q_{min}}^{q} B_{\text{smooth}}(q') dq'][L - y^*] + \sum_{\alpha^2 \in \mathbb{N}} [C_{\alpha,\text{dry}} + \int_{q_{min}}^{q} C_{\alpha,\text{smooth}}(q') dq'] \varphi_\alpha(v) \exp(\alpha y^*) \tag{76}$$

and we can find the functions $C_{\alpha,\text{smooth}}$ by solving the previous equation 76 for each $\int_{q_{min}}^{q} C_{\alpha,\text{smooth}}(q') dq'$ and then differentiating with respect to q.

4. Finally, once all the coefficients have been determined, we need to add the saturated peak to the northwards PDF (cf C):

$$\rho(q, y, v) = B(q)[y - v] + \sum_{\alpha^2 \in \mathbb{N}} C_\alpha(q) \varphi_\alpha(v) \exp(\alpha y) + \mathcal{W}(y, v > 0) \delta^-(q - q^*)$$

where $\mathcal{W}(y, v)$ is an undetermined weight function such that $\mathcal{W}(y, v < 0) = 0$. We can find its positive part by applying the normalization condition 72 to the northwards PDF:

$$\forall y, \forall v > 0,\ \mathcal{W}(y, v > 0) = 1 - \int_{q_{min}}^{q^*(y)} \{B(q)[y - v] + \sum_{\alpha^2 \in \mathbb{N}} C_\alpha(q) \varphi_\alpha(v) \exp(\alpha y)\} dq \tag{77}$$

4.1.4 Global distributions

For the Ornstein-Uhlenbeck process, the assumptions at the beginning of paragraph 2.3 are verified, giving us the global PDFs in y defined in 24and the global distribution in q defined in 33:

$$\boxed{P_Y(y) = \frac{1}{L}} \tag{78}$$

$$\boxed{2P_Q(q) = \delta^+(q - q_{min}) - \frac{1}{L} \frac{d}{dq}[\Lambda y^*]} \tag{79}$$

The global PDF in v can be found by integrating 68 in y:

$$\boxed{P_V(v) \stackrel{\text{def}}{=} \int_0^L dy p(y) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{v^2}{2})} \tag{80}$$

To check the validity of the previous analytical expressions, we simulate the global PDFs in the classical case of the complete remoistening at the Southern boundary $\Phi_I(q) = \delta^-(q - q_{max})$ and an exponentially decreasing profile of saturation specific humidity $q^*(y) = q_{max}\exp(-\alpha y)$. We choose the following parameters:

$$\begin{cases} N_{tot} = 10^6 \\ q_{max} = 1 \\ q_{min} = q_{max}\exp(-\alpha L) = 0.1 \\ L = \frac{(L_{\text{dim}}=1)}{\sqrt{(\kappa_V=1)(\tau=1)}} = 1 \end{cases}$$

and compare the analytical expressions found for the global distributions $P_Y(y)$, $P_Q(q)$ and $P_V(v)$ to the numerical simulation of the Ornstein-Uhlenbeck SDEs:

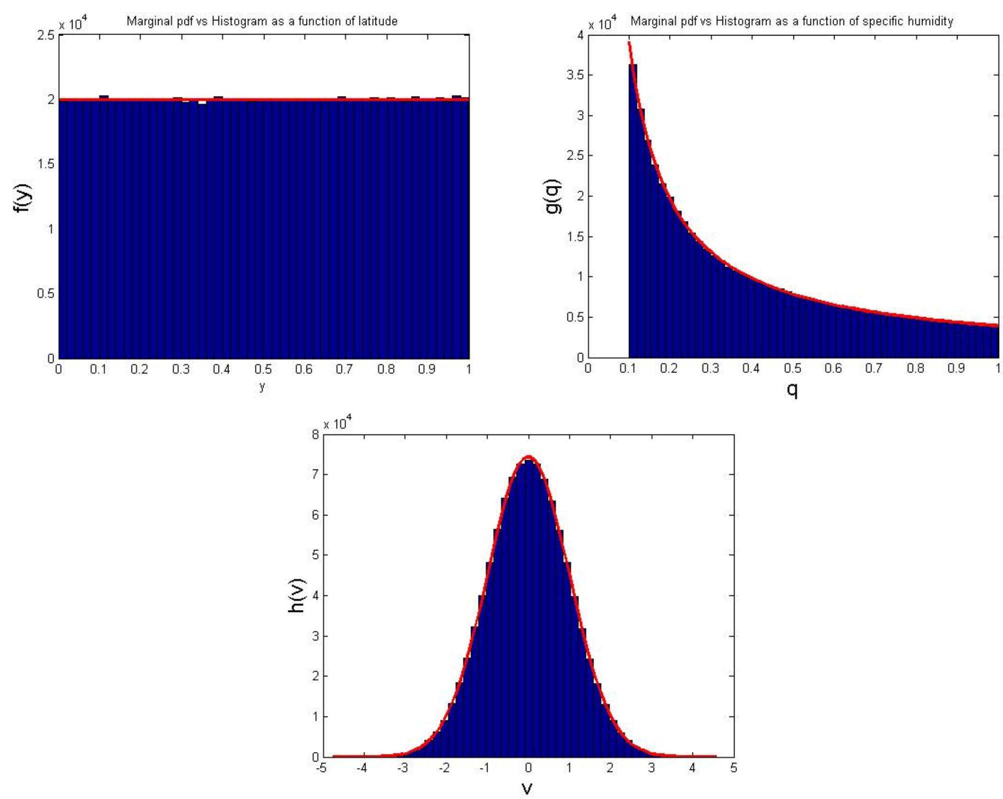

Figure 5: Numerical (blue) vs analytical (red) global PDFs for the Ornstein-Uhlenbeck model in case I of a complete remoistening

4.2 The two-stream model

4.2.1 Preliminary assumptions

A discrete version of the Ornstein-Uhlenbeck model (4.1) is the n-stream model (appendix D), where it is easier to compute exact analytical solutions for the PDF for small n. The simplest $n-$stream model is the $2-$stream model, defined by the following assumptions:

1. We work on a bounded domain $y \in [0, L]$, where the saturation specific humidity $q^*(y)$ is a strictly decreasing function of latitude.

2. The parcels are remoistened at the equator by a distribution $\Phi(q)$ normalized to 1.

3. We have two ensembles of moist parcels with an exchange frequency $\frac{\beta}{2}$:

- The parcels moving Northward, which have a velocity $+V$, and are described by the PDF $N(q, y)$.

- The parcels moving Southward, which have a velocity $-V$, and are described by the PDF $S(q, y)$.

The equations for (N, S) are thus:

$$\begin{cases} \frac{\partial N}{\partial t} + V\frac{\partial N}{\partial y} = \frac{\beta}{2}(S - N) \\ \frac{\partial S}{\partial t} - V\frac{\partial S}{\partial y} = \frac{\beta}{2}(N - S) \end{cases} \tag{81}$$

4.2.2 Steady southwards PDF

We now assume the northwards and southwards PDF (N, S) to be steady. Their normalization conditions can be deduced from 26:

$$\forall y, \int_{q_{min}}^{q^*(y)} N(q, y)dq = \int_{q_{min}}^{q^*(y)} S(q, y)dq = \frac{1}{2L} \tag{82}$$

We change variables to make our equations dimensionless:

$$y \mapsto \frac{\beta y}{V} \tag{83}$$

$$(N, S) \mapsto (\frac{V}{\beta})(N, S) \tag{84}$$

to obtain the steady dimensionless version of 81:

$$\begin{cases} \frac{\partial N}{\partial y} = \frac{\partial S}{\partial y} \\ \frac{\partial}{\partial y}(N + S) = S - N \end{cases} \tag{85}$$

Note that the system is governed by a single dimensionless number, the dimensionless extent of the domain:

$$L \stackrel{\text{def}}{=} \frac{\beta L_{\text{dim}}}{V} \tag{86}$$

The boundary condition along with the normalization condition 82 yields:

$$N(q,0) = \Phi(q)\int_{q_{min}}^{q_{max}} S(q,0)dq = \frac{\Phi(q)}{2L} \tag{87}$$

Integrating the previous system 85 using 87 yields:

$$\begin{cases} N(q,y) = yA(q) + \frac{\Phi(q)}{2L} \\ S(q,y) = (y+2)A(q) + \frac{\Phi(q)}{2L} \end{cases}$$

The PDF comprises a dry peak, a smooth peak and a saturated spike (cf appendix C). Here, we decompose A into a dry peak and a smooth part:

$$A(q) = A_{\text{smooth}}(q) + A_{\text{dry}}\delta^+(q - q_{min})$$

The normalization condition 82 for S at $y = L$ where $q = q_{min} = q^*(y)$ yields:

$$A_{\text{dry}} = \frac{1}{2L(2+L)}$$

The southwards PDF S satisfies the normalization condition for all $y \in [0, L]$, and does not have any saturated peak by definition (cf C):

$$1 = 2L\int_{q_{min}}^{q^*(y)} S(q,y)dq = (y+2)(\frac{1}{2+L} + 2L\int_{q_{min}}^{q^*(y)} A_{\text{smooth}}(q)dq) + \int_{q_{min}}^{q^*(y)} \Phi(q)dq$$

$$2L\int_{q_{min}}^{q^*(y)} A_{\text{smooth}}(q)dq = \frac{\int_{q^*(y)}^{q_{max}} \Phi(q)dq}{y+2} - \frac{1}{2+L} \tag{88}$$

Switching the independent variable from y to q , changing the bounds of the integrals following figure 2, and differentiating the previous equation 88 with respect to q, we obtain:

$$2LA_{\text{smooth}} = \frac{d}{dq}\frac{\Lambda(q)}{2+y^*(q)}$$

which allows us to express the PDF S as a function of (Λ, y^*):

$$\boxed{2LS(q,y) = (y+2)[\frac{d}{dq}\frac{\Lambda(q)}{2+y^*(q)} + \frac{\delta^+(q-q_{min})}{2+L}] - \frac{d\Lambda}{dq}} \tag{89}$$

In the two previous equations, we have introduced the Cumulative Distribution Function of the remoistening distribution Φ:

$$\Lambda(q) \stackrel{\text{def}}{=} \int_{q}^{q_{max}} \Phi(q')dq' \tag{90}$$

4.2.3 Steady northwards PDF

We incorporate a saturated spike $\mathcal{W}(y)\delta^-[q - q^*(y)]$ in the total northwards PDF $\mathcal{N}$:

$$\mathcal{N}(q,y) = N(q,y) + \mathcal{W}(y) \cdot \delta^-[q - q^*(y)]$$

The total normalization condition for $\mathcal{N}$ (82) can now be written:

$$1 = 2L \int_{q_{min}}^{q^*(y)} \mathcal{N}(q,y)dq = 2L \int_{q_{min}}^{q^*(y)} N(q,y)dq + 2L\mathcal{W}(y)$$

which allows us to compute the weight $\mathcal{W}(y)$ of the saturated spike in this specific case:

$$1 = 1 - \frac{2}{2+y}\Lambda[q^*(y)] + 2L\mathcal{W}(y) \;\Rightarrow\; 2L\mathcal{W}(y) = \frac{2}{2+y}\Lambda[q^*(y)]$$

The generalized northwards PDF is then:

$$\boxed{2L\mathcal{N}(q,y) = y[\frac{d}{dq}\frac{\Lambda(q)}{2+y^*(q)} + \frac{\delta^+(q-q_{min})}{2+L}] - \frac{d\Lambda}{dq} + \frac{2\Lambda[q^*(y)]}{2+y}\delta^-[q - q^*(y)]} \quad (91)$$

We can verify a posteriori that the value found for $\mathcal{W}$ in this paragraph agrees with the differential equation we derive for the saturated weight $\mathcal{W}$ in appendix 119:

$$2L(\frac{d\mathcal{W}}{dy} + \frac{\mathcal{W}}{2}) = \frac{2}{2+y}\frac{dq^*}{dy}\frac{d\Lambda}{dq}(q^*) + \frac{y\Lambda(q^*)}{(2+y)^2}$$

$$2LN(q^*,y) = -\frac{d\Lambda}{dq}(q^*) + y[\frac{\frac{d\Lambda}{dq}(q^*)}{2+y} - \frac{\frac{dy^*}{dq}\Lambda(q^*)}{(2+y)^2}]$$

$$-2L\frac{dq^*}{dy}N(q^*,y) = 2\frac{dq^*}{dy}\frac{\frac{d\Lambda}{dq}(q^*)}{2+y} + \frac{y\Lambda(q^*)}{(2+y)^2} = 2L(\frac{d\mathcal{W}}{dy} + \frac{\mathcal{W}}{2}) \quad (92)$$

Equation 92 is just the weight equation 119, multiplied by the dimensionless extent of the domain L, which proves that it indeed represents the amount of supersaturated parcels. Note that adding the saturated peak in the PDF might prevent it to satisfy the remoistening boundary condition at $y = 0$ 87 in the special case where:

$$\mathcal{W}(0) = \Lambda[q_{max}] = \int_{q_{max}}^{q_{max}} \Phi(q)dq \neq 0 \;\Leftrightarrow\; \Phi(q) = \delta^-(q - q_{max})$$

This special case is treated in appendix F of this report. However, if we are to simulate this special case numerically (which we will do in 4.2.8), we will have a peak of finite amplitude $\varepsilon \neq 0$, which will result in $\mathcal{W}_{numerical}(0) = 0$.

4.2.4 Steady global distributions of specific humidity

The global distribution of specific humidity are by definition:

- For the southwards parcels:

$$2Ls(q) \stackrel{\text{def}}{=} 2L\int_0^{y^*(q)} S(q,y)dy$$

Using equation 89:

$$2Ls(q) = \frac{y^*(4+y^*)}{2}\frac{\delta^+(q-q_{min})}{2+L} + \frac{y^*(4+y^*)}{2}\frac{d}{dq}\frac{\Lambda(q)}{2+y^*(q)} - y^*\frac{d\Lambda}{dq}$$

$$\boxed{4Ls(q) = \frac{L(L+4)}{2+L}\delta^+(q-q_{min}) - \frac{(y^*)^2}{2+y^*}\frac{d\Lambda}{dq} - \frac{dy^*}{dq}\frac{y^*(4+y^*)}{(2+y^*)^2}\Lambda} \tag{93}$$

- For the northwards parcels:

$$2Ln(q) \stackrel{\text{def}}{=} 2L\int_0^{y^*(q)} \mathcal{N}(q,y)dy$$

Using equation 91:

$$2Ln(q) = \frac{(y^*)^2}{2}\frac{\delta^+(q-q_{min})}{2+L} + \frac{(y^*)^2}{2}\frac{d}{dq}\frac{\Lambda(q)}{2+y^*(q)} - y^*\frac{d\Lambda}{dq} + |\frac{dy^*}{dq}|\frac{2\Lambda(q)}{2+y^*(q)}$$

$$\boxed{4Ln(q) = \frac{L^2}{2+L}\delta^+(q-q_{min}) - \frac{y^*(4+y^*)}{2+y^*}\frac{d\Lambda}{dq} - \frac{dy^*}{dq}\frac{8+4y^*+(y^*)^2}{(2+y^*)^2}\Lambda} \tag{94}$$

- For all the parcels, following definition 33, the global distribution of specific humidity is given by:

$$2LP_Q(q) \stackrel{\text{def}}{=} 2L(n+s)(q)$$

$$\boxed{2P_Q(q) = \delta^+(q-q_{min}) - \frac{1}{L}\frac{d}{dq}[\Lambda y^*]} \tag{95}$$

Result 95 is equal to the universal form of $P_Q(q)$ 35 found in paragraph 2.3, as the two-stream model verifies the assumptions listed at the beginning of this paragraph.

4.2.5 Averages

By definition the average of a function $f(q)$ is:

$$\overline{f}(y) \stackrel{\text{def}}{=} L\int_{q_{min}}^{q^*(y)} f(q)(N+S)dq \tag{96}$$

Using the expressions 91 and 89 we computed for (N,S):

$$L(N+S) = [y+1][\frac{d}{dq}\frac{\Lambda(q)}{2+y^*(q)} + \frac{\delta^+(q-q_{min})}{2+L}] - \frac{d\Lambda}{dq} + \frac{\Lambda[q^*(y)]}{2+y}\delta^-[q-q^*(y)]$$

We can use integration by parts on integral 96 to obtain $\overline{f}$ as a function of $(\Lambda, q^*(y), f, \frac{df}{dq})$:

$$\boxed{\overline{f}(y) = \{\Lambda f\}[q^*(y)] - (y+1)\int_{q_{min}}^{q^*(y)} \frac{\Lambda(q)}{2+y^*(q)}\frac{df}{dq}dq - \int_{q_{min}}^{q^*(y)} \frac{d\Lambda}{dq}f(q)dq} \tag{97}$$

The meridional gradient of average 97 can be computed using Leibniz's formula to differentiate integrals, giving a fairly simple expression once simplified:

$$\boxed{\frac{d\overline{f}}{dy} = \frac{dq^*}{dy}\frac{\{\Lambda\frac{df}{dq}\}[q^*]}{2+y} - \int_{q_{min}}^{q^*(y)} \{\frac{\Lambda}{2+y^*}\frac{df}{dq}\}(q)dq} \tag{98}$$

We can use expressions 97 and 98 to compute the average moisture content and its meridional gradient as functions of latitude:

$$\overline{q}(y) = q^*\Lambda(q^*) - (y+1)\int_{q_{min}}^{q^*(y)} \frac{\Lambda(q)}{2+y^*(q)}dq - \int_{q_{min}}^{q^*(y)} \frac{d\Lambda}{dq}qdq \tag{99}$$

$$\frac{d\overline{q}}{dy} = \frac{dq^*}{dy}\frac{\Lambda(q^*)}{2+y} - \int_{q_{min}}^{q^*(y)} \{\frac{\Lambda}{2+y^*}\}(q)dq \tag{100}$$

4.2.6 Diffusivity of parcels in the two-stream model

The diffusivity of parcels in this model can be most easily computed by coming back to the initial equations 4.2 and working with dimensional variables. We define the dimensional northward flux of parcels as:

$$\mathcal{F}_c \stackrel{\text{def}}{=} V(N-S) \tag{101}$$

From the initial equations 4.2, we can relate the meridional gradient of the concentration of parcels c to the flux $\mathcal{F}_c$ defined in 101:

$$V\frac{\partial}{\partial y}(N+S) = V\frac{\partial c}{\partial y} = \beta(S-N) = -\frac{\beta}{V}\mathcal{F}_c$$

giving Fick's law for the parcel's flux:

$$\mathcal{F}_c = -\kappa\frac{\partial c}{\partial y}$$

where we have defined the dimensional diffusivity of parcels:

$$\boxed{\kappa \stackrel{\text{def}}{=} \frac{V^2}{\beta}} \tag{102}$$

In this case, we can prove that the diffusivity is consistent with its more fundamental definition. First, let's compute the correlation function for the velocities of the system in two different ways to better understand the system. To avoid the trivial case with no/weak exchanges of parcels, we release all the parcels at $t = 0$ at the Southern boundary, which means they all initially have northwards velocities:

1. We can directly solve the transient regime between $t' = 0$ and $t' = t$:

$$\begin{cases} \frac{dN}{dt} = \frac{\beta}{2}(S - N) \\ \frac{dS}{dt} = \frac{\beta}{2}(N - S) \end{cases} \Rightarrow \begin{cases} 2N = 1 + \exp(-\beta t) \\ 2S = 1 - \exp(-\beta t) \end{cases}$$

   where $1 = (N - S)(t = 0)$. The correlation at time t is then:

$$C(t) = V(0)V(t) = V^2(N - S)(0)(N - S)(t) = V^2 \exp(-\beta t)$$

2. We can also note that from the exchange equations, the probability of n signs reversal is given by the Poisson distribution of coefficient $\frac{\beta}{2}$:

$$\mathbb{P}(n) = \exp(-\frac{\beta t}{2})\frac{(\beta t)^n}{2^n n!}$$

   The correlation at time t is the expected value of $V(0)V(t)$:

$$C(t) = \mathbb{E}[V(0)V(t)]$$

$$C(t) = V \cdot V[\mathbb{P}(n = 2k) - \mathbb{P}(n = 2k + 1)] = V^2 \exp(-\frac{\beta t}{2}) \sum_{k \in \mathbb{N}} \frac{(\beta t)^{2k}}{2^{2k} 2k!} - \frac{(\beta t)^{2k+1}}{2^{2k+1}(2k + 1)!}$$

$$C(t) = V^2 \exp(-\frac{\beta t}{2}) \cdot [\cosh(\frac{\beta t}{2}) - \sinh(\frac{\beta t}{2})] = V^2 \exp(-\beta t)$$

In both cases, the diffusivity of the system is then equal to Taylor's diffusivity defined in 2:

$$\boxed{\kappa = \int_{\mathbb{R}_+} C(t)dt = \int_{\mathbb{R}_+} V^2 \exp(-\beta t)dt = \frac{V^2}{\beta}} \tag{103}$$

which is consistent with the fact that the parcels are purely passive tracers.

4.2.7 Diffusivity of moisture in the two-stream model

If we now want to look at the diffusivity of moisture, we need to compute:

1. The dimensional northward flux of moisture:

$$\mathcal{F}_q \stackrel{\text{def}}{=} L \int_{q_{min}}^{q^*(y)} q(VN - VS)(q)dq$$

$$\frac{\mathcal{F}_q}{V} = \int_{q_{min}}^{q^*(y)} qL(N - S)(q)dq$$

   We use the dimensionless expressions found for the PDF 91 and 89 so that integration by parts yields:

$$\frac{\mathcal{F}_q}{V} = \int_{q_{min}}^{q^*(y)} \frac{\Lambda(q)dq}{2 + \frac{\beta y^*}{V}}$$

2. The dimensionless mean meridional gradient of moisture computed in 100:

$$\frac{V}{\beta}\frac{d\overline{q}}{dy} = \frac{d\overline{q}}{d(\frac{\beta y}{V})} = \frac{V}{\beta}\frac{dq^*}{dy}\frac{\Lambda(q^*)}{2+\frac{\beta y}{V}} - \int_{q_{min}}^{q^*(y)} \frac{\Lambda(q)}{2+\frac{\beta y^*}{V}} dq$$

leading to a dimensional moisture flux which is lower than its Fick's value:

$$\boxed{\frac{\mathcal{F}_q}{\kappa} = -\frac{d\overline{q}}{dy} - |\frac{dq^*}{dy}|\frac{\Lambda(q^*)}{2+\frac{\beta y}{V}}} \tag{104}$$

$$\mathcal{F}_q < \mathcal{F}_{q,\text{Fickian}} = -\kappa\frac{d\overline{q}}{dy}$$

The difference between $\mathcal{F}_q$ and $\mathcal{F}_{q,\text{Fickian}}$ increases as the saturation specific humidity profile $\frac{dq^*}{dy}$ (and thus the mean specific humidity profile $\frac{d\overline{q}}{dy}$) becomes sharper and sharper. This difference is 0 for a flat profile $\frac{dq^*}{dy} = 0$, and the flux is perfectly Fickian in this limit.

4.2.8 Application to an exponentially decreasing $q^*(y)$ and a given remoistening $\Phi(q)$

We make the following assumptions:

1. We assume that the saturation specific humidity profile exponentially decreases with latitude:
$$q^*(y) \approx q_{max}\exp(-\alpha y) \;\Leftrightarrow\; y^*(q) \approx -\frac{1}{\alpha}\ln(\frac{q}{q_{max}})$$

2. We assume that the "Southern boundary resetting" produces complete saturation at $y = 0$ (case I of complete remoistening):
$$\Phi_I(q) = \delta^-(q - q_{max})$$
or remoistens the parcels from a uniform distribution on $[q_{min}, q_{max}]$ (case II of uniform remoistening):
$$\Phi_{II}(q) = \frac{1}{q_{max} - q_{min}}$$

To check the validity of our general solution in this specific case, we numerically simulate the SDE with:

$$\begin{cases} N_{tot} = 10^6 \\ q_{min} = 0.1 \\ q_{max} = 1 \\ L = \frac{(\beta=1)(L_{\text{dim}}=2)}{(V=1)} = 2 \end{cases}$$

and compare the numerical PDF to the marginal distributions $P_Y(y)$ and $(P_Q, n, s)(q)$:

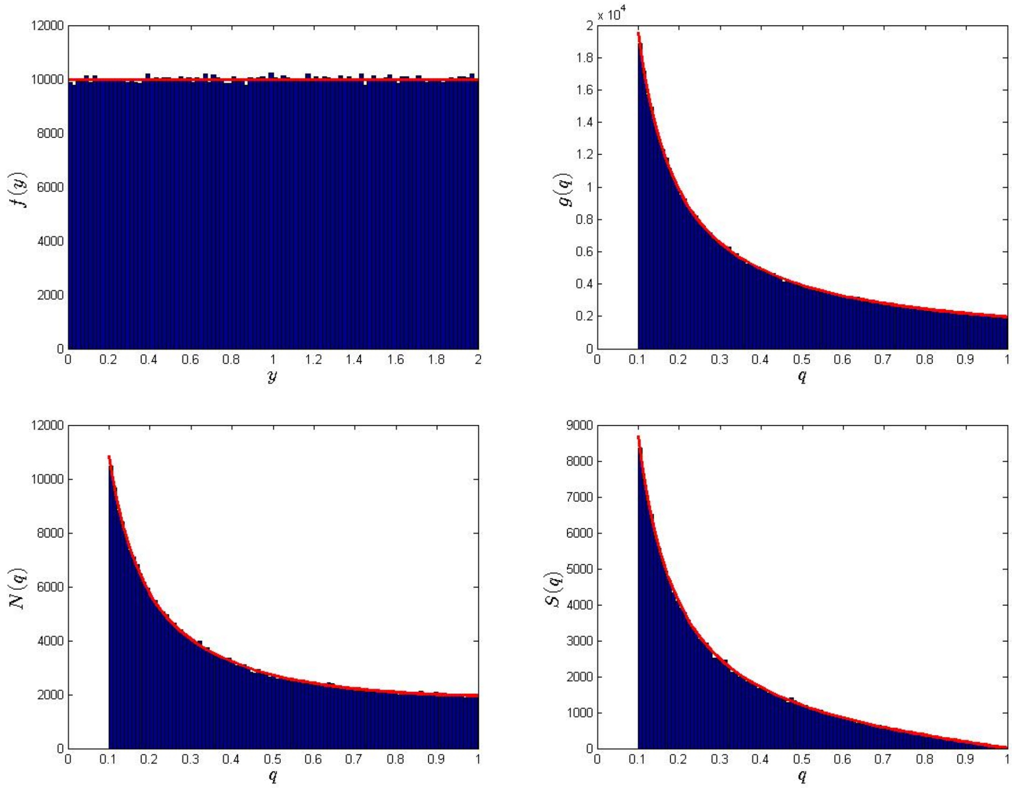

Figure 6: Numerical simulation of the SDE (blue) compared to the theoretical PDF (red) for $f(y)$ (top left), $g(q)$ (top right), $N(q)$ (bottom left) and $S(q)$ (bottom right) for a complete remoistening (case I)

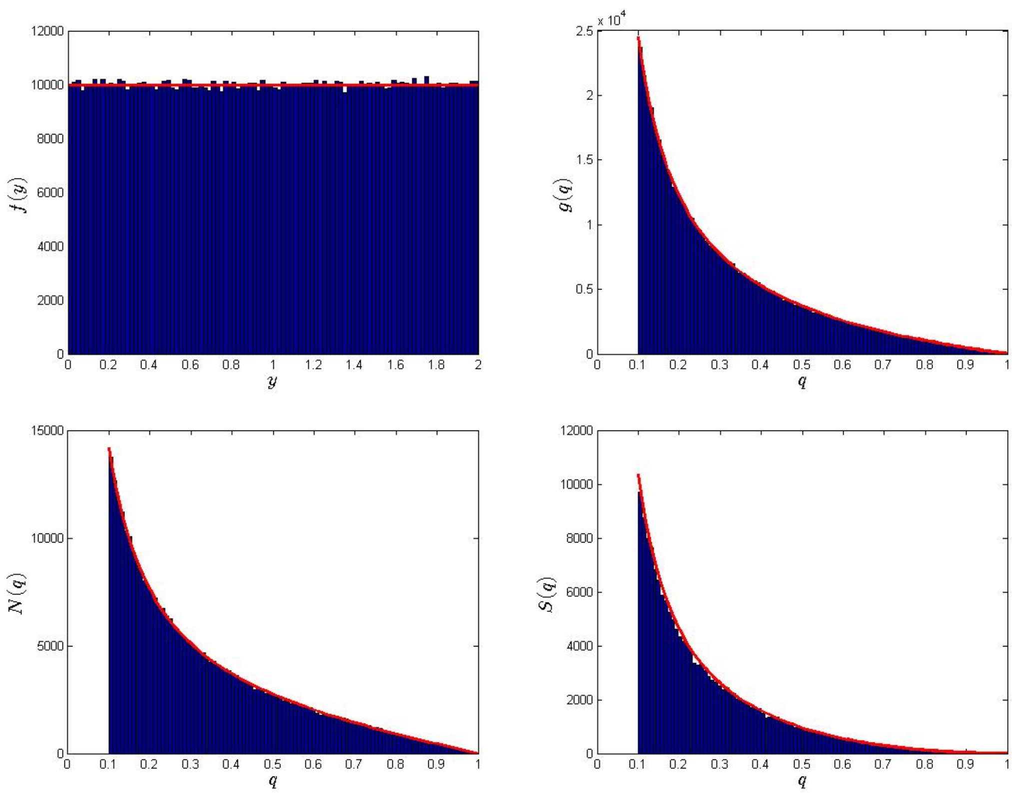

Figure 7: Numerical simulation of the SDE (blue) compared to the theoretical PDF (red) for $f(y)$ (top left), $g(q)$ (top right), $N(q)$ (bottom left) and $S(q)$ (bottom right) for a uniform remoistening (case II)

When L is increased, the SDE takes a longer time to reach a statistically steady state, as it takes longer for the parcels to reach the Southern and the Northern boundaries, making the numerical simulations of the steady PDF more costful. A good metric to see if a simulation has reached the statistically steady state, based on the steady expression of $P_Q(q)$, is based on the relative intensity of the dry peak/smooth part of $P_Q(q)$:

$$\frac{\|P_{Q,\mathrm{smooth}}(q)\|}{\|P_Q(q)\|} = 1 - \frac{\|P_{Q,\mathrm{dry}}(q)\|}{\|P_Q(q)\|}$$

At the beginning of the simulation, this ratio is close to 1, as not a lot of parcels have hit the Northern dry boundary. As the simulation reaches the steady state, we have proven in 95 that this ratio tends towards $\frac{1}{2}$. Using this ratio, we can see that if we want to obtain the steady PDF in the following case:

$$\begin{cases} N_{tot} = 10^4 \\ q_{min} = 0.1 \\ q_{max} = 1 \\ L = \frac{(\beta=10)(L_{\mathrm{dim}}=2)}{(V=1)} = 20 \end{cases}$$

we need to run the simulation at least 10 times longer than in the previous case to obtain the steady PDF:

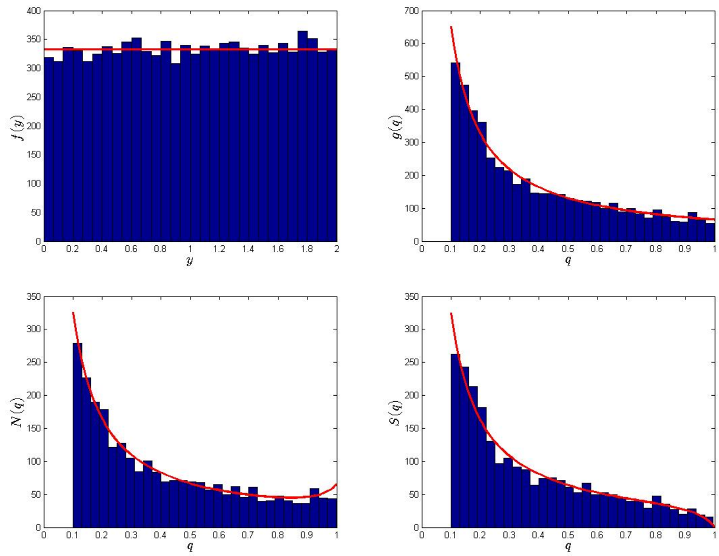

Figure 8: Numerical simulation of the SDE (blue) compared to the theoretical PDF (red) for $f(y)$ (top left), $g(q)$ (top right), $N(q)$ (bottom left) and $S(q)$ (bottom right) in case I (complete remoistening)

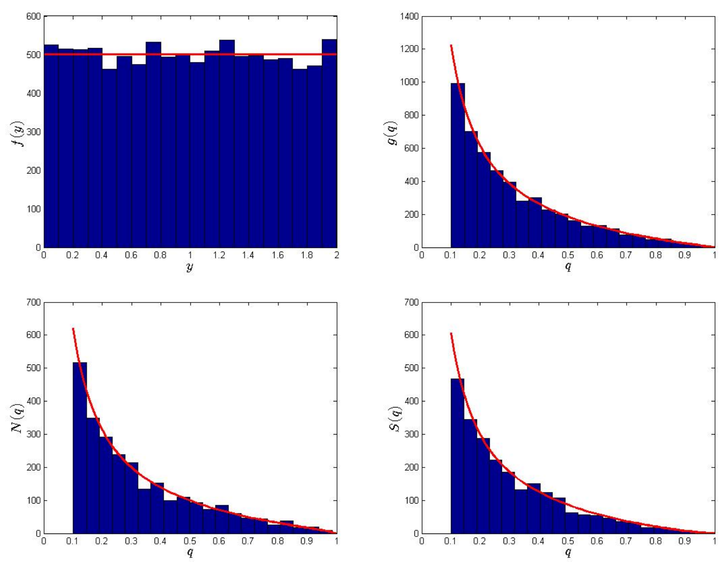

Figure 9: Numerical simulation of the SDE (blue) compared to the theoretical PDF (red) for $f(y)$ (top left), $g(q)$ (top right), $N(q)$ (bottom left) and $S(q)$ (bottom right) in case II (uniform remoistening)

Fortunately, in the case $L \to +\infty$, the system reaches the white noise limit (cf 4.2.9) and we can use the white noise SDEs (cf appendix A.2) to numerically simulate the steady PDF rather than waiting for an infinite amount of time. The global steady PDFs are very good quantities to check the validity of our analytical solutions because of their universal aspect, but because of this universality, they tell us very little about the two-stream model itself. The key quantities we are interested in are the degree of sub-saturation $\overline{q_{\text{sub}}} \stackrel{\text{def}}{=} \overline{q^* - q}$, the moisture flux $\mathcal{F}_q$, and the average condensation rate $\overline{\mathcal{C}} = -\frac{d\mathcal{F}_q}{dy}$. Their dimensional analytical expression in the two-stream model are respectively:

$$\boxed{\overline{q_{\text{sub}}}(y) = (y+1)\int_{q_{min}}^{q^*(y)} \frac{\Lambda(q)dq}{2+\frac{\beta y^*}{V}} - \int_{q_{min}}^{q^*(y)} \frac{d\Lambda}{dq} q_{sub} dq} \tag{105}$$

$$\boxed{\frac{\mathcal{F}_q}{\kappa} = -\frac{d\overline{q}}{dy} - |\frac{dq^*}{dy}|\frac{\Lambda(q^*)}{2+\frac{\beta y}{V}}} \tag{106}$$

$$\boxed{\frac{\overline{\mathcal{C}}(y)}{\kappa} = \frac{d^2 < q >}{dy^2} - \frac{1}{\sqrt{\tau\kappa}}\frac{dq^*}{dy}\frac{\Lambda(q^*)}{(2+\frac{\beta y}{V})^2} - (\frac{dq^*}{dy})^2 \frac{\frac{d\Lambda}{dq}(q^*)}{2+\frac{\beta y}{V}} - \frac{d^2 q^*}{dy^2}\frac{\Lambda(q^*)}{2+\frac{\beta y}{V}}} \tag{107}$$

We compare them in case I of a complete remoistening and for the numerical simulations with dimensionless extents $L \in \{2, 20\}$ studied previously:

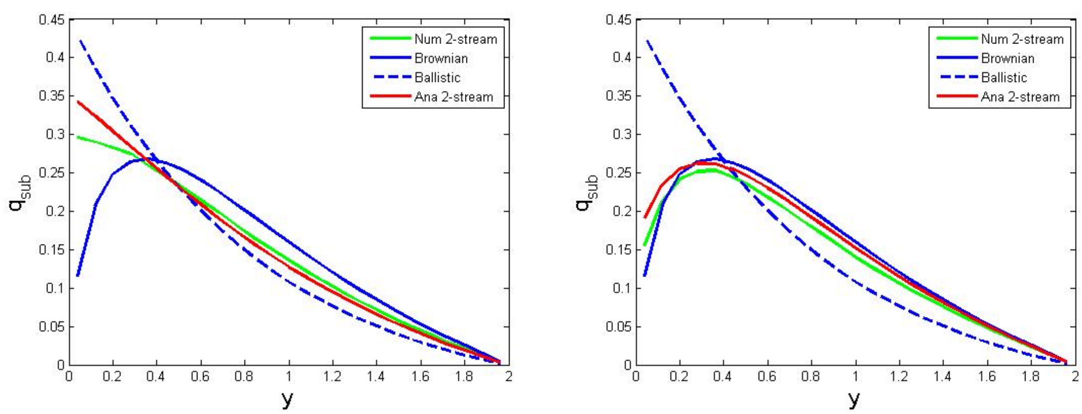

Figure 10: $< q_{sub} >$ vs y for $\hat{L} = 1$ (left) and $\hat{L} = 10$ (right); the analytical expression (red) is compared to the numerical flux (green)

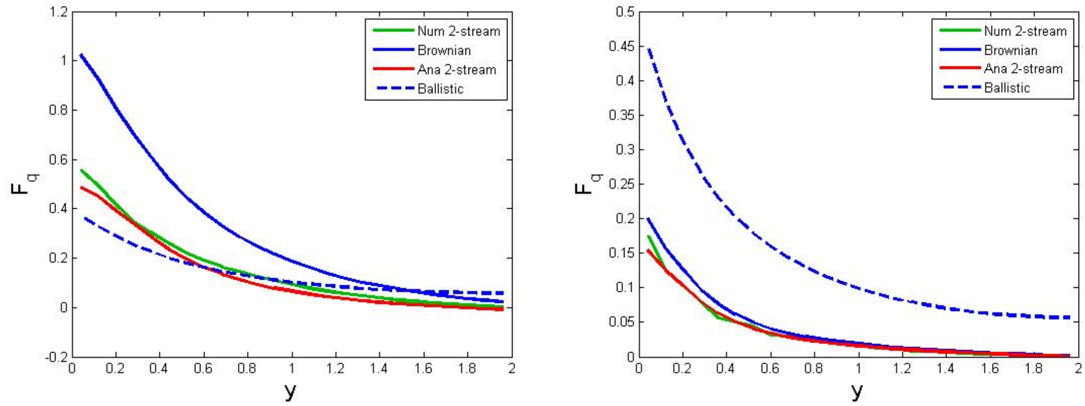

Figure 11: $\mathcal{F}_q$ vs y for $\hat{L} = 1$ (left) and $\hat{L} = 10$ (right); the analytical expression (red) is compared to the numerical flux (green)

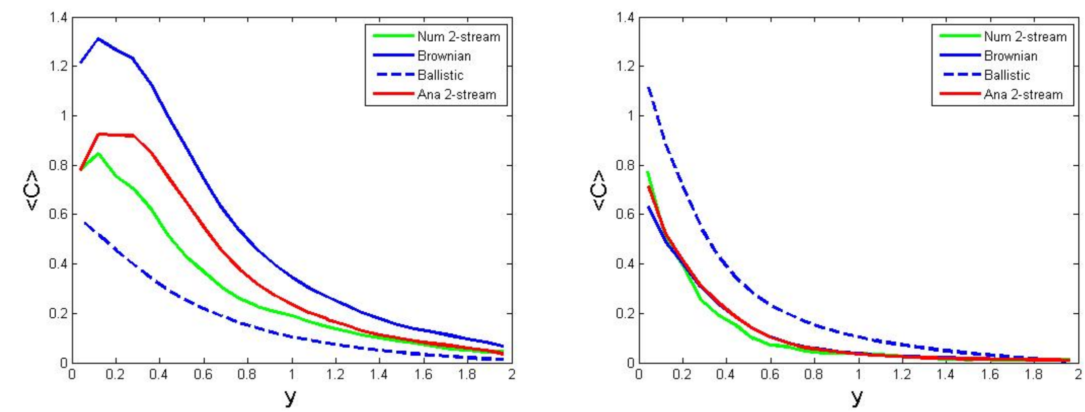

Figure 12: $< \mathcal{C} >$ vs y for $\hat{L} = 1$ (left) and $\hat{L} = 10$ (right); the analytical expression (red) is compared to the numerical flux (green)

Note that the analytical and numerical expressions of $\overline{\mathcal{C}}$ do not agree as well, as they imply the computation of second derivatives, which is not extremely precise due to the fact that we have only 25 discrete points to compute it in y.

4.2.9 White noise limit

The white noise limit can be obtained by keeping the diffusivity $\kappa = \frac{V^2}{\beta}$ and the dimensional length of the domain L_{dim} constant while letting $(V, \beta) \to +\infty$. As a consequence:

$$L = \frac{\beta L_{\text{dim}}}{V} = L_{\text{dim}} \sqrt{\frac{\beta}{\kappa}} \to +\infty$$

It follows that except very close to the equator (and the proportion of parcels with y close to 0 decreases very fast as $L \to +\infty$):

$$y \gg 1$$

The behavior of the PDF of the northwards and southwards parcels (91 and 89) follows:

$$2LS(q,y) = (y+2)[\frac{d}{dq}\frac{\Lambda(q)}{2+\frac{\beta y^*}{V}} + \frac{\delta^+(q-q_{min})}{2+L}] - \frac{d\Lambda}{dq} \sim_{L\to+\infty} y[\frac{d}{dq}\frac{\Lambda(q)}{y^*(q)} + \frac{\delta^+(q-q_{min})}{L}] - \frac{d\Lambda}{dq}$$

$$2L\mathcal{N}(q,y) = y[\frac{d}{dq}\frac{\Lambda(q)}{2+y^*(q)} + \frac{\delta^+(q-q_{min})}{2+L}] - \frac{d\Lambda}{dq} + \frac{2\Lambda[q^*(y)]}{2+\frac{\beta y}{V}}\delta^-[q-q^*(y)] \sim_{L\to+\infty} 2LS(q,y)$$

The total PDF is then asymptotically given by:

$$\boxed{L(\mathcal{N}+S) \sim_{L\to+\infty} y[\frac{d}{dq}\frac{\Lambda(q)}{y^*(q)} + \frac{\delta^+(q-q_{min})}{L}] - \frac{d\Lambda}{dq}} \tag{108}$$

which means that as expected, we recover the PDF found in the Brownian case (cf appendix A.2) assuming a white noise behavior in this limit.

4.3 Analytical approximation to the OU model

The n-stream model is the natural generalization of the two-stream model, where we consider n ensembles of parcels with n different velocities, that are exchanged at a rate proportional to $\frac{\beta}{2}$ (cf D). As $n \to +\infty$, the n-stream model converges to the OU model. Here, we use the analytical expressions derived for the average sub-saturation and the moisture flux in the case of the 2/3/4-stream model. We study how well they approximate the same quantities in the OU model, that we obtain from running Monte-Carlo simulations. The simulations are very similar to those described in 4.2.8 except that:

- Each parcel is advected by a velocity that follows an OU process with time-correlation τ.
- The parcels do not need to be exchanged between different processes anymore.

We have seen in 4.2.8 that the 2-stream model exactly reproduces the OU global distribution of moisture $P_Q(q)$. In figure 13, the average sub-saturation is very-well captured by the 2-stream model, except for the boundary layer near $y = 0$ which requires higher-order models, such as the 4-stream model. The values chosen for $(\kappa, \tau, L_{\text{dim}})$ give $L = 1$, and the OU model is quantitatively closer to the ballistic limit $L \ll 1$. As L increases, the OU average sub-saturation approaches the Ballistic limit, and a maximum appears in the distribution, corresponding to a zone of minimal relative humidity. In figure 14, the moisture flux is also well-approximated by the 2/3/4-stream model, confirming the important of taking into account the condensation in 1. Indeed, approximating the moisture flux as Fickian overestimates it by a factor ten near $y = 0$ for $L = 1$.

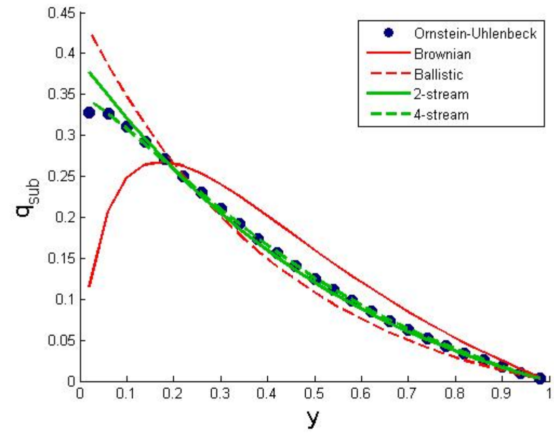

Figure 13: $\overline{q_{\text{sub}}}(y)$ vs y for $(\kappa, \tau, L_{\text{dim}}) = (1, 1, 1)$

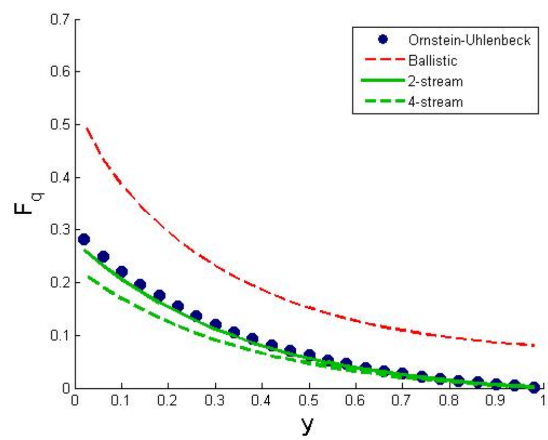

Figure 14: $\mathcal{F}_q(y)$ vs y for $(\kappa, \tau, L_{\text{dim}}) = (1, 1, 1)$

5 Conclusion

In conclusion, we have studied the main physics of the time-correlated advection-condensation model:

- The Ornstein-Uhlenbeck model for the velocity process is very well analytically approximated by the $n-$stream models.
- These models naturally produce a bimodal PDF of specific humidity, with a dry spike, a saturated spike, and a smooth part that decreases with q.
- The degree of sub-saturation and the condensation rate both decrease monotonically from the Equator to the Pole.
- In this model, time-correlation does not affect the global distribution for moisture $P_Q(q)$.

We have also obtained new insights in the diffusivity of a condensing scalar:

- The moisture flux is smaller than if Fick's law applied.
- The reduction of this moisture flux is proportional to the number of condensed parcels and the saturation specific humidity gradient to first approximation.

There are two important next steps for this project. First, it would be interesting to apply the 2/3/4-stream models to reanalysis data:

- Should latent heating be added to the model? The answer to this question would tell us when the moist parcels evolve on a dry or a moist isentropic surface.
- It is important to identify the limiting latitudes for this model: most likely, the model will not extend into the Tropics where microphysics and convection play a capital role in determining the distribution of specific humidity. As a consequence, the Tropics will most likely be taken as a Southern boundary condition for the model, producing a given remoistening distribution $\Phi(q)$.

In order to apply the model to reanalysis data, it is important to extent the model to a longitudinally symmetric sphere:

- An extension of the model to a longitudinally averaged sphere in the case of a white noise velocity process can be found in appendix E.
- It will be necessary to generalize the 2/3-stream model on a symmetric sphere.
- It would be ideal to also generalize the Ornstein-Uhlenbeck process to a symmetric sphere.

Finally, for geophysical applications, it will probably be important to generalize the model to a molecular diffusivity coefficient κ that varies with latitude.

A Limits of the OU model

A.1 The Ballistic limit

In the Ballistic limit $L \ll 1$, we rescale the latitude $Y = Ly$ to resolve the boundary layer, and obtain the following steady FPE in dimensionless form:

$$Lv\frac{\partial \rho}{\partial Y} = \frac{\partial}{\partial v}(v\rho + \frac{\partial \rho}{\partial v}) \tag{109}$$

To first approximation $L \to 0$; using 68 and 109:

$$\rho(q, y, v) \approx \frac{r(q, y, v)}{\sqrt{2\pi}L} \exp(-\frac{v^2}{2}) \tag{110}$$

where r is a function satisfying $\frac{\partial r}{\partial v} = 0$ almost everywhere. Physically, the velocity of a parcel remains unchanged from one boundary to another, which allows us write r as a sum of:

- A northwards part, understanding that the parcels saturate continuously as they move northwards:

  $$\{\Phi(q)\mathcal{H}[q^*(y) - q] + \delta^-[q - q^*(y)]\Lambda(q)\}\mathcal{H}(v)$$

  where we have used the definition 90.

- A southwards part, only constituted by a dry spike:

  $$\delta^+(q - q_{\min})\mathcal{H}(-v)$$

In this limit, the average of a function $f(q)$ is defined by:

$$\overline{f}(y) \stackrel{\text{def}}{=} L \int_{-\infty}^{+\infty} dv \int_{q_{\min}}^{q_s(y)} f(q)\rho(q, y, v)dq \tag{111}$$

From 111 and 110, we can approximate:

- The sub-saturation:

  $$\overline{q_{sub}}(y) \approx q^*(y) - q_{\min} - \frac{1}{2}\int_{q_{\min}}^{q^*(y)} \Lambda$$

- The moisture flux:

  $$\mathcal{F}_q \approx \sqrt{\frac{2\kappa}{\pi\tau}}\overline{q}(y)$$

A.2 The Brownian limit

In the Brownian limit $L \gg 1$, the stochastic differential equations for the displacement collapses to the model developed in [23]:

$$dY(t) = \sqrt{2\kappa}dW(t)$$

making the FPE a Laplace equation for the PDF:

$$\frac{\partial^2 \rho}{\partial y^2} = 0$$

The detailed solution of the Brownian problem can be found in [Sukhatme]; the main results of interest in our case are:

- The PDF:

$$\rho(q,y) = \frac{1}{L}\{\Phi(q) + y[\frac{\delta^+(q-q_{\min})}{L} + \frac{d}{dq}\frac{\Lambda(q)}{y^*(q)}]\}$$

- The average sub-saturation:

$$\overline{q_{sub}}(y) = \int_{q_{\min}}^{q^*(y)} \Phi q_{sub} + y\int_{q_{\min}}^{q^*(y)} \frac{\Lambda(q)dq}{y^*(q)}$$

- The Fickian moisture flux:

$$\mathcal{F}_q = -\kappa\frac{d\overline{q}}{dy}$$

B The $\varphi_\alpha-$functions

B.1 Definition from the second order differential equation

We start with the following second order differential equation, which is an eigenvalue problem in α:

$$\alpha v\varphi_\alpha = \mathcal{D}\varphi_\alpha \stackrel{\text{def}}{=} \frac{d^2\varphi_\alpha}{dv^2} - v\frac{d\varphi_\alpha}{dv} \tag{112}$$

Note that $\mathcal{D}$ is a self-adjoint operator for the scalar product defined for two well-behaved functions (f,g) by:

$$< f|g > \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp(-\frac{v^2}{2})f(v)g(v)dv \tag{113}$$

The eigenfunctions, ie the bounded solution of this differential equation for $\alpha^2 \in \mathbb{N}$, are a special kind of parabolic cylinder functions, that we call the $\varphi_\alpha-$functions, defined by:

$$\boxed{\varphi_\alpha(v) \stackrel{\text{def}}{=} \exp(-\alpha v - \alpha^2)He_{\alpha^2}(v + 2\alpha)} \tag{114}$$

where He_{α^2} is the α^2-th "probabilistic" Hermite polynomial in v. *To prove that the functions 114 are solutions to the eigenvalue problem 112, we can change variables from φ_α to ψ:*

$$\varphi_\alpha(v) = \psi(v)\exp(-\alpha v - \alpha^2)$$

$$\frac{d\varphi_\alpha}{dv} = [\frac{d\psi}{dv} - \alpha\psi]\exp(-\alpha v - \alpha^2)$$

$$\frac{d^2\varphi_\alpha}{dv^2} = [\frac{d^2\psi}{dv^2} - 2\alpha\frac{d\psi}{dv} + \alpha^2\psi]\exp(-\alpha v - \alpha^2)$$

leading to the differential equation:

$$\frac{d^2\psi}{dv^2} - (v + 2\alpha)\frac{d\psi}{dv} + \alpha^2\psi = 0$$

An obvious change of variable is $w = v + 2\alpha$, leading to:

$$\frac{d^2\psi}{dw^2} - w\frac{d\psi}{dw} + \alpha^2\psi = 0$$

which solutions can be express using the well-known "deterministic" Hermite polynomials of α^2-order:

$$\psi = He_{\alpha^2}(w)$$

where the "probabilistic" Hermite polynomials are defined by:

$$He_{\alpha^2}(v) \stackrel{\text{def}}{=} (-1)^{\alpha^2}\exp(\frac{v^2}{2})\frac{d^{\alpha^2}}{dv^{\alpha^2}}\exp(-\frac{v^2}{2}) = (v - \frac{d}{dv})^{\alpha^2}\cdot 1 \tag{115}$$

$$\psi = H_{\alpha^2}(\frac{w}{\sqrt{2}})$$

It is also possible to use "deterministic" Hermite polynomials, defined by:

$$H_{\alpha^2}(v) \stackrel{\text{def}}{=} (-1)^{\alpha^2}\exp(v^2)\frac{d^{\alpha^2}}{dv^{\alpha^2}}\exp(-v^2) = (2v - \frac{d}{dv})^{\alpha^2}\cdot 1 \tag{116}$$

The solution ψ is then given by:

$$\psi = 2^{-\frac{\alpha^2}{2}}H_{\alpha^2}(\frac{w}{\sqrt{2}})$$

Finally, to make sure we have solved the complete eigenvalue problem, we have to check for degenerate eigenvalues, which we do by differentiating the initial differential equation with respect to α:

$$\mathcal{D}\frac{\partial\varphi_\alpha}{\partial\alpha} = v\varphi_\alpha + \alpha v\frac{\partial\varphi_\alpha}{\partial\alpha}$$

Since $\mathcal{D}$ is self-adjoint for $< | >$:

$$0 =< \frac{\partial\varphi_\alpha}{\partial\alpha}|(\mathcal{D} - \alpha v)\varphi_\alpha >=< (\mathcal{D} - \alpha v)\varphi_\alpha|\varphi_\alpha >=< v\varphi_\alpha|\varphi_\alpha >$$

This equation is only satisfied for $\alpha = 0$, which means that we may take a generalized eigenfunction at this point:

$$\varphi_g(v) \stackrel{\text{def}}{=} v$$

Since:

$$\forall\alpha,\ < \varphi_\alpha|\varphi_g > \neq 0$$

by differentiating a second time with respect to α, we can show that the multiplicity of the eigenvalue $\alpha = 0$ is exactly 2. This proves that we only need φ_0 and φ_g to construct a general solution to the initial differential equation.

B.2 Orthogonality of the φ_α−functions

Let φ_α and φ_β be 2 eigenfunctions of 112 with eigenvalues $\alpha \neq \beta$:

$$(\alpha - \beta) < v\varphi_\alpha|\varphi_\beta >=< \mathcal{D}\varphi_\alpha|\varphi_\beta > - < \varphi_\alpha|\mathcal{D}\varphi_\beta >= 0$$

so that $(\varphi_\alpha, \varphi_\beta)$ satisfy the orthogonality relation:

$$< v\varphi_\alpha|\varphi_\beta >= \delta_{\alpha\beta} \tag{117}$$

where $\delta_{\alpha\beta}$ is the Kronecker symbol for (α, β). From the properties of Hermite polynomials, it can be proven that:

$$< v\varphi_0|\varphi_g >= 1$$

$$< v\varphi_\alpha|\varphi_\beta >= -2\alpha(\alpha^2!)\delta_{\alpha\beta} \cdot 1\{\alpha\beta > 0\}$$

$$< v\varphi_0|\varphi_0 >=< v\varphi_g|\varphi_g >=< v\varphi_0|\varphi_{\alpha\neq 0} >=< v\varphi_g|\varphi_{\alpha\neq 0} >= 0$$

meaning that the expansion of a well-behaved function $f(v) = o_{v\to+\infty}[\exp(\frac{v^2}{2})]$ satisfying the initial differential equation can be written:

$$f(v) = \frac{< v\varphi_g|f >}{< v\varphi_g|\varphi_0 >}\varphi_0 + \frac{< v\varphi_0|f >}{< v\varphi_0|\varphi_g >}\varphi_g + \sum_{\alpha^2\in\mathbb{N}^*} \frac{< v\varphi_\alpha|f >}{< v\varphi_\alpha|\varphi_\alpha >}\varphi_\alpha$$

$$\boxed{f(v) =< v^2|f > + < v|f > v + \sum_{\alpha^2\in\mathbb{N}^*} \frac{< v\varphi_\alpha|f >}{< v\varphi_\alpha|\varphi_\alpha >}\varphi_\alpha} \tag{118}$$

From the properties of the Hermite polynomials, we also obtain:

$$< 1|\varphi_{\alpha>0} >= \alpha^{\alpha^2} \exp(-\frac{\alpha^2}{2})$$

$$< 1|\varphi_{\alpha<0} >= (-1)^{\alpha^2}\alpha^{\alpha^2} \exp(-\frac{\alpha^2}{2})$$

B.3 Characteristics of the φ_α−functions

The four lowest orders functions can be found by setting:

$$\alpha \in \pm\{1, \sqrt{2}\}$$

From definition 114, we find:

$$\varphi_{-\sqrt{2}}(v) = \{v^2 - 4\sqrt{2}v + 7\} \exp(\sqrt{2}v - 2)$$

$$\varphi_{-1}(v) = (v - 2) \exp(v - 1)$$

$$\varphi_1(v) = (v + 2) \exp(-v - 1)$$

$$\varphi_{\sqrt{2}}(v) = \{v^2 + 4\sqrt{2}v + 7\} \exp(-\sqrt{2}v - 2)$$

Note that we actually only need to study $\alpha > 0$ or $\alpha < 0$ as from the evenness/oddness of the Hermite polynomials:

$$\varphi_{-\alpha}(v) = (-1)^{\alpha^2}\varphi_\alpha(v)$$

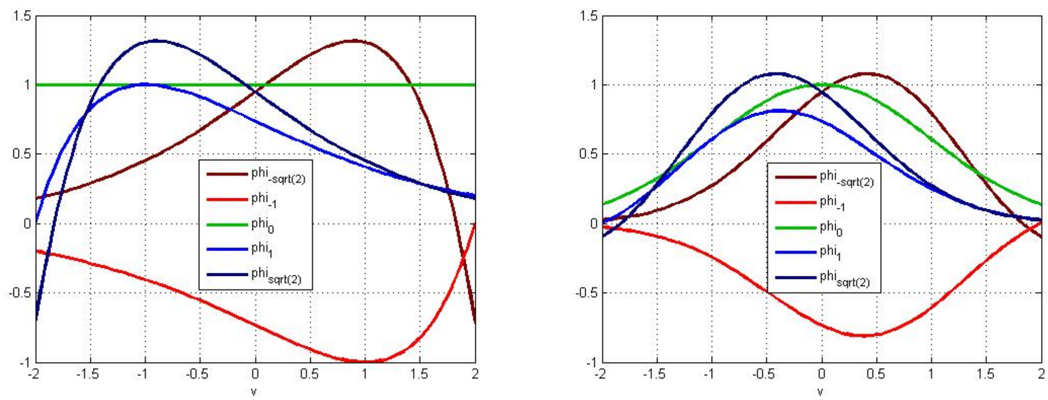

Figure 15: $\varphi_\alpha(\hat{v})$ (left) and $\exp(-\frac{\hat{v}^2}{2})\varphi_\alpha(\hat{v})$ (right) for $\alpha \in \{-\sqrt{2}, -1, 0, 1, \sqrt{2}\}$

C Dry and saturated spikes of the PDF

C.1 Definition

By construction of the n-stream (D) and OU models (4.1), the distribution comprises three "group" of parcels:

1. The parcels moving northwards in a zone where $y > y^*(q)$ are supersaturated and condensate as they move Northwards, constituting the saturated spike $\delta^-[q - q^*(y)]$.
2. The moisture of the parcels which have last hit the Northern boundary $y = L$ where $q = q^*(L) = q_{\min}$ cannot be changed until the parcels hit the Southern boundary $y = 0$. They constitute the dry spike $\delta^+(q - q_{\min})$.
3. The remaining part of the PDF is referred as the smooth part.

C.2 Saturated spike in the two-stream model

To understand how condensation occurs in the two-stream model, we relax the assumption of instant condensation; in dimensionless form:

$$\frac{\partial}{\partial y}\begin{pmatrix} N \\ -S \end{pmatrix} - \frac{\partial}{\partial q}[\mathcal{C}(q,y)\begin{pmatrix} N \\ S \end{pmatrix}] = \frac{1}{2}\begin{pmatrix} S - N \\ N - S \end{pmatrix}$$

Using the notations introduced in 4, we define a dimensionless condensation time that we assume small $\lambda^{-1}\tau \ll 1$, and use:

$$\mathcal{C}(q,y) = \lambda^{-1}\tau[q - q^*(y)]\mathcal{H}[q - q^*(y)]$$

In the supersaturated region $q \geq q^*(y)$, the parcels condensate fast enough for their velocities not to change sign, and we can assume $S \ll N$. The second equation gives $S \sim \lambda^{-1}\tau N$ making the approximation self-consistent, while integrating the first equation in the super-saturated region gives:

$$\frac{dW}{dy} + \frac{W(y)}{2} = -\frac{dq^*}{dy} N[q^*(y), y] \tag{119}$$

where we have defined the total density of supersaturated parcels:

$$W(y) \stackrel{\text{def}}{=} L \int_{q^*(y)}^{+\infty} N(q, y) dq \tag{120}$$

D The n-stream model

D.1 Definition

The n-stream model is a natural discretization of the OU model and thus a generalization of the two-stream model. It is easy to think about it as n bits which can take the value ± 1. If we consider one combination of bits, the sum of the bits gives the velocity of the corresponding parcel's ensemble, and allows us to build the advection matrix of the process$\boldsymbol{A}$. The exchange of parcels between ensembles happens when one bit's value is modified, and the probability of switching from one velocity to another allows us to build the exchange matrix of the process $\boldsymbol{E}$. Finally, the normalization condition is obtained by considering the probability of a combination of bits. Mathematically, defining the vectorial PDF $\underline{\rho}(q, y)$:

$$\frac{\partial}{\partial t}\underline{\rho} + \frac{V}{\sqrt{n-1}} \boldsymbol{A} \frac{\partial}{\partial y}\underline{\rho} = \frac{\beta}{2} \boldsymbol{E} \underline{\rho} \tag{121}$$

$$\boldsymbol{A}_{ij} \stackrel{\text{def}}{=} (n + 1 - 2i)\delta_{ij} \tag{122}$$

$$\boldsymbol{E}_{ij} \stackrel{\text{def}}{=} -n\delta_{ij} + j\delta_{i+1,j} + (n - j)\delta_{i-1,j} \tag{123}$$

where δ_{ij} is the Kronecker symbol. The normalization condition for $\underline{\rho}(q, y)$ yields:

$$\int_{q_{\min}}^{q^*(y)} \rho_i(q, y) dq = \frac{(n-1)!}{(i-1)!(n-i)!} \frac{1}{2^{n-1} L} \tag{124}$$

By construction of the n-stream model, the diffusivity of the parcels is given by:

$$\kappa \stackrel{\text{def}}{=} \frac{V^2}{\beta} \tag{125}$$

D.2 The three-stream model

The steady three-stream equations can be written:

$$\frac{V}{\sqrt{2}} \frac{\partial}{\partial y} \begin{pmatrix} 2N \\ 0 \\ -2S \end{pmatrix} = \frac{\beta}{2} \begin{pmatrix} -2 & 1 & 0 \\ 2 & -2 & 2 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} N \\ M \\ S \end{pmatrix}$$

where (N, M, S) respectively represent the northwards, motionless and southwards parcel's PDFs. We now work with the dimensionless variables defined in the two-stream model: 83 and 84. The normalization condition for the vectorial PDF is:

$$\int_{q_{\min}}^{q^*(y)} \begin{pmatrix} N \\ M \\ S \end{pmatrix} dq = \frac{1}{4L} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Following the same steps as for the two-stream model, we find that the vectorial PDF is the sum of:

- A smooth part:

$$\frac{\Phi(q)}{4L} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + C_{\text{smooth}}(q) \begin{pmatrix} y \\ 2(y+\sqrt{2}) \\ y+2\sqrt{2} \end{pmatrix}$$

- A dry spike:

$$C_{\text{dry}} \begin{pmatrix} y \\ 2(y+\sqrt{2}) \\ y+2\sqrt{2} \end{pmatrix}$$

- A saturated spike:

$$\frac{W(y)}{4L} \delta^{-}[q - q^*(y)] \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

where from the boundary and normalization conditions:

$$C_{\text{dry}} = \frac{1}{4L(L+2\sqrt{2})}$$

$$C_{\text{smooth}}(q) = \frac{1}{4L} \frac{d}{dq} \frac{\Lambda(q)}{y^*(q)+2\sqrt{2}}$$

$$W(y) = \frac{\sqrt{2}\Lambda[q_s(y)]}{y+2\sqrt{2}}$$

Defining the average of a function $f(q)$ by:

$$\overline{f}(y) \stackrel{\text{def}}{=} L \int_{q_{\min}}^{q^*(y)} f(q)(N+M+S)(q)dq$$

the average sub-saturation is:

$$\overline{q_{\text{sub}}} = \int_{q_{\min}}^{q^*(y)} \Phi q_{\text{sub}} + (y+\sqrt{2}) \int_{q_{\min}}^{q^*(y)} \frac{\Lambda(q)dq}{y^*(q)+2\sqrt{2}}$$

Defining the moisture flux as:

$$\mathcal{F}_q \stackrel{\text{def}}{=} L_{\text{dim}} \int_{q_{\min}}^{q^*(y)} \sqrt{2} V q(N-S)(q)dq$$

we can once again relate it to the moisture gradient in dimensional form:

$$\mathcal{F}_q = -\kappa(\frac{d\overline{q}}{dy} - W|\frac{dq^*}{dy}|)$$

There is very little difference between the 2/3-stream models, except for the presence of motionless parcels, which explains why the moisture flux is smaller in the three-stream case.

D.3 The four-stream model

The steady four-stream equations can be written:

$$\frac{V}{\sqrt{3}}\frac{\partial}{\partial y}\begin{pmatrix} 3N_3 \\ N_1 \\ -S_1 \\ -3S_3 \end{pmatrix} = \frac{\beta}{2}\begin{pmatrix} -3 & 1 & 0 & 0 \\ 3 & -3 & 2 & 0 \\ 0 & 2 & -3 & 3 \\ 0 & 0 & 1 & -3 \end{pmatrix}\begin{pmatrix} N_3 \\ N_1 \\ S_1 \\ S_3 \end{pmatrix}$$

where (N_3, N_1, S_1, S_3) respectively represent the fast northwards, slow northwards, slow southwards and fast southwards parcel's PDFs. We now work with the dimensionless variables defined in 83 and 84. The normalization condition for the vectorial PDF is:

$$\int_{q_{\min}}^{q^*(y)} \begin{pmatrix} N_3 \\ N_1 \\ S_1 \\ S_3 \end{pmatrix} dq = \frac{1}{8L}\begin{pmatrix} 1 \\ 3 \\ 3 \\ 1 \end{pmatrix}$$

Following the same steps as for the 2/3-stream models, we find that the vectorial PDF is the sum of:

- A smooth part:

$$\frac{\Phi(q)}{8L}\begin{pmatrix} 1 \\ 3 \\ 3 \\ 1 \end{pmatrix} + 147\sqrt{3}C_{\text{smooth},\pm}(q)\underline{c}_{\pm}(y)$$

- A dry spike:

$$147\sqrt{3}C_{\text{dry},\pm}\underline{c}_{\pm}(y)$$

- A saturated spike:

$$\frac{\delta^-[q - q^*(y)]}{8L}\begin{pmatrix} W_3(y) \\ W_1(y) \\ 0 \\ 0 \end{pmatrix}$$

where a sum over $\pm$ is implied and $\underline{c}_{\pm}(y)$ is given by:

$$\begin{pmatrix} (1 \mp 6\sqrt{2}) \pm \sqrt{6}(y + 2\sqrt{3}) - \exp(\pm\sqrt{6}y) \\ 3(1 \mp 6\sqrt{2}) \pm \sqrt{6}(3y + 8\sqrt{3}) - 3(1 \pm \sqrt{2})\exp(\pm\sqrt{6}y) \\ 3(1 \mp 6\sqrt{2}) \pm \sqrt{6}(3y + 10\sqrt{3}) - 3(5 \pm 4\sqrt{2})\exp(\pm\sqrt{6}y) \\ (1 \mp 6\sqrt{2}) + \pm\sqrt{6}(y + 4\sqrt{3}) + (3 \pm 2\sqrt{2})\exp(\pm\sqrt{6}y) \end{pmatrix}$$

From the boundary and normalization conditions, we can solve for the six unknowns $(C_{\text{smooth},\pm}(q), C_{\text{dry},\pm}, W_1(y), W_3(y))$:

$$C_{\text{dry},\pm} = -\frac{1}{8L}\frac{\sqrt{3}}{1764}\Gamma(L)$$

$$C_{\text{smooth},\pm}(q) = -\frac{1}{8L}\frac{\sqrt{3}}{1764}\frac{d(\Lambda\Gamma_\pm[y^*])}{dq}$$

$$\begin{pmatrix} W_3(y) \\ W_1(y) \end{pmatrix} = \mathcal{W}(y)\begin{pmatrix} -\sqrt{2}+5\sqrt{2}\cosh(\sqrt{6}y)+8\sinh(\sqrt{6}y) \\ 3[\sqrt{2}+3\sqrt{2}\cosh(\sqrt{6}y)+4\sinh(\sqrt{6}y)] \end{pmatrix}$$

where:

$$\Gamma_\pm(y) = \frac{\sqrt{2}+(3\sqrt{2}\mp 4)\exp(\mp\sqrt{6}y)}{-\sqrt{2}+\sqrt{2}(9+2\sqrt{3}y)\cosh(\sqrt{6}y)+(14+3\sqrt{3}y)\sinh(\sqrt{6}y)}$$

$$\mathcal{W}(y) = \frac{2\Lambda[q^*(y)]}{-\sqrt{2}+\sqrt{2}(9+2\sqrt{3}y)\cosh(\sqrt{6}y)+(14+3\sqrt{3}y)\sinh(\sqrt{6}y)}$$

Defining the average of a function $f(q)$ by:

$$\overline{f}(y) \stackrel{\text{def}}{=} L\int_{q_{\min}}^{q^*(y)} f(q)(N_3+N_1+S_1+S_3)(q)dq$$

the average sub-saturation is:

$$\overline{q_{\text{sub}}} = \int_{q_{\min}}^{q^*(y)} \Phi q_{\text{sub}} - \mathcal{Q}(y)$$

where:

$$\mathcal{Q}(y) = \frac{1\pm\sqrt{6}(\sqrt{3}+y)-2(1\pm\sqrt{2})\exp(\pm\sqrt{6}y)}{4}\int_{q_{\min}}^{q^*(y)}\Lambda\Gamma_\pm[y^*]$$

where a sum over $\pm$ is implied. Defining the moisture flux as:

$$\mathcal{F}_q \stackrel{\text{def}}{=} L_{\text{dim}}\int_{q_{\min}}^{q^*(y)} \sqrt{3}V q(N_3+\frac{N_1-S_1}{3}-S_3)(q)dq$$

we need to use the first and second meridional derivative of the average moisture to write the gradient/flux relation:

$$\mathcal{F}_q = -\kappa(\frac{d\overline{q}}{dy} - W_{tot}|\frac{dq^*}{dy}|)(1-\frac{2}{D})$$

$$+\kappa\mathcal{L}(\frac{d^2\overline{q}}{dy^2} - W_{tot}\frac{d^2q^*}{dy^2})(1-\frac{2+(\sqrt{2}-1)\exp(-\frac{\sqrt{6}\beta y}{V})}{D})$$

where $W_{tot} = W_1 + W_3$ is the total amount of parcels which have condensed, $\mathcal{L} = \frac{V}{\sqrt{6}\beta}$ is the boundary layer decay-length of the 4-stream model and the denominator in dimensionless form is:

$$D(y) = 2+\cosh(\sqrt{6}y)+\sqrt{2}\sinh(\sqrt{6}y)$$

Qualitatively, the 4-stream model adds a boundary layer structure to the solution, which explains the better agreement of the analytical 4-stream moisture flux with the OU flux for small y (cf figure 14). Furthermore, the boundary layer provides a positive contribution depending on $\frac{d^2q}{dy^2}$ to the moisture flux, once again decreased by the amount of condensed parcels.

E Brownian motion with condensation a longitudinally averaged sphere

E.1 Preliminary assumptions

We are now interested in the advection-condensation problem on a isentropic sphere of radius a, parametrized by longitude λ and latitude θ. We make the following simplifying assumptions:

1. By taking a longitudinal average of the model, we reduce our problem to a 1D process on the surface of a sphere.
2. The domain we work on extends from the Equator ($\theta = 0$) to a given latitude $\theta_L \in]0, \frac{\pi}{2}[$.
3. We suppose that the saturation specific humidity monotonically decreases from the Southern boundary of the domain ($\theta = 0$) where $q^* = q_{max}$ to the Northern boundary of the domain ($\theta = \theta_L$) where $q^* = q_{min} = \min_\theta q(\theta)$.
4. We model the source term $\mathcal{S}$ following Sukhatme et al.. The model "resets" the specific humidity q to a random value chosen from a specified distribution $\Phi(q)$ when it encounters the Southern boundary of the domain ($\theta = 0$). Physically, we could think of the fictive domain $\theta \leq 0$ as the Tropics, that remoisten the dry parcels that are advected southward by the mid-latitude eddies. The normalization condition on Φ yields:
$$\int_{q_{min}}^{q_{max}} \Phi = 1$$
5. We assume that the moist parcels on this sphere have a constant diffusivity coefficient κ.

E.2 Normalization of the concentration of moist parcel

On the sphere of constant radius, we define $c(\lambda, \theta, t)$ [m^{-2}] as the physical concentration of moist parcels. It can be computed from the generalized concentration γ in (q, λ, θ, t) space by integrating over the specific humidity q:

$$c(\lambda, \theta, t) \stackrel{\text{def}}{=} \int_{q_{min}}^{q^*(\theta)} \gamma(q, \lambda, \theta, t) dq$$

By definition, the concentration of moist parcels c verifies the following normalization condition:

$$N_{tot} = a^2 \int_0^{2\pi} d\lambda \int_0^{\theta_L} d\theta \cos(\theta) c(\lambda, \theta, t) = a^2 \int_{q_{min}}^{q^*(\theta)} dq \int_0^{2\pi} d\lambda \int_0^{\theta_L} d\theta \cos(\theta) \gamma(q, \lambda, \theta, t) \tag{126}$$

Defining the zonally averaged generalized condensation as:

$$< \gamma > (q, \theta, t) \overset{\text{def}}{=} \frac{1}{2\pi} \int_0^{2\pi} d\lambda \gamma(q, \lambda, \theta, t)$$

and introducing the distribution:

$$\nu(q, \theta, t) \overset{\text{def}}{=} \frac{2\pi a^2}{N_{tot}} < \gamma > (q, \theta, t) = \frac{a^2}{N_{tot}} \int_0^{2\pi} d\lambda \gamma(q, \lambda, \theta, t)$$

the previous normalization condition 126 yields:

$$\boxed{\int_0^{\theta_L} \int_{q_{min}}^{q^*(\theta)} \nu(q, \theta, t) \cos\theta d\theta dq = 1}$$

Integrating $\nu(q, \theta)$ over (q) produces the marginal density:

$$p(\theta, t) \overset{\text{def}}{=} \int_{q_{min}}^{q^*(\theta)} \nu(q, \theta) dq$$

In the stationary limit $\lim_{t \to +\infty} p(t) = p_\infty$ and $\underline{\nabla p_\infty} = \underline{0}$, which allows us to compute p_∞ through its normalization condition:

$$\int_0^{\theta_L} p_\infty \cos\theta d\theta = 1 \ \Rightarrow \ p_\infty = \sin^{-1}(\theta_L)$$

As a consequence, if we seek a long-time equilibrium solution $\nu_\infty = \lim_{t \to +\infty} \nu$, we require:

$$\boxed{1 = \sin\theta_L \cdot \int_{q_{min}}^{q^*(\theta)} \nu_\infty(q, \theta) dq} \tag{127}$$

E.3 FPE and SDE of the PDF

The generalized concentration γ of moist parcels verifies the evolution-condensation-diffusion equation on this sphere:

$$\frac{\partial \gamma}{\partial t} + \frac{\partial}{\partial q}[(\mathcal{S} - \mathcal{C})\gamma] = \kappa \nabla^2 \gamma = \frac{\kappa}{a^2} \{ \frac{1}{\cos\theta} \frac{\partial}{\partial \theta} [\cos\theta \frac{\partial \gamma}{\partial \theta}] + \frac{1}{\cos^2\theta} \frac{\partial^2 \gamma}{\partial \lambda^2} \}$$

Integrating the previous relation from $\lambda = 0$ to $\lambda = 2\pi$ and using the fact that:

- $\frac{\partial \gamma}{\partial \lambda}(\lambda = 0) = \frac{\partial \gamma}{\partial \lambda}(\lambda = 2\pi)$ because $\frac{\partial \gamma}{\partial \lambda}$ is a continuous function of longitude.

- $\frac{\partial\gamma}{\partial q}(\lambda = 0) = \frac{\partial\gamma}{\partial q}(\lambda = 2\pi)$ because $\frac{\partial\gamma}{\partial q}$ is a continuous function of longitude.
- $\frac{\partial}{\partial\lambda}(\mathcal{S} - \mathcal{C}) = 0$, because the source and the sinks in the system are longitudinally invariant.

we obtain:

$$\frac{\partial <\gamma>}{\partial t} = \frac{\kappa}{a^2\cos\theta}\frac{\partial}{\partial\theta}[\cos\theta\frac{\partial<\gamma>}{\partial\theta}]$$

Multiplying the previous equation by $\frac{2\pi a^2}{N_{tot}}$, we show that ν verifies the following evolution-diffusion equation:

$$\boxed{\frac{\partial\nu}{\partial t} = \frac{\kappa}{a^2\cos\theta}\frac{\partial}{\partial\theta}[\cos\theta\frac{\partial\nu}{\partial\theta}]} \tag{128}$$

We are now interested in the PDF $n(q,\theta,t)$ of Θ, which is the random variable associated to the latitudinal motion of the moist parcel. From the normalization condition on $n(q,\theta,t)$:

$$\int_0^{\theta_L}\int_{q_{min}}^{q^*(\theta)} n(q,\theta,t)d\theta dq = 1$$

and the normalization condition on ν_∞ (127) we can identify n as:

$$\boxed{n(q,\theta,t) = \cos\theta\cdot\nu(q,\theta,t)}$$

It is then immediate to derive the FPE for n from the equation that ν satisfies:

$$\frac{\partial n}{\partial t} = \frac{\kappa}{a^2}\frac{\partial}{\partial\theta}[\tan\theta\cdot n + \frac{\partial n}{\partial\theta}]$$

By identifying the drift and the diffusion terms, we recognize that the previous FPE corresponds to the Ito's SDE for a random walk on a sphere with diffusivity κ:

$$\boxed{d\Theta(t) = -\frac{\kappa\tan[\Theta(t)]}{a^2}dt + \frac{\sqrt{2\kappa}}{a}dW(t)} \tag{129}$$

We can see that the random walk is biased towards small values of Θ as the surface of the sphere decreases with the latitude θ. The SDE for the specific humidity stays the same:

$$dQ(t) = \{\mathcal{S}[\Theta(t)] - \mathcal{C}[\Theta(t), Q(t)]\}dt$$

E.4 Steady PDF

From equation 128, we can prove that the steady Markov diffusive process $\nu_\infty = \lim_{t\to+\infty}\nu$ verifies Laplace's equation:

$$\frac{\partial}{\partial\theta}[\cos\theta\frac{\partial\nu_\infty}{\partial\theta}] = 0$$

Integrating this equation is immediate and gives:

$$\nu_\infty(q,\theta) = A(q)\mathcal{L}(\theta) + B(q)$$

where we have defined the function:

$$\mathcal{L}(\theta) \stackrel{\text{def}}{=} \int_0^\theta \frac{d\theta'}{\cos(\theta')} = \ln[\frac{1+\sin(\theta)}{\cos(\theta)}]$$

and (A, B) need to be determined:

- First, we consider the origin $\theta = 0$ where $\nu_\infty(q, 0) = B(q) = \beta\Phi(q)$ where the second equality comes from the resetting condition at $(\theta = 0)$. We combine the normalization condition on Φ and the integral condition on ν_∞ to find β:

$$\begin{cases} \int_{q_{min}}^{q_{max}} \beta\Phi(q)dq = \beta \\ \int_{q_{min}}^{q_{max}} \nu_\infty dq = \sin^{-1}\theta_L \end{cases} \quad \Rightarrow \ \beta = \sin^{-1}\theta_L$$

- Because the only moistening mechanism at $(\theta = 0)$, the driest parcel $q = q_{min}$ will keep their dryness until they hit the Southern boundary. Consequently, ν_∞ will be peaked at q_{min}, and we need to take into account the "dry spike" $\delta^+(q - q_{min})$ in the distribution:

$$\nu_\infty(q, \theta) = [A_1 \cdot \delta(q - q_{min}) + A_2(q)] \cdot \mathcal{L}(\theta) + \frac{\Phi(q)}{\sin\theta_L}$$

where (A_1, A_2) are to be determined. At $\theta = \theta_L$ where $q^* = q_{min}$:

$$\int_{q_{min}}^{q_{max}} \nu_\infty(q, \theta_L)dq = A_1\mathcal{L}_L = \sin^{-1}\theta_L \ \Rightarrow \ A_1 = \{\sin\theta_L \cdot \mathcal{L}_L\}^{-1}$$

where we have defined :

$$\mathcal{L}_L = \mathcal{L}(\theta_L) = \ln[\frac{1+\sin(\theta_L)}{\cos(\theta_L)}]$$

which verifies:

$$\begin{cases} \mathcal{L}(\theta_L = 0) = 0 \\ \mathcal{L}(\theta_L \to \frac{\pi}{2}) \to +\infty \end{cases}$$

- Finally, we substitute ν_∞ in the normalization condition 127:

$$1 = \sin\theta_L \int_{q_{min}}^{q^*(\theta)} \nu_\infty dq = \frac{\mathcal{L}(\theta)}{\mathcal{L}_L} + \sin\theta_L\mathcal{L}(\theta) \int_{q_{min}}^{q^*(\theta)} A_2(q)dq + \int_{q_{min}}^{q^*(\theta)} \Phi(q)dq$$

$$\frac{1}{\mathcal{L}(\theta)} \int_{q^*(\theta)}^{q_{max}} \Phi(q)dq = \frac{1}{\mathcal{L}(\theta)} - \frac{1}{\mathcal{L}(\theta)} \int_{q_{min}}^{q^*(\theta)} \Phi(q)dq = \frac{1}{\mathcal{L}_L} + \sin\theta_L \int_{q_{min}}^{q^*(\theta)} A_2(q)dq$$

We solve for A_2 by using q rather than θ as the independent variable; defining θ^* as the reciprocal function of q^*, we can rewrite the previous equation as:

$$\frac{1}{\mathcal{L}[\theta^*(q)]} \int_q^{q_{max}} \Phi(q')dq' = \frac{1}{\mathcal{L}_L} + \sin\theta_L \int_{q_{min}}^{q} A_2(q')dq'$$

We obtain A_2 by taking the derivative of the previous equation:

$$\sin\theta_L \cdot A_2(q) = \frac{d}{dq}\frac{\Lambda(q)}{\mathcal{L}[\theta^*(q)]}$$

where once again Λ is the CDF of Φ which is defined in 90 so that $\Lambda(q_{min}) = 1$.

In conclusion, we can write ν_∞ as a function of (Λ, θ^*):

$$\boxed{\sin\theta_L \cdot \nu_\infty = [\frac{\delta^+(q - q_{min})}{\mathcal{L}_L} + \frac{d}{dq}\frac{\Lambda(q)}{\mathcal{L}[\theta^*(q)]}]\mathcal{L}(\theta) - \frac{d\Lambda}{dq}} \tag{130}$$

In the limit $\theta_L \to \frac{\pi}{2}$ (full Northern hemisphere), the distribution reduces to:

$$\nu_\infty = \mathcal{L}(\theta)\frac{d}{dq}\frac{\Lambda(q)}{\mathcal{L}[\theta^*(q)]} - \frac{d\Lambda}{dq}$$

In that limit, no parcels can reach the Northern pole ($\theta \to \frac{\pi}{2}$), which surface is 0, which suppresses the dry spike in the distribution. Note that the previous expressions are only valid for $q \leq q^*(\theta)$:

$$\boxed{\nu_\infty[q > q^*(\theta)] = 0}$$

which means that we have completely solved the FPE in this particular case. The steady PDF in spherical coordinates $n_\infty = \lim_{t\to+\infty} n$ can be directly computed from the previous solution:

$$n_\infty(q, \theta) = \cos(\theta) \cdot \nu_\infty(q, \theta)$$

E.5 Global steady PDF

By definition, the global PDF of specific humidity is:

$$P_Q(q) \stackrel{\text{def}}{=} \int_0^{\theta^*(q)} n_\infty(q, \theta)d\theta = \int_0^{\theta^*(q)} \cos\theta \cdot \nu_\infty(q, \theta)d\theta$$

Using the solution computed previously for ν_∞ (130), we can evaluate this integral:

$$\boxed{\sin\theta_L \cdot g(q) = [\frac{\delta^+(q - q_{min})}{\mathcal{L}_L} + \frac{d}{dq}\frac{\Lambda(q)}{\mathcal{L}[\theta^*(q)]}]\mathcal{K}[\theta^*(q)] - \frac{d\Lambda}{dq}\sin[\theta^*(q)]}$$

where we have defined the function:

$$\mathcal{K}(\theta) \stackrel{\text{def}}{=} \int_0^{\theta} \cos\theta' \cdot \mathcal{L}(\theta')d\theta' = \ln(\cos\theta) + \sin(\theta) \cdot \mathcal{L}(\theta)$$

E.6 Averages

With the knowledge of ν_∞, we can compute the average of any function $f(q)$, which we define by:

$$\overline{f}(\theta) = \sin\theta_L \int_{q_{min}}^{q^*(\theta)} f(q)\nu_\infty(q, \theta)dq$$

Using the expression we computed for ν_∞ (130), we can use integration by parts to obtain $\overline{f}$ as a function of $(\Lambda, q^*(\theta), f, \frac{df}{dq})$:

$$\boxed{\overline{f}(\theta) = \{\Lambda f\}[q^*(\theta)] - \mathcal{L}(\theta)\int_{q_{min}}^{q^*(\theta)} \{\frac{\Lambda}{\mathcal{L}[\theta^*]}\frac{df}{dq}\}(q)dq - \int_{q_{min}}^{q^*(\theta)} \{f\frac{d\Lambda}{dq}\}(q)dq} \tag{131}$$

Once again, we can compute the meridional gradient using Leibniz's formula to differentiate integrals:

$$\boxed{\frac{d\overline{f}}{d\theta} = -\frac{1}{\cos\theta}\int_{q_{min}}^{q^*(\theta)}\{\frac{\Lambda}{\mathcal{L}[\theta^*]}\frac{df}{dq}\}(q)dq} \tag{132}$$

We can use the previous expressions 131 and 132 to compute the average moisture content $\overline{q}(\theta)$ and its meridional gradient:

$$\overline{q}(\theta) = q^*\Lambda(q^*) - \mathcal{L}(\theta)\int_{q_{min}}^{q^*(\theta)}\frac{\Lambda(q)}{\mathcal{L}[\theta^*(q)]}dq - \int_{q_{min}}^{q^*(\theta)} q\frac{d\Lambda}{dq}dq$$

$$\frac{d\overline{q}}{d\theta} = -\frac{1}{\cos\theta}\int_{q_{min}}^{q^*(\theta)}\frac{\Lambda(q)}{\mathcal{L}[\theta^*(q)]}dq$$

E.7 Application to an exponentially decreasing $q^*(\theta)$ and a given remoistening $\Phi(q)$

We make the following assumptions:

1. We assume that the saturation specific humidity profile exponentially decreases with latitude:
$$q^*(\theta) \approx q_{max}\exp(-\alpha\theta) \;\Leftrightarrow\; \theta^*(q) \approx -\frac{1}{\alpha}\ln(\frac{q}{q_{max}})$$

2. We assume that the "Southern boundary resetting" produces complete saturation at $\theta = 0$ (case I of complete remoistening):
$$\Phi_I(q) = \delta^-(q - q_{max})$$
or that it remoistens the parcels following a uniform distribution (case II of uniform remoistening):
$$\Phi_{II}(q) = \frac{1}{q_{max} - q_{min}}$$

In case I, the steady distribution ν_∞ can be written:

- For $\theta = 0$ (Southern edge):
$$\nu_\infty(q, 0) = \frac{\delta^-(q - q_{max})}{\sin\theta_L}$$

- For $\theta \in]0, \theta_L]$ (Interior of the domain and Northern edge):
$$\frac{\sin\theta_L}{\mathcal{L}(\theta)}\nu_\infty(q,\theta) = \frac{\delta^+(q - q_{min})}{\mathcal{L}_L} + \frac{1}{(\alpha q)\cdot[\mathcal{L}^2\cos][\theta^*(q)]}$$

Similarly, the distribution g is given by:

- For $q = q_{max}$ (Southern edge):
$$g(q_{max}) = 0$$

- For $q \in [q_{min}, q_{max}[$ (Interior of the domain and Northern edge):

$$\frac{\sin\theta_L}{\mathcal{K}[\theta^*(q)]} g(q) = \frac{\delta^+(q - q_{min})}{\mathcal{L}_L} + \frac{1}{(\alpha q)\cdot[\mathcal{L}^2\cos][\theta^*(q)]}$$

To check the validity of our general solution in this specific case, we numerically simulate the SDE on a sphere with:

$$\begin{cases} N_{tot} = 3.10^4 \\ q_{min} = 0.1 \\ q_{max} = 1 \\ \kappa = 1 \\ a = 1 \end{cases}$$

and compare the numerical PDF to the marginal distributions $f(\theta) = \cos\theta p_\infty$ and $g(q)$ for different values of the latitudinal extent of the domain θ_L.

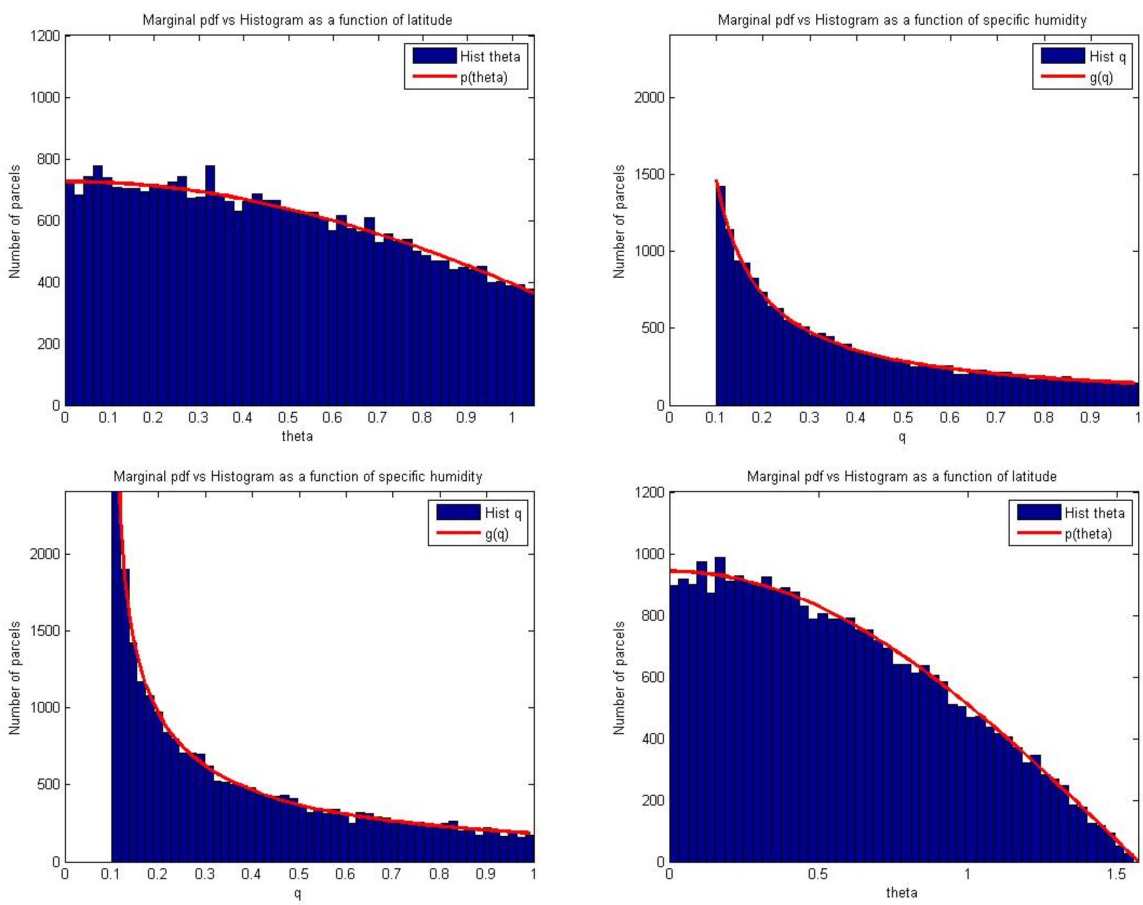

Figure 16: $\cos\theta \cdot p_\infty$ (left) and $g(q)$ (right) for $\theta_L = \frac{\pi}{3}$ (top) and $\theta_L = \frac{\pi}{2}$ (bottom) in case I of a complete remoistening

In case II, we obtain:

$$\sin\theta_L \cdot \nu_\infty = [\frac{\delta^+(q-q_{min})}{\mathcal{L}_L} + \frac{q_{max}-q}{q_{max}-q_{min}} \frac{1}{\alpha q[\mathcal{L}^2 \cos][\theta^*(q)]}]\mathcal{L}(\theta) + \frac{1}{q_{max}-q_{min}}\{1 - \frac{\mathcal{L}(\theta)}{\mathcal{L}[\theta^*(q)]}\}$$

$$\sin\theta_L \cdot g(q) = [\frac{\delta^+(q-q_{min})}{\mathcal{L}_L} + \frac{q_{max}-q}{q_{max}-q_{min}} \frac{1}{\alpha q[\mathcal{L}^2 \cos][\theta^*(q)]}]\mathcal{K}[\theta^*(q)] + \frac{1}{q_{max}-q_{min}}\{\sin[\theta^*(q)] - \frac{\mathcal{K}[\theta^*(q)]}{\mathcal{L}[\theta^*(q)]}\}$$

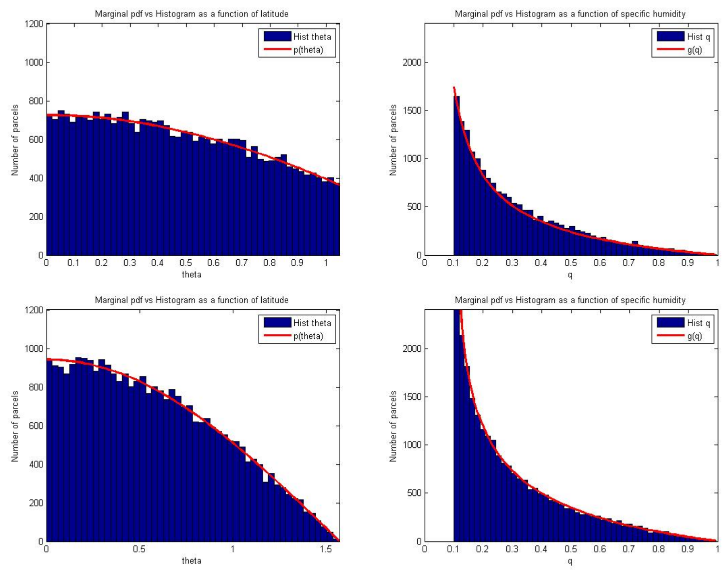

Figure 17: $\cos\theta \cdot p_\infty$ (left) and $g(q)$ (right) for $\theta_L = \frac{\pi}{3}$ (top) and $\theta_L = \frac{\pi}{2}$ (bottom) in case II of a uniform remoistening

F Incorporation of the saturated peak in the special case $\Phi_I(q) = \delta^-(q - q_{max})$

Here we will take into account the saturated peak in the initial conditions of the two-stream model 81, and prove that only the moist peak at $q = q_{max}$ is modified. The argument can be easily generalized to the Ornstein-Uhlenbeck process 4.1. Keeping the same notations, we add the dry peak and the saturated peak to the solution without applying any boundary condition:

$$\begin{cases} \mathcal{N}(q,y) = y[A_{\text{smooth}}(q) + A_{dry}\delta^+(q-q_{min})] + B(q) + \mathcal{W}(y)\delta^-(q-q*) \\ S(q,y) = (y+2)[A_{\text{smooth}}(q) + A_{dry}\delta^+(q-q_{min})] + B(q) \end{cases}$$

The normalized boundary condition at $y = 0$ (87) gives:

$$\Phi(q) = 2LN(q,0) = 2LB(q) + 2L\mathcal{W}(0)\Phi(q) \;\Rightarrow\; 2LB(q) = [1 - 2L\mathcal{W}(0)]\Phi(q)$$

where we have used the fact that $\Phi(q) = \delta^-(q - q_{max})$ and $q^*(0) = q_{max}$. The dry peak intensity doesn't change as it is determined by the normalization condition for the southwards

PDF S 82 at $y = L$ where $q = q^*(L) = q_{min}$:

$$2LA_{\text{dry}} = \frac{1}{2+L}$$

Again, the southwards PDF S satisfies the normalization condition 82 for all $y \in [0, L]$, and does not have any saturated peak:

$$1 = 2L\int_{q_{min}}^{q^*(y)} S(q,y)dq = [y+2][\frac{1}{2+L}+2L\int_{q_{min}}^{q^*(y)} A_{\text{smooth}}(q)dq]+[1-2L\mathcal{W}(0)]\int_{q_{min}}^{q^*(y)} \Phi(q)dq$$

$$2L\int_{q_{min}}^{q^*(y)} A_{\text{smooth}}(q)dq = \frac{1}{y+2}[\int_{q^*(y)}^{q_{max}} \Phi(q)dq + 2L\mathcal{W}(0)\int_{q_{min}}^{q^*(y)} \Phi(q)dq] - \frac{1}{2+L}$$

Switching the independent variable from y to q , changing the bounds of the integrals carefully according to figure 2, and differentiating the previous equation with respect to q, we obtain:

$$2LA_{\text{smooth}} = \frac{d}{dq}\frac{1+2L\mathcal{W}(0)\delta_{qq_{max}}}{2+y^*(q)}$$

where we have used the fact that:

$$\begin{cases} \int_q^{q_{max}} \Phi(q')dq' = \int_q^{q_{max}} \delta^-(q'-q_{max})dq' = 1 \\ \int_{q_{min}}^{q} \Phi(q')dq' = \int_{q_{min}}^{q} \delta^-(q'-q_{max})dq' = \delta_{qq_{max}} \end{cases}$$

where $\delta_{qq_{max}}$ is the Kronecker symbol for $q = q_{max}$. Finally, we find the weight of the saturated parcels $\mathcal{W}$ by applying the normalization condition 82 for $\mathcal{N}$:

$$1 = 2L\int_{q_{min}}^{q^*(y)} \mathcal{N}(q,y)dy$$

$$1 = \frac{y[1+2L\mathcal{W}(0)\delta_{q^*q_{max}}]}{2+y} + 2L\mathcal{W}(y)$$

$$2L\mathcal{W}(y) = \frac{2}{2+y}$$

leading to a PDF that only comprises a dry and a saturated peak in this special case:

$$\boxed{2LS(q,y) = (2+y)[\frac{\delta^+(q-q_{min})}{2+L} + \frac{d}{dq}\frac{1+\delta_{qq_{max}}}{2+y^*(q)}]}$$

$$2LS(q,y) = (2+y)[\frac{\delta^+(q-q_{min})}{2+L} + \frac{\delta^-(q-q_{max})}{2+y^*(q)} - \frac{dy^*}{dq}\frac{1+\delta_{qq_{max}}}{(2+y^*)^2}]$$

$$\boxed{2L\mathcal{N}(q,y) = y[\frac{\delta^+(q-q_{min})}{2+L} + \frac{d}{dq}\frac{1+\delta_{qq_{max}}}{2+y^*(q)}] + \frac{2\delta^-(q-q^*)}{2+y}}$$

$$2L\mathcal{N}(q,y) = y[\frac{\delta^+(q-q_{min})}{2+L} + +\frac{\delta^-(q-q_{max})}{2+y^*(q)} - \frac{dy^*}{dq}\frac{1+\delta_{qq_{max}}}{(2+y^*)^2}] + \frac{2\delta^-(q-q^*)}{2+y}$$

In conclusion, the distributions agree with the ones computed in 91 and 89 everywhere except at $q = q_{max}$, which means that:

- To compute the analytical solution, we can safely apply the boundary condition on N directly without having to worry about the saturated peak.
- The limit $\Phi(q) \to \delta^{-}(q - q_{max})$ is singular at $q = q_{max}$, where a moist peak $\delta^{-}(q - q_{max})$ linearly increasing with y appears.

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