Lecture 5

Upper Bounds for Turbulent Transport

F. H. Busse

Notes by Tomoki Tozuka and Huiquin Wang

1 Introduction

Malkus [1] observed kinks in Nu (Nusselt number) - Ra (Rayleigh number) relationship of turbulent Rayleigh-Benard convection and formulated a mean field theory for superposition of convective modes using hypothesis of maximum transport in 1954. In 1963, Howard [2] derived rigorous upper bound for Nusselt number, $Nu \leq c\ Ra^{\frac{1}{2}}$. Then, Busse [3, 4] improved bounds through incorporation of the continuity equation constraint, introduced multi-alpha solutions of variational problem, and derived upper bound $M \leq cRe^2$ (Re: Reynolds number) for an momentum transports in shear layers in 1969. On the other hand, Doering and Constantin [5] extended the method of Hopf to derive bounds on dissipation by turbulent flows in 1994 (see Lecture 6 for the detail). Nicodemus et al. [6] optimized Doering-Constantin approach in 1997 and Kerswell [7] proved the equivalence of Doering-Constantin and Howard-Busse methods in 1998 (see Lecture 10 for the proof) . This lecture is focused on the Howard-Busse method.

The theory of upper bounds for functionals of turbulent flows provides rigorous bounds for transport properties. It also indicates characteristic properties of extremalizing vector fields, which are reflected in observations of turbulent flows and thus can provide some insights into properties of turbulence.

2 Upper Bounds on Momentum Transport Between Two Moving Parallel Plates

In this section, we consider a flow between two moving parallel plates as shown in Fig. 1. Using the distance d between two plates as length scale, and d^2/μ as time scale, we write the Navier-Stokes equation for the incompressible fluid in the form

$$\frac{\partial}{\partial t}\mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + 2\Omega \times \mathbf{v} = -\nabla p \ + \nabla^2 \mathbf{v} \tag{1}$$

$$\nabla \cdot \mathbf{v} = 0\,. \tag{2}$$

We separate the velocity field $\mathbf{v}$ into a mean and a fluctuating part:

$$\mathbf{v} = \mathbf{U} + \check{\mathbf{v}} \text{ with } \overline{\check{\mathbf{v}}} = 0\ ,\ \overline{\mathbf{v}} \equiv \mathbf{U}(z,t) \tag{3}$$

where

$$\overline{\cdots} \equiv \lim_{L\to\infty} \frac{1}{4L^2} \int_{-L}^{L} \int_{-L}^{L} \cdots d\mathbf{x}\,. \tag{4}$$

We also separate the fluctuating part of the velocity field $\breve{\mathbf{v}}$ into components perpendicular and parallel to the plates as

$$\breve{\mathbf{v}} \equiv \breve{\mathbf{u}} + \mathbf{k}\breve{w} \text{ with } \breve{\mathbf{u}} \cdot \mathbf{k} = 0\,. \tag{5}$$

[width=]fig1.eps

Figure 1: Schematic sketch of a flow between two moving parallel plates.

For $\Omega \cdot \mathbf{k} = 0$ (e.g. Taylor-Couette case), since $\mathbf{U}$ does not have a z-component because of the continuity equation, the average over planes z=constant of (1) yields

$$\frac{\partial}{\partial t}\mathbf{U} + \overline{\breve{\mathbf{v}} \cdot \nabla \breve{\mathbf{u}}} = \frac{\partial^2}{\partial z^2}\mathbf{U} \tag{6}$$

$$\overline{\breve{\mathbf{v}} \cdot \nabla \breve{w}} = -\frac{\partial}{\partial z}\overline{p} - 2\Omega \times \mathbf{U}\,. \tag{7}$$

Subtracting (6) and (7) from the corresponding components of (1), we obtain the following equation for the fluctuating velocity field $\breve{\mathbf{v}}$:

$$\frac{\partial}{\partial t}\breve{\mathbf{v}} + \breve{\mathbf{v}} \cdot \nabla \breve{\mathbf{v}} - \overline{\breve{\mathbf{v}} \cdot \nabla \breve{\mathbf{v}}} + \mathbf{U} \cdot \nabla \breve{\mathbf{v}} + \breve{\mathbf{v}} \cdot \nabla \mathbf{U} + 2\Omega \times \breve{\mathbf{v}} = -\nabla \breve{p} + \nabla^2 \breve{\mathbf{v}}\,. \tag{8}$$

After multiplying the above equation with $\breve{\mathbf{v}}$, taking the average over the entire fluid layer, and using the boundary conditions that $\breve{\mathbf{v}}$ vanishes at $z = \pm\frac{1}{2}$, we have the energy relationship

$$\frac{1}{2}\frac{d}{dt}\langle|\breve{\mathbf{v}}|^2\rangle + \langle|\nabla\breve{\mathbf{v}}|^2\rangle + \langle\breve{\mathbf{u}} \cdot (\breve{w}\frac{\partial}{\partial z})\mathbf{U}\rangle = 0 \tag{9}$$

where

$$\langle\cdots\rangle = \int_{-\frac{1}{2}}^{\frac{1}{2}} \overline{\cdots} dz\,. \tag{10}$$

The above energy relationship (9) can be further simplified if we restrict our attention to the fluid flow under stationary conditions:

$$\frac{\partial}{\partial t}\mathbf{U} = 0; \frac{d}{dt}\langle|\breve{\mathbf{v}}|^2\rangle = 0\,. \tag{11}$$

The equation (6) under above condition yields

$$\frac{d}{dz}\mathbf{U} = \overline{\breve{w}\breve{\mathbf{u}}} - \langle|\breve{w}\breve{\mathbf{u}}|\rangle - Re\cdot\mathbf{i}\,. \tag{12}$$

Hence, using the above equation, we obtain the final form of the energy balance

$$\langle|\nabla\breve{\mathbf{v}}|^2\rangle + \langle\breve{\mathbf{u}}\breve{w}\cdot(\overline{\breve{\mathbf{u}}\breve{w}} - \langle\breve{\mathbf{u}}\breve{w}\rangle)\rangle - Re\langle\breve{u}_x\breve{w}\rangle = 0\,. \tag{13}$$

Here, the identity

$$\langle\overline{\breve{\mathbf{u}}\breve{w}}^2\rangle - \langle\breve{\mathbf{u}}\breve{w}\rangle^2 = \langle|\overline{\breve{\mathbf{u}}\breve{w}} - \langle\breve{\mathbf{u}}\breve{w}\rangle|^2\rangle \tag{14}$$

has been used.

The momentum transport between two moving plates is obtained from its value at the boundary

$$M \equiv -\frac{\partial U_x}{\partial z}|_{z=\frac{1}{2}} = \langle\breve{w}\breve{u}_x\rangle + Re\,. \tag{15}$$

Since $\langle\breve{w}\breve{u}_x\rangle \geq 0$, the momentum transport is bounded from below by the value of the laminar solution and increases by $\langle\breve{w}\breve{u}_x\rangle$ for turbulent flow. Thus, the goal here is to derive an upper bound for $\langle\breve{u}_x\breve{w}\rangle$ at a given value of Re and this leads us to the formulation of the following variational problem. For a given μ, find the minimum $R(\mu)$ of the functional

$$R(\mathbf{v},\mu) \equiv \frac{\langle|\nabla\mathbf{v}|^2\rangle}{\langle u_x w\rangle} + \mu\frac{\langle|\overline{\mathbf{u}w} - \langle\mathbf{u}w\rangle|^2\rangle}{\langle u_x w\rangle^2} \tag{16}$$

among all vector fields with $\mathbf{v} = 0$ at $z = \pm\frac{1}{2}$ where

$$\mathbf{v} \equiv \mathbf{u} + \mathbf{k}w \;,\; \mathbf{u}\cdot\mathbf{k} \equiv 0\,. \tag{17}$$

Thus, the Euler-Lagrange equations for a stationary value of $R(\mathbf{v},\mu)$ are

$$\nabla^2\mathbf{v} - \nabla\pi = w\frac{d}{dz}\mathbf{U}^* + \mathbf{k}\mathbf{u}\cdot\frac{d}{dz}\mathbf{U}^* \tag{18}$$

where

$$\frac{d}{dz}\mathbf{U}^* = \overline{w\mathbf{u}} - \langle w\mathbf{u}\rangle - (R - \frac{\langle|\nabla\mathbf{v}|^2\rangle}{2\langle u_x w\rangle})\mathbf{i} \tag{19}$$

Since the functional is homogeneous, the normalization $\langle\breve{u}_x\breve{w}\rangle = \mu$ can be assumed.

- -

The proof for $\frac{dR(\mu)}{d\mu} = \frac{\langle|\overline{w\mathbf{u}} - \langle w\mathbf{u}\rangle|^2\rangle}{\langle wu_x\rangle^2}$ is as follows:

$$\begin{aligned}(\mu^* - \mu')\frac{\langle|\overline{w^*\mathbf{u}^*} - \langle w^*\mathbf{u}^*\rangle|^2\rangle}{\langle w^*u_x^*\rangle^2} &= R(\mathbf{v}^*,\mu^*) - R(\mathbf{v}^*,\mu') \\ &\leq R(\mu^*) - R(\mu') \\ &\leq R(\mathbf{v}',\mu^*) - R(\mathbf{v}',\mu') \\ &\leq (\mu^* - \mu')\frac{\langle|\overline{w'\mathbf{u}'} - \langle w'\mathbf{u}'\rangle|^2\rangle}{\langle w'u_x'\rangle^2}\end{aligned}$$

where $\mathbf{v}^*$ and $\mathbf{v}'$ are the extremalizing vector fields for μ^* and μ', respectively. For $\mu^* \to \mu'$, the above result follows.

[width=]fig2.eps

Figure 2: Schematic schetch of a thermal convection in a porous medium.

3 Upper Bounds on the Heat Transport in a Porous Layer

In this section, we consider a thermal convection in a porous medium as shown in Fig. 2. Using the distance d between two plates as length scale, d^2/κ as time scale, κ/d as velocity scale, and $(T_2$-$T_1)/R$ as temperature scale, we write dimensionless equations based on Darcy-Law as

$$-\mathbf{u}+\mathbf{k}T-\nabla p=B(\frac{\partial}{\partial t}\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u})\approx 0 \tag{20}$$

$$\nabla\cdot\mathbf{u}=0 \tag{21}$$

$$\nabla^2 T=(\frac{\partial}{\partial t}+\mathbf{u}\cdot\nabla)T \tag{22}$$

where

$$B\equiv\frac{\kappa K}{d^2\nu} \tag{23}$$

$$R\equiv\frac{\gamma g K d(T_2-T_1)}{\nu\kappa}\,. \tag{24}$$

and K is the Darcy permeability coefficient.

We separate the temperature field T into a mean and a fluctuating part

$$T=\overline{T}+\theta\ ,\ \text{with}\ \overline{\theta}=0 \tag{25}$$

By subtracting the horizontal average of (22) from (22), we obtain

$$\frac{\partial}{\partial t}\overline{T}+\overline{\mathbf{u}\cdot\nabla\theta}=\frac{\partial^2}{\partial z^2}\overline{T} \tag{26}$$

$$(\frac{\partial}{\partial t}+\mathbf{u}\cdot\nabla\theta)+w\frac{\partial\overline{T}}{\partial z}-\frac{\partial}{\partial z}\overline{w\theta}=\nabla^2\theta \tag{27}$$

Assuming the statistically stationary turbulence, we integrate (26) and obtain

$$\frac{\partial}{\partial z}\overline{T} = \overline{w\theta} - \langle w\theta\rangle - R \tag{28}$$

By multiplying (20) by $\mathbf{u}$ and (27) by θ, taking the average over the whole porous layer, and using (28), we obtain two dissipation integral relationships:

$$\langle|\mathbf{u}|^2\rangle = \langle w\theta\rangle \tag{29}$$

$$\langle|\nabla\theta|^2\rangle + \langle|\overline{w\theta} - \langle w\theta\rangle|^2\rangle = R\langle w\theta\rangle\,. \tag{30}$$

The dimensionless heat transport across the porous layer can be obtained from its value at the boudary:

$$H = -\frac{\partial\overline{T}}{\partial z}|_{z=\pm\frac{1}{2}} = R + \langle w\theta\rangle \geq R\,. \tag{31}$$

Since $\langle w\theta\rangle$ is always positive from (29), the heat transport for the turbulent flow is always greater than that for the laminar flow, and it is bounded from below by the value of the laminar solution.

The goal here is to find an upper bound on the heat transport or $\langle w\theta\rangle$ at a given value of R. We are thus led to the formulation of the following variational problem. For given $\mu > 0$, find the minimum $P(\mu)$ of the functional

$$P(\mathbf{u},\theta,\mu) \equiv \frac{\langle|\mathbf{u}|^2\rangle\langle|\nabla\theta|^2\rangle + \mu\langle|\overline{w\theta} - \langle w\theta\rangle|^2\rangle}{\langle w\theta\rangle^2} \tag{32}$$

for all fields $\mathbf{u}$ and θ, which satisfy the constraint $\nabla\cdot\mathbf{u} = 0$ and the boundary condition $w = \theta = 0$ at $z = \pm\frac{1}{2}$. First, from the general form of the dissipation integral

$$\langle|\mathbf{u}|^2\rangle \equiv \langle\nabla^2 v\Delta_2 v\rangle + \langle|\mathbf{k}\times\nabla\psi|^2\rangle \tag{33}$$

and the property

$$w = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)v \equiv -\Delta_2 v \tag{34}$$

it is clear that the minimum of the functional is obtained for $\nabla\times\mathbf{k}\psi = 0$. Hence, the variational problem now depends only on the scalar variables v and θ.

The Euler-Lagrange equations for a stationary value can be thus written as

$$\langle|\nabla\theta|^2\rangle\nabla^2 w - [P\langle w\theta\rangle + \mu(\langle w\theta\rangle - \overline{w\theta})]\Delta_2\theta = 0 \tag{35}$$

$$\langle\nabla^2 v\Delta_2 v\rangle\nabla^2\theta + [P\langle w\theta\rangle + \mu(\langle w\theta\rangle - \overline{w\theta})]w = 0 \tag{36}$$

Now, the nonlinearity is only through z-dependence and the equations are linear with respect to the x, y dependence. This property allows us to write solutions in the form of superposition of waves. Because of the homogenity with respect to x and y in w and θ, we can impose the following normalization conditions:

$$\langle|\nabla\theta|^2\rangle = 1 \tag{37}$$

$$\langle\nabla^2 v\Delta_2 v\rangle = \langle|\mathbf{k}\times\nabla\nabla v|^2\rangle = 1 \tag{38}$$

Then, we introduce the following general solutions for w and θ

$$w = w^{(N)} \equiv \sum_{k=1}^{N} \alpha_n^{\frac{1}{2}} w_n(z)\Phi_n(x,y) \tag{39}$$

$$\theta = \theta^{(N)} \equiv \sum_{k=1}^{N} \alpha_n^{-\frac{1}{2}} \theta_n(z)\Phi_n(x,y) \tag{40}$$

where Φ_n satisfies the equation:

$$\Delta_2\Phi_n = -\alpha_n^2\Phi_n \tag{41}$$

and the orthonormalization condition

$$\overline{\Phi_n\Phi_m} = \delta_{mn}\,. \tag{42}$$

Then, the Euler-Lagrangian equations can be reduced to

$$(\frac{\partial^2}{\partial z^2} - \alpha_n^2)w_n + \alpha_n\Psi\theta_n = 0 \tag{43}$$

$$(\frac{\partial^2}{\partial z^2} - \alpha_n^2)\theta_n + \alpha_n\Psi w_n = 0 \tag{44}$$

where

$$\Psi \equiv P\sum_{n=1}^{N}\langle w_n\theta_n\rangle + \mu\sum_{n=1}^{N}(\langle w_n\theta_n\rangle - \overline{w_n\theta_n})\,. \tag{45}$$

The above equations have the following properties [8]:
(1) By considering the equations for $w_n+\theta_n$ and $w_n-\theta_n$, we can obtain

$$w_n = \theta_n\,. \tag{46}$$

Thus, the problem can be reduced to

$$(\frac{\partial^2}{\partial z^2} - \alpha_n^2)\theta_n + \alpha_n \Psi \theta_n = 0 \tag{47}$$

(2) The functions $\theta_n(z)$ are either symmetric or antisymmetric in z.
(3) Since $\theta_n \equiv \theta_m$ follows from $\alpha_n = \alpha_m$, it can be assumed that all α_n are different.
(4) For $m \neq n$, by subtracting the n-th equation of (47) multiplied by $\alpha_n^{-1}\theta_m$ from the m-th equation multiplied by $\alpha_m^{-1}\theta_n$, and averaging it using the partial integration, we obtain an important property

$$\langle \theta'_m \theta'_n \rangle - \alpha_m \alpha_n \langle \theta_m \theta_n \rangle = 0 \tag{48}$$

where θ'_m denotes the z-derivative of θ_m.
(5) Minimization of $P(\theta_n, \alpha_n, \mu)$ with respect to α_n yields

$$\frac{\partial}{\partial \alpha_n} I = 0 \tag{49}$$

$$\alpha_n^2 = \frac{\langle \theta'_n \theta'_m \rangle}{\langle \theta_n \theta_m \rangle} \,. \tag{50}$$

References

[1] W. Malkus, "The heat transport and spectrum of thermal turbulence," Proc. Roy. Soc. London **225**, 185 (1954).

[2] L. N. Howard, "Heat transport by turbulent transport," J. Fluid Mech. **17**, 405 (1963).

[3] F. H. Busse, "Bounds on the transport of mass and momentum by turbulent flow beteen parallel plates," J. Applied Math. Phys. **20**, 1 (1969).

[4] F. H. Busse, "On Howard's upper bound for heat transport by turbulent convection," J. Fluid Mech. **37**, 457 (1969).

[5] C. R. Doering and P. Constantin, "Variational bounds on energy dissipation in incompressible flows; shear flow," Phys. Rev. E **49**, 4087 (1994).

[6] S. G. R. Nicodemus and M. Holthaus, "Variational bound on energy dissipation in plate Couette flow," Phys. Rev. E **56**, 6774 (1997).

[7] R. R. Kerswell, "Unification of variational principles for turbulent shear flows: The background method of Doering-Constantin and Howard-Busse's mean-fluctuation formulation," Physica D **121**, 175 (1998).

[8] F. H. Busse and D. D. Joseph, "Bounds for heat transport in a porous layer," J. Fluid Mech. **54**, 521 (1972).

References

[1] W. Malkus, "The heat transport and spectrum of thermal turbulence," Proc. Roy. Soc. London **225**, 185 (1954).

[2] L. N. Howard, "Heat transport by turbulent transport," J. Fluid Mech. **17**, 405 (1963).

[3] F. H. Busse, "Bounds on the transport of mass and momentum by turbulent flow beteen parallel plates," J. Applied Math. Phys. **20**, 1 (1969).

[4] F. H. Busse, "On Howard's upper bound for heat transport by turbulent convection," J. Fluid Mech. **37**, 457 (1969).

[5] C. R. Doering and P. Constantin, "Variational bounds on energy dissipation in incompressible flows; shear flow," Phys. Rev. E **49**, 4087 (1994).

[6] S. G. R. Nicodemus and M. Holthaus, "Variational bound on energy dissipation in plate Couette flow," Phys. Rev. E **56**, 6774 (1997).

[7] R. R. Kerswell, "Unification of variational principles for turbulent shear flows: The background method of Doering-Constantin and Howard-Busse's mean-fluctuation formulation," Physica D **121**, 175 (1998).

[8] F. H. Busse and D. D. Joseph, "Bounds for heat transport in a porous layer," J. Fluid Mech. **54**, 521 (1972).