Bound on the Heat Transport through a Layer from the Boundary Layer Theory Perspective: Fixed Heat Flux

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1 Introduction

Bounds on the heat transport in the Rayleigh-Bénard convection problem is a fundamental problem. The first mathematical formulation of the problem was given in the pioneer paper by Howard[1]. The problem was formulated as an optimization problem. The motivation was that nature might "choose" to realize a process that maximizes the heat transport when the flow is turbulent. This is only a hypothesis, but in certain cases the features of the solution to the so formulated problem has qualitative agreement with real physical flows. In any case, the solution of the maximization problem gives a bound on the quantity of transport of heat flux.

The problem of thermal convection can be realized in different experimental settings. In some, the boundaries can have infinitely bigger thermal conductivity that the fluid in the layer. This is the so called "fixed temperature" problem. In other experimental settings we might have the opposite case: The fluid's thermal conductivity might be much larger than the thermal conductivity of the boundaries; the latter is "fixed heat flux" problem. The physical basis for this nomination is that in the former case the fluid on the boundary has the temperature of the boundary, whereas in the latter, this is not required. Instead, the heat flux through the boundary, which is proportional to the gradient of the temperature on the boundary, is fixed. In this study we are concerned with the fixed heat flux problem.

Despite the different boundary conditions arising from different experimental settings, the physics behind both phenomena is similar. When the temperature difference between the upper and lower plates (or analogously, the temperature gradient) is small, the fluid is in a pure conductive state so that the velocity throughout the layer is zero. As we start increasing the temperature difference (or the heat flux), the system becomes unstable and the fluid starts to move. There is a critical parameter that describes when this first happens—a control parameter. This is the Rayleigh number. As we keep increasing the Rayleigh number, the fluid sets into turbulent motion.

In turbulent regime it is believed that quantities reach asymptotic behavior and have certain scaling determined by the Rayleigh number. For example, the quantity that describes how much bigger the heat flux in a turbulent regime is, compared to that in a pure conductive state, is the Nusselt number Nu. In the fixed temperature problem the scaling derived in the paper by Howard[1]— maximizing over fields with one horizontal wavenumber—is $Nu \sim Ra^{\frac{3}{8}}$, while the scaling derived by Busse[5]—maximizing over multiple horizontal wavenumbers—is $Nu \sim Ra^{\frac{1}{2}}$. In a recent paper by Otero *et al.* [3], using a different method, it is shown that the scaling for the fixed heat flux problem is $Nu \sim Ra^{\frac{1}{2}}$. However, for a single horizontal wave number C. Doering and J. Otero[4], following a method by Howard[2], have derived an estimate that leads to a scaling $Nu \sim Ra^{\frac{5}{12}}$. We will try to follow the Howard-Busse approach and derive a scaling for the Nusselt number in the fixed heat flux problem, using a single wave number approximation. We will comment on applying the multi-alpha approach developed by Busse[5].

2 Formulation of the Problem

We describe the physical setup in this section and then formulate the problem. Consider a fluid between two horizontal infinite plates. The lower plate is heated so the fluid at the bottom is hotter than the fluid at the top. There are two opposing forces that act on the fluid—the buoyancy force and the gravity. The equations that describe this convection problem are the Boussinesq equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + 1\rho \nabla p - \alpha g T \hat{\mathbf{k}} = \nu \nabla^2 \mathbf{u}, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{2}$$

$$\frac{\partial T^*}{\partial t} + \mathbf{u} \cdot \nabla T^* = \kappa \nabla^2 T^*. \tag{3}$$

The meaning of the quantities in the above equations is the following. $\mathbf{u} = (u, v, w)$ is the velocity vector, p is the deviation of the pressure from the hydrostatic pressure, corresponding to the horizontal average of the temperature, α is the coefficient of thermal expansion, g is the acceleration of gravity, T^* is the temperature, κ is the thermal conductivity of the fluid. The fluid occupies the space in the direction of z from 0 to d. The boundary conditions are

$$\mathbf{u}(0) = \mathbf{u}(d) = 0, \quad \kappa \frac{\partial \overline{T^*}}{\partial z}\bigg|_0 = \kappa \frac{\partial \overline{T^*}}{\partial z}\bigg|_d = -\kappa\beta = \text{const.} \tag{4}$$

We split the temperature in a horizontal averaged part $\overline{T^*}$, and a deviating part T, so that

$$T^* = \overline{T^*} + T. \tag{5}$$

Our notation is: An over-bar denotes horizontal average, and angle brackets—volume average. Then we can write

$$\langle wT \rangle = \frac{1}{d} \int_0^d \overline{wT}\, dz. \tag{6}$$

If we multiply Eq. (1) by $\mathbf{u}$ and average over the volume, we get

$$\alpha g \langle wT \rangle = \nu \left\langle |\nabla \mathbf{u}|^2 \right\rangle . \tag{7}$$

This equation expresses the balance between the rate of generation of energy motion in the field of the buoyancy force $\alpha g T\mathbf{k}$, and the rate of dissipation of energy by viscosity. If we average Eq. (3) horizontally, using the boundary conditions for $\mathbf{u}$ and T^*, we obtain

$$\frac{d\overline{wT^*}}{dz} = \kappa \frac{d^2\overline{T^*}}{dz^2}. \tag{8}$$

Since $\overline{w} = 0$, $\overline{wT^*} = \overline{wT}$. From Eq. (8) we see that

$$\frac{d}{dz}\left(\overline{wT} - \kappa \frac{d\overline{T^*}}{dz}\right) = 0 \tag{9}$$

and therefore the sum in the braces in Eq. (9) is constant equal to the its average over the volume

$$\begin{aligned} \overline{wT} - \kappa \frac{d\overline{T^*}}{dz} &= \langle wT \rangle - \kappa \frac{1}{d} \int_0^d \kappa \frac{d\overline{T^*}}{dz} \\ &= \langle wT \rangle + \kappa \frac{\Delta T}{d}, \end{aligned} \tag{10}$$

where by definition

$$-\Delta T = \overline{T^*}(d) - \overline{T^*}(0). \tag{11}$$

Finally from (10) we obtain

$$-\kappa \frac{d\overline{T^*}}{dz} = \kappa \frac{\Delta T}{d} + \langle wT \rangle - \overline{wT}. \tag{12}$$

Multiplying Eq. (3) by T, averaging and using (12) we obtain

$$\kappa^{-1} \left[\langle wT \rangle^2 - \left\langle \overline{wT}^2 \right\rangle \right] + \frac{\Delta T}{d} \langle wT \rangle = \kappa \left\langle |\nabla T|^2 \right\rangle . \tag{13}$$

Putting (7) and (13) into a dimensionless form with d as a length scale, κ/d as velocity scale, and βd as a temperature scale have the two "power integrals" (called so in the paper by Howard[1])

$$R \langle wT \rangle = \left\langle |\nabla \mathbf{v}|^2 \right\rangle , \tag{14}$$

$$\Delta T \langle wT\rangle + \langle wT\rangle^2 - \langle \overline{wT}\rangle = \langle |\nabla T|^2\rangle, \tag{15}$$

where $R = \alpha g\beta d^4/\kappa\nu$ is the Rayleigh number based on the given constant heat flux at the boundary. Its relation to the Rayleigh number based on the temperature difference between the plates is

$$Ra = R\Delta T. \tag{16}$$

Define the Nusselt number as the ratio of the total heat flux and the conductive heat flux through the layer. The total heat flux is given by $-\kappa\beta$, whereas the conductive heat flux by $-\kappa\Delta T$. In dimensionless form we have

$$Nu = \frac{1}{\Delta T}, \tag{17}$$

where now ΔT is dimensionless temperature difference. The problem we will try to solve is to find a bound on the Nusslet number (17) i.e., we will try to find a relation between Nu and the Rayleigh number R (or, Ra.) of the form $Nu \sim R^p$ for some p. We do this in the following sections.

3 Bounding as a Minimization Problem

Multiplying Eq.(8) by z, integrating by parts and using the boundary conditions for T we obtain for the left-hand and the right-hand sides

$$\kappa \int_0^d z\frac{d^2\overline{T^*}}{dz^2}\,dz = -\kappa d\beta + \kappa\Delta T,$$
$$\int_0^d z\frac{d\overline{wT^*}}{dz}\,dz = -d\,\langle wT\rangle$$

and after putting those in a dimensionless form we have

$$\Delta T = 1 - \langle wT\rangle. \tag{18}$$

Using (18), we rewrite the power integrals (14) and (15) in the form

$$R\langle wT\rangle = \langle |\nabla \mathbf{v}|^2\rangle, \tag{19}$$

$$\langle wT\rangle - \left\langle \overline{wT}^2\right\rangle = \langle |\nabla T|^2\rangle. \tag{20}$$

Using (19) in the right-hand side of (21) by $\langle wT \rangle$ and regrouping we get

$$\langle wT \rangle = \frac{\langle wT \rangle^2 - \frac{1}{R} \left\langle |\nabla \mathbf{v}|^2 \right\rangle \left\langle |\nabla T|^2 \right\rangle}{\left\langle \overline{wT}^2 \right\rangle} \tag{21}$$

Substituting (21) into (17) and using (18) we can write

$$\frac{1}{N} = \frac{\left\langle \overline{wT}^2 \right\rangle - \langle wT \rangle^2 + \frac{1}{R} \left\langle |\nabla \mathbf{u}|^2 \right\rangle \left\langle |\nabla T|^2 \right\rangle}{\left\langle \overline{wT}^2 \right\rangle} \tag{22}$$

Maximizing the Nusselt number is equivalent to minimizing (22). The maximal Nu Will provide a bound on the total heat transport throughout the layer of fluid. Therefore, we will look for a minimum of the functional

$$\mathcal{F}[\mathbf{v}, T] = \frac{\left\langle \overline{wT}^2 \right\rangle - \langle wT \rangle^2 + \lambda \left\langle |\nabla \mathbf{u}|^2 \right\rangle \left\langle |\nabla T|^2 \right\rangle}{\left\langle \overline{wT}^2 \right\rangle}, \tag{23}$$

where $\lambda = 1/R$.

The so derived functional is to be minimized among functions that satisfy the boundary conditions

$$\mathbf{v}(0) = \mathbf{v}(1) = dT/dz|_{z=0} = dT/dz|_{z=1} = 0, \tag{24}$$

the continuity equation

$$\nabla \cdot \mathbf{v} = 0, \tag{25}$$

and the power integrals (19) and (20).

We continue the analysis in the following section by making a certain simplification. We will assume a single wave number horizontal dependence of the test functions.

4 Bound with a Single Wave Number

We assume the following form of the functions w and T

$$w(x, y, z) = \omega(z)\phi(x, y)$$
$$T(x, y, z) = \theta(z)\phi(x, y)$$

and the function $\phi(x, y)$ having the properties

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi(x,y)=-a^2\phi(x,y),\quad \overline{\phi^2}=1. \tag{26}$$

Now we can write

$$\left\langle|\nabla T|^2\right\rangle=\left\langle\theta'^2+a^2\theta^2\right\rangle \tag{27}$$

We note that the continuity equation alone is not enough to determine the relation between ω and $\left\langle|\nabla \mathbf{v}|^2\right\rangle$, but if only the minimum of the latter is requested, we can write

$$\left\langle|\nabla \mathbf{v}|^2\right\rangle=\left\langle a^{-2}\omega''^2+2\omega'^2+a^2\omega^2\right\rangle. \tag{28}$$

With (27) and (28) we can express the functional (23) in terms of only ω and θ only

$$\mathcal{F}[\omega,\theta]=\frac{\left\langle\omega^2\theta^2\right\rangle-\left\langle\omega\theta\right\rangle^2+\lambda\left\langle\theta'^2+a^2\theta^2\right\rangle\left\langle a^{-2}\omega''^2+2\omega'^2+a^2\omega^2\right\rangle}{\left\langle\omega^2\theta^2\right\rangle}. \tag{29}$$

The boundary conditions for the functions ω and θ are

$$\omega=\omega'=\theta'=0\quad\text{at}\quad z=0,1. \tag{30}$$

Since the functional (29) is homogeneous of degree zero in ω and θ, we can choose the amplitudes of the test functions so that they satisfy two conditions

$$\begin{aligned}\left\langle\omega\theta\right\rangle&=1,\\ \left\langle\omega^2\right\rangle&=\left\langle\theta^2\right\rangle.\end{aligned} \tag{31}$$

The Euler-Lagrange equations following from the functional (29) are

$$Eq1\omega\theta^2(1-\mathcal{F})-\left\langle\omega\theta\right\rangle\theta+\lambda\left\langle\theta'^2+a^2\theta^2\right\rangle\left[a^{-2}\omega^{iv}+2\omega''+a^2\omega\right]=0 \tag{32}$$

$$Eq2\omega^2\theta(1-\mathcal{F})-\left\langle\omega\theta\right\rangle\omega+\lambda\left[-\theta''+a^2\theta\right]\left\langle a^{-2}\omega''^2+2\omega'^2+a^2\omega'^2\right\rangle=0 \tag{33}$$

Since these equations are difficult to solve analytically, we resort to numerical methods to solve them. The results are given in the next section.

In the rest of this section we apply the boundary layer approximation. It consists of the following. The form of the functional (29) suggests that the minimizing functions should be nearly constant throughout a large portion of the interval $[0,1]$. To satisfy the boundary conditions, ω will need to drop to zero together with its derivative, and so will the derivative of θ. Therefore we expect that there will be a narrow interval around the two boundaries where the derivatives of the functions will have large values—boundary layers. The values of the functions will change fast within these boundary layers and the thickness of the latter will determine the magnitude of their growth. The contributions to the integrals are expected to come predominantly from the boundary layers. Therefore, we will derive equations that describe the functions ω, θ only in the boundary layer—the boundary layer equations—and will approximate the full interval of integration by integration over the thin boundary layers (there are two of them.)

Note that the first two terms in (29) are of order one, see (31). To minimize the functional and comply with (31), both of them will approach the value of 1 so that their difference approaches zero. The two tendencies—the terms with the derivatives approaching zero and the difference between the first two terms approaching zero—must occur simultaneously and have the same order of magnitude. In mathematical form, the above reasoning can be formulated as follows. Assume the following scaling of the boundary layer thickness (as a small parameter we choose $\lambda = 1/R$)

$$\omega = \lambda^p \omega_1, \quad \theta = \lambda^{-p}\theta_1, \quad z = \lambda^r \zeta, \quad a^2 = \lambda^{-q} b^2, \tag{34}$$

where the functions ω_1, θ_1 are of order one inside the boundary layers. From the argument above

$$rel \left\langle \omega^2\theta^2 \right\rangle \to 1 \quad \text{as} \quad \lambda \to 0. \tag{35}$$

Because of the relation $\left\langle \omega^2\theta^2 \right\rangle = \left\langle (1+(1-\omega\theta))^2 \right\rangle$ we see that as $\omega\theta \to 1$ we must have (35). Substitution of (34) into (29), and taking into account (35), we obtain

$$\begin{aligned} FF\mathcal{F} = {} & 2\lambda^r \int_0^\infty (1-\omega_1\theta_1)^2 d\zeta + \lambda \left[2\lambda^{-2p-r} \int_0^\infty \left(\frac{d\theta_1}{d\zeta} \right)^2 d\zeta + b^2 \lambda^{-q} \right] \\ & \times \left[2b^{-2}\lambda^{q+2p-3r} \int_0^\infty \left(\frac{d^2\omega_1}{d\zeta^2} \right)^2 d\zeta + 4\lambda^{2p-r} \int_0^\infty \left(\frac{d\omega_1}{d\zeta} \right)^2 d\zeta + b^2\lambda^{-q} \right]. \end{aligned} \tag{36}$$

After expansion of the terms in the square brackets we obtain the following exponents

$$r, \quad 1+q-4r, \quad 1-2r, \quad 1-2p-r-q, \quad 1+2p-3r, \quad 1-q+2p-r, \quad 1-2q. \tag{37}$$

We need to maximize the minimal possible exponent among (37). Let e be the minimal of all exponents. Of all 7 inequalities, consider

$$r \ \geq \ e, \tag{38}$$
$$1+q-4r \ \geq \ e, \tag{39}$$
$$1-2q \ \geq \ e. \tag{40}$$

Multiplying (38) by 8, (39) by 2, and adding to (40) we get $3 \geq 11$ from which we deduce that $e = 3/11$. By adding 4 times (38) to (39), and using (40) we get $4/11 = 5e - 1 \leq q \leq 1/2(1-e) = 4/11$, so that $q = 4/11$. Similarly, from (38) and (39) we get $3/11 = e \leq r \leq 1/4(1+q-e) = 3/11$, and thus $q = 3/11$. From the fourth and fifth of (37) we find $1/11 = e - 1 + 3r \leq 2p \leq 1 - r - q - e = 1/11$ which shows that $p = 1/22$. For these values of p, q, r all exponents in (37) take the same maximal value of $3/11$ except the third and sixth, which become 5/11 (we neglect those.) Hence the maximum value is uniquely determined. If we set $\mathcal{F} = \lambda^{\frac{3}{11}}\mathcal{F}_1$, we obtain within the boundary layer approximation

$$\mathcal{F}_1 = 2\int_0^\infty (1-\omega_1\theta_1)^2 d\zeta + \left[2\int_0^\infty \left(\frac{d\theta_1}{d\zeta}\right)^2 d\zeta + b^2\right]\left[2b^{-2}\int_0^\infty \left(\frac{d^2\omega_1}{d\zeta^2}\right)^2 d\zeta + b^2\right]. \tag{41}$$

We need to minimize $\mathcal{F}_1$ among functions ω_1 and θ_1 that satisfy the boundary conditions

$$\omega_1(0) = \omega'(0) = \theta'(0) = 0,$$
$$\theta_1 \to 0, \quad \omega_1\theta_1 \to \quad \text{as} \quad \zeta \to \infty.$$

Varying $\mathcal{F}_1$ with respect to ω_1, θ_1 , and b^2 we find the following equations

$$b^{-2}\left[2\int_0^\infty \theta_1^2 d\zeta + b^2\right]\frac{d^4\omega_1}{d\zeta^4} - (1-\omega_1\theta_1)\theta_1 = 0, \tag{42}$$

$$\left[2b^{-2}\int_0^\infty \omega''^2 d\zeta + b^2\right]\frac{d^2\theta_1}{d\zeta^2} + (1-\omega_1\theta_1)\omega_1 = 0, \tag{43}$$

$$\frac{2}{b^2}\int_0^\infty \omega''^2 d\zeta + b^2 + \left[2\int_0^\infty {\theta_1'}^2 d\zeta + b^2\right]\left[-2b^{-4}\int_0^\infty \omega''^2 d\zeta + 1\right] = 0. \tag{44}$$

From (42) and (43) we obtain

$$\begin{aligned} b^{-2}\left[2\int_0^\infty {\theta_1'}^2 d\zeta + b^2\right]\int_0^\infty \omega''^2 d\zeta &= \int_0^\infty (1-\omega_1\theta_1)\omega_1\theta_1 d\zeta = \\ &= \left[2b^{-2}\int_0^\infty \omega''^2 d\zeta + b^2\right]\int_0^\infty {\theta_1'}^2 d\zeta \end{aligned}$$

and therefore

$$\int_0^\infty \omega''^2 d\zeta \quad = \quad b^2\int_0^\infty {\theta_1'}^2 d\zeta \equiv b^2\mu$$

a which defines μ. Substituting these into (44) we find

$$2\mu + b^2 + (2\mu + b^2)(-2\mu b^{-2} + 1) \equiv 2(2\mu + b^2)(1 - \mu b^{-2}) = 0 \tag{45}$$

which shows that $\mu = b^2$. Using this in (42) and (43) we find

$$3(d^4\omega_1/d\zeta^4) - (1-\omega_1\theta_1)\,\theta_1 = 0, \tag{46}$$

$$3b^2(d^2\theta_1/d\zeta^2) + (1-\omega_1\theta_1)\,\omega_1 = 0. \tag{47}$$

Setting

$$\omega_1 = (3b^4)^{\frac{1}{6}}\Omega, \quad \theta_1 = (3b^4)^{-\frac{1}{6}}\Theta, \quad \zeta = (3b)^{\frac{1}{3}}\xi, \tag{48}$$

equations(46) and (47) become

$$d^4\Omega/d\xi^4 - (1-\Omega\Theta)\Theta \quad = \quad 0, \tag{49}$$
$$d^2\Theta/d\xi^2 + (1-\Omega\Theta)\Omega \quad = \quad 0. \tag{50}$$

These equations have boundary conditions $\Omega(0) = \Omega'(0) = \Theta'(0) = 0$, $\Theta \to 0$ and $\Omega\Theta \to 1$ as $\xi \to \infty$. The functions Ω and Θ can be determined independently of knowing b. Once we know the solutions of (49), (50), we can determine b

$$\textit{relation} \qquad b^2 = \mu = \int_0^\infty {\theta_1'}^2 d\zeta = (3b^4)^{-\frac{1}{3}}(3b)^{-\frac{1}{3}} \int_0^\infty \left(\frac{d\Theta}{d\xi}\right)^2 d\xi$$

from which

$$b^{\frac{11}{3}} = 3^{-\frac{2}{3}} \int_0^\infty \left(\frac{d\Theta}{d\xi}\right)^2 d\xi \tag{51}$$

We can see from (49) and (50) that

$$\int_0^\infty \Omega''^2 d\xi = \int_0^\infty \Theta'^2 d\xi. \tag{52}$$

Using the renormalization (eq:renorm) and the relation (51) in the functional (23) we find for the minimal value of F_1

$$\mathcal{F}_1 = 33b^4. \tag{53}$$

herefore, the minimal value of the functional (**??**) becomes $\mathcal{F} = 33b^4\lambda^{\frac{3}{11}}$ or

$$\mathcal{F} = 33b^4 R^{-\frac{3}{11}}. \tag{54}$$

We note that the equations (1) and (2) follow from the following functional

$$\mathcal{J} = \frac{1}{6}\int_0^\infty \left(\Theta'^2 + \Omega''^2 + (1 - \Omega\Theta)^2\right) d\xi. \tag{55}$$

Relations between different quantities are given below

$$\begin{aligned}
Nu &= (33b^4)^{-1}R^{\frac{3}{11}} = (33b^4)^{-\frac{11}{8}}Ra^{\frac{3}{8}}, && (56)\\
R &= (33b^4)^{-\frac{11}{8}}Ra^{\frac{11}{8}}, && (57)\\
a &= bR^{\frac{2}{11}} = (\frac{Ra}{33})^{\frac{1}{4}}, && (58)\\
z &= (3b)^{\frac{1}{3}}R^{-\frac{3}{11}}, && (59)\\
\omega(z) &= (33b^4)^{\frac{1}{6}}R^{-\frac{1}{22}}, \Omega && (60)\\
\theta(z) &= (33b^4)^{-\frac{1}{6}}R^{\frac{1}{22}}\Theta. && (61)
\end{aligned}$$

As seen from (56), from the boundary layer theory we have the scaling $Nu \sim Ra^{\frac{3}{8}}$, the same as in the fixed temperature problem. To further test this scaling, we do some numerical computations. The results are given in the next section.

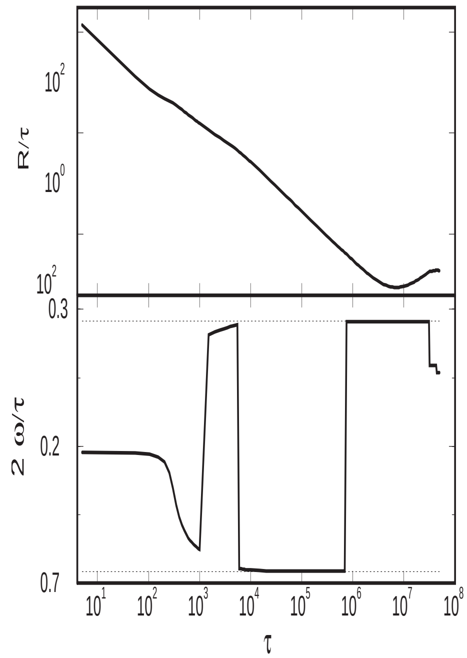

Figure 1: Velocity and temperature deviation: Fixed flux. $R = 10^7$, $a = 8.5456$.

5 Numerical Results and Discussion

In the previous section we used asymptotic methods to derive a particular scaling of the Nusselt number in the limit of large R. In this section we solve the complete equations (46), (47)(the single alpha approximation) for finite R. We will try to verify as much as we can the asymptotic theory's prediction of the 3/8 scaling.

First, in Figure 1 we present the plots of the velocity and the temperature profiles for $R = 10^7$. The minimizing wave number has value $a = 8.5456$. For comparison, we give the similar plot for the fixed temperature problem in Fig. 2. We note the following difference.

In the rising and falling parts of the velocity profile there is a slight bend which is absent in the analogous plot for the velocity profile for fixed temperature. Our investigation showed that this reflects the different boundary conditions of the fixed heat flux problem (to see that, we solved the the Euler-Lagrange equations (46), (47) with zero boundary condition for the temperature deviation θ; also, we solved the equations for the fixed temperature problem with boundary condition $\theta' = 0$ and we observed the bend appear.) As we increase the Rayleigh number R, this bend becomes more and more pronounced: In Fig. 3 we present a plot of the velocity for $R = 10^9$.

The difference between the fixed heat flux and the fixed temperature problems is also shown in Fig. 4which is compared with its fixed temperature analogue shown in Fig. 5.

In Fig. 6 we show the product of ω and θ and, again, compare that with its fixed temperature analogue in Fig. 7. The two curves have very similar behavior.

Next we show our results for the dependence of the Nusselt number on R on a log–log plot, Fig. 8.

Our data (pluses) is compared to the data points (dotted line) kindly provided by

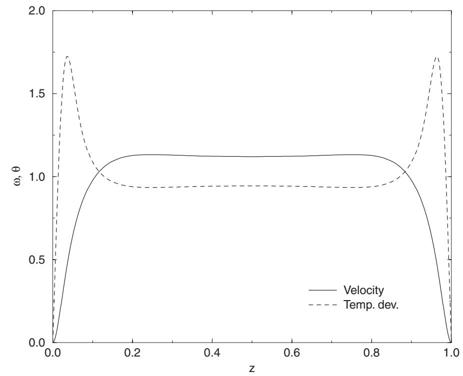

Figure 2: Velocity and temperature deviation: Fixed temperature. $R = 10^7$, $a = 11.1778$.

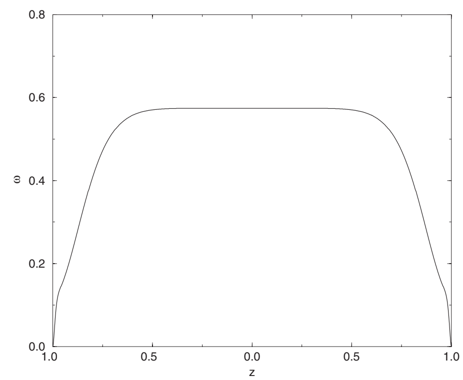

Figure 3: Velocity (this run was made with a different normalization of the velocity and temperature deviation.) $R = 10^9$, $a = 19.3072$.

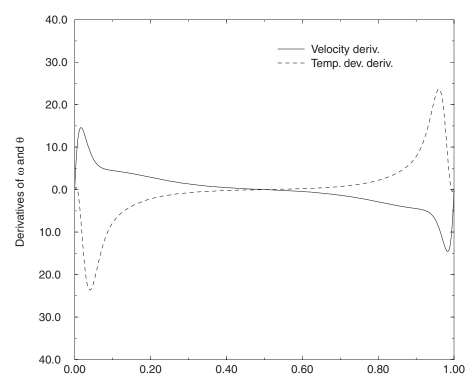

Figure 4: Derivatives of ω and θ: Fixed flux. $R = 10^7$, $a = 8.5456$.

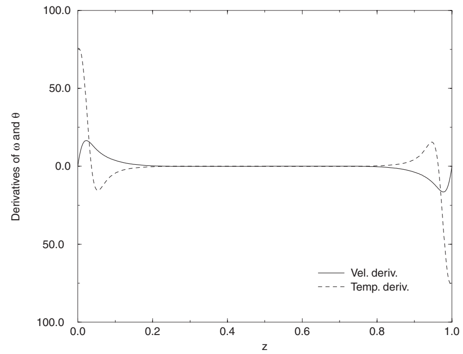

Figure 5: Derivatives of ω and θ: Fixed temperature. $R = 10^7$, $a = 11.1778$.

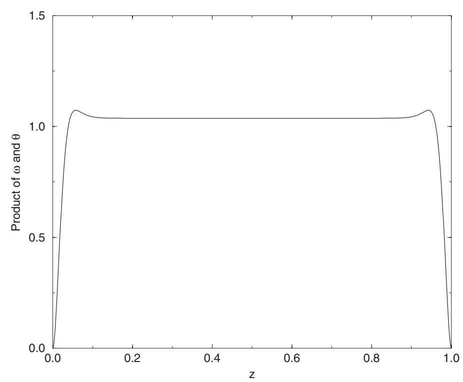

Figure 6: Product of ω and θ: Fixed flux. $R = 10^7$, $a = 8.5456$.

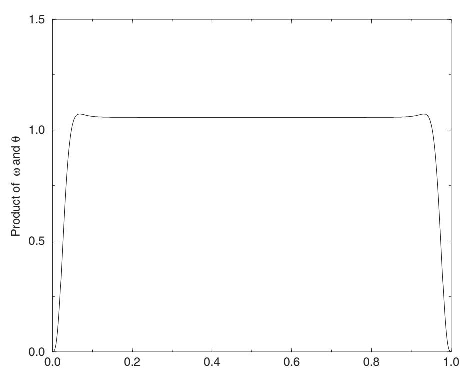

Figure 7: Product of ω and θ: Fixed temperature. $R = 10^7$, $a = 11.1778$.

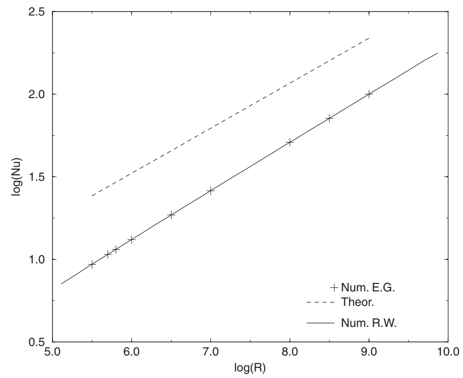

Figure 8: $\log(Nu)$ vs. $\log(R)$.

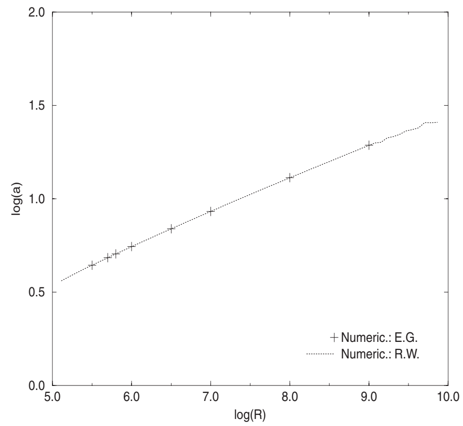

Figure 9: Optimal wave number a.

Rodney Worthing. Having found the minimizing values for the wave number a numerically, from relation (58) we can determine the value of b. For $R = 10^9$ it is $b = 0.446$. With this value and with relation (56) we plot the bounding curve suggested by the theory in the preceding section (dashed line.)

Finally we give plots for the dependence of the wave number on R. First we compare our data points to those of Rodney Worthing in Fig. 9.

Then in Fig. 10 we compare that with the theoretical prediction of C. Doering and J. Otero[4] who derived the scaling $Nu \sim Ra^{\frac{5}{12}}$. They predict the dependence $a \sim R^{\frac{3}{17}}$, whereas we deduced in the preceding section the dependence $a \sim R^{\frac{2}{11}}$. We have transformed the curves so that our theoretical curve be a horizontal line. From this plot we see that the Doering–Otero theory agrees somewhat better than what follows from the prediction of the preceding section.

We wanted to see if we really capture the asymptotic behavior—and the scaling—with our numerical data that extends up to $R = 10^9$. We calculated the slope for the analogous dependence $\log(Nu)$ vs. $\log(R)$ for the fixed temperature problem. We ended up with similar values for the slope at $R = 10^{7.5}$: 0.41. It has been proved that the single wave

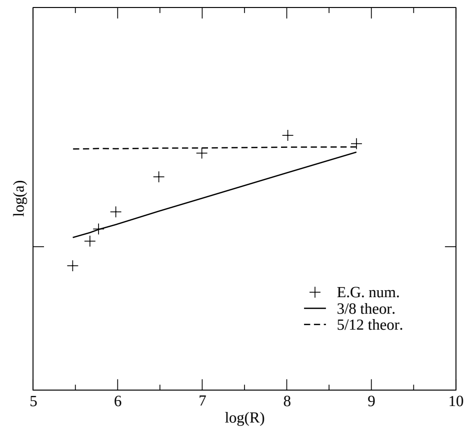

Figure 10: Optimal wave number a.

number bound gives a scaling $Nu \sim R^{\frac{3}{8}}$ (3/8 = 0.375.) This leads us to the thought that we would have to go to much higher values of the Rayleigh number R than 10^9.

Another way of estimating the value of b is by solving the boundary layer equations 49) and (50). We approached this problem in a couple of different ways. In one way we tried to minimize the functional (55) by truncating the upper boundary to some finite value (say 4), solving (49) and (50) up to this value and evaluating the functional (55). We considered the boundary conditions for Θ, Ω'' and Ω''' at the upper limit of (55) as parameters and so tried to minimize $\mathcal{J}$ with respect to those parameters. In the other way we used a boundary value problem solver provided by Matlab, again, assuming that we have reached asymptotic behavior of the solutions for some finite value of the independent variable ξ. Then we changed this value and solved the problem again. In both ways we encountered some problems. In the first approach we were able to do several iterations (using gradient methods) in the course of minimizing $\mathcal{J}$. However our results were very sensitive to the initial point we chose and were not very consistent. In the second approach we observed extreme sensitivity on the truncation limit. Generally we would expect that increasing the the truncation value would lead to convergence of the solution. Unfortunately that was not the case: The solution changed dramatically even for small changes of the truncation limit (e.g. from 4 to 4.5.) In fact, beyond some point we were not able to find a solution at all. Our second approach worked very successfully for the fixed temperature case which differs only by the boundary condition for the temperature deviation Θ.

The difference in the velocity profile suggests the possibility of different structure of the boundary layer in the fixed heat flux problem. It may be the reason for our difficulties in solving the boundary layer equations. It also suggests the possibility of two boundary layers, or signifies of the importance of an intermediate region (between the boundary layer and the interior part where the functions are predominantly constant.) This question could be clarified by a successful attempt in solving the boundary layer equations and comparing their solution to the solution of the complete Euler-Lagrange equations in the single alpha approximation.

Finally we comment on applying to the problem of bounding the heat transport the multi-alpha approach developed by Busse[5]. If we assume that the scaling 3/8 is correct, we find that in this general case the Nusselt number scales as $Nu \sim R^{\frac{1}{3}}$ or, equivalently $Nu \sim Ra^{\frac{1}{2}}$. The latter result has also been found by Otero *et al.*[3] by applying the background method developed by Doering and Constantin.

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