Equatorial Quasi-Geostrophy

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1 Introduction

The changing sign of the Coriolis force across the equator causes the dynamics of equatorial waves to be quite different from that of the mid- to high-latitudes. The waves that are found in this region play an important role in the El Niño - Southern Oscillation (ENSO) [23; 24; 29], the Madden-Julian Oscillation [17; 30], and in the exchange of energy from the equatorial region to the midlatitudes. Hence, it is important that their dynamics and evolution are well understood. The inviscid, linearized shallow water equations on an equatorial β-plane have been used to study equatorially trapped Kelvin, inertio-gravity and Rossby waves [1; 8; 19; 21]. Furthermore, the effect of nonlinearity on equatorial waves in the context of the shallow water equations has been described by Boyd, in a series of papers in the 1980s [2; 3; 4; 5]. Boyd found solitary wave solutions for the weakly dispersive long Rossby modes as well as for the strongly dispersive Rossby, inertio-gravity and mixed Rossby-gravity modes and also characterized the effect of nonlinearity on the weakly-dispersive (where the effects of nonlinearity dominate over dispersion) Kelvin mode.

While the shallow water equations have been quite successful in understanding the dynamics of the equatorial region, they are nonetheless an approximation to the full nonlinear primitive equations and neglect some physics that are potentially important in the equatorial region. In the present work, we investigate a nonlinear, equatorial quasi-geostrophic model that includes the vertical component of momentum as well as the non-hydrostatic effect, with the aim of extending the work of Boyd in understanding the dynamics and evolution of nonlinear equatorial waves.

The quasi-geostrophic approximation [6; 7] is useful for studying flows with characteristic timescales of a day or more, thus filtering out high-frequency motions. One underlying assumption of the quasi-geostrophic approximation is that the Rossby number, which measures the relative importance of inertial to rotational motion in the momentum equations, is small. The Rossby number is defined as $Ro = U/2\Omega L$, where U and L are characteristic velocities and lengths, respectively. In the "traditional approximation", the rotation vector $\mathbf{\Omega}$ is assumed to depend only on the local vertical component, namely the Coriolis parameter $f = 2\Omega \sin\theta$, where θ is the latitude. However, in the "traditional approximation", the Rossby number is infinite at the equator due to the vanishing of the Coriolis parameter there, rendering the quasi-geostrophic approximation invalid. While the "traditional approximation" is justifiable for problems based in the mid-latitudes [8; 28], the contribution of the local horizontal component of rotation, $2\Omega\cos\theta$, is important in applications in the tropics [26; 28] and may be necessary for the simulation of some equatorial phenomena [10] (e.g. the Madden-Julian Oscillation).

[13] derived a nonlinear quasi-geostrophic model of the equatorial region, which takes into account the local horizontal component of rotation. Their model, denoted SNH-QGE III (Sideways Non-Hydrostatic Quasi-Geostrophic Equations type III), differs from the classical quasi-geostrophic model of fluid that is thin in the vertical, by assuming the meridional (y-direction) length scale is large compared to the zonal (x-direction) and vertical (z-direction) length scales. It follows that the leading balance in this model is between the zonal and vertical components of momentum, rather than the zonal and meridional components in the thin layer approach.

In the following section, we examine the linearized primitive equations on an equatorial β-plane and the equatorial waves they describe. Section 3 outlines the derivation of the quasi-geostrophic model SNH-QGE III from [13]. Section 4 treats the linearized version of this model, comparing to the results of the linearized primitive equations. Finally, in section 5, the nonlinear version of this quasi-geostrophic model is analyzed, following a similar procedure to that of [3]. In the case when the Brunt-Väisälä frequency is constant the waves are found to behave according to a generalized Swift-Hohenberg equation. By contrast, when the Brunt-Väisälä frequency varies with height, the waves behave according to a generalized Kadomtsev-Petviashvili equation.

2 Primitive equations

2.1 Equatorial waves

Consider the linearized, primitive equations in the following form:

$$\partial_t u' - \beta y v' + 2\Omega w' = -\partial_x p', \tag{1}$$

$$\partial_t v' + \beta y u' = -\partial_y p', \tag{2}$$

$$\partial_t w' - 2\Omega u' = -\partial_z p' - \frac{\rho' g}{\rho_r}, \tag{3}$$

$$\partial_x u' + \partial_y v' + \partial_z w' = 0, \tag{4}$$

$$\partial_t \rho' + \partial_z \overline{\rho} w' = 0, \tag{5}$$

where x is in the zonal direction, y is in the latitudinal direction, z is in the vertical direction, ρ_r is the reference density, $\overline{\rho}$ is the mean density field and ρ' is the perturbation of the density about the mean field. The zonal, latitudinal (or meridional) and vertical components of the 3-dimensional velocity field $\mathbf{u}$ are u', v', and w', respectively and p' is pressure. Apostrophes denote variables that are a function of x, y, z and t, and subscripts denote partial derivatives. The earth's rotation vector $\boldsymbol{\Omega}$ is assumed to depend on both the vertical and horizontal components, namely the Coriolis parameter $f = 2\Omega \sin\theta$ and $f_h = 2\Omega\cos\theta$, respectively. In Eqs. (1) - (5), we have assumed that the horizontal component is constant, $f_h = 2\Omega$, and f varies with latitude, $f \approx \beta y$, for constant β (this is known as the rational β-plane approximation [9]). Eqs. (3) and (5) can be combined by differentiating the former with respect to t. Taking the square of the Brunt-Väisälä frequency, $N^2 = -g\partial_z\overline{\rho}/\rho_r$, to be constant and assuming solutions of the form $u' = u(y)\mathrm{e}^{ikx+i\ell z-i\omega t}$, and similarly for v', w' and p', Eqs. (1) - (5) can be simplified to

$$-i\omega u - \beta y v + 2\Omega w = -ikp, \tag{6}$$
$$-i\omega v + \beta y u = -\partial_y p, \tag{7}$$
$$-i\omega w - 2\Omega u = -i\ell p + \frac{N^2 w}{i\omega}, \tag{8}$$
$$iku + \partial_y v + i\ell w = 0. \tag{9}$$

In this section we are interested in waves with frequencies less than the Brunt-Väisälä frequency, $|\omega| < N$. We proceed by eliminating u, w and p in Eqs. (6) - (9), to derive the following equation for v

$$(\omega^2 - N^2 - 4\Omega^2)\frac{d^2v}{dy^2} - 4\Omega i\ell\beta y\frac{dv}{dy} - k^2\left(\omega^2\left[1+\frac{\ell^2}{k^2}\right] - N^2\right)v + \ell^2\beta^2y^2v - \frac{k\beta}{\omega}\left(\omega^2 - N^2 + \frac{2\Omega\omega i\ell}{k}\right)v = 0. \tag{10}$$

The standard transformation $v(y) = V(y)\mathrm{e}^{-\lambda y^2/2}$ is employed, with $\lambda = -\frac{2\Omega i\ell\beta}{\omega^2-N^2-4\Omega^2}$, to eliminate the first-order derivative, and by rescaling y with $\hat{y} = y\sqrt{N^2-\omega^2+4\Omega^2}$, the following simplified expression for the y-dependent component of the meridional velocity is obtained, namely

$$\frac{d^2V}{d\hat{y}^2} - \ell^2\beta^2(N^2-\omega^2)\hat{y}^2V + \sigma V = 0, \tag{11}$$

where $\sigma = \ell^2\omega^2 - (N^2-\omega^2)\left(k^2 + \frac{k\beta}{\omega}\right)$. If σ is defined as

$$\sigma = \sigma_n = (2n+1)\ell\beta\sqrt{N^2-\omega^2}, \quad \text{for } n = 0, 1, 2, ..., \tag{12}$$

then the solutions to Eq. (11) are given by parabolic cylinder functions of order n that decay exponentially as $|y| \to \infty$, namely

$$V_n((\beta/\alpha)^{1/2}\hat{y}) = 2^{-n/2}\mathrm{e}^{-\beta\hat{y}^2/2\alpha}H_n((\beta/\alpha)^{1/2}\hat{y}), \tag{13}$$

where H_n is the nth physicists Hermite polynomial and $\alpha^2 = \frac{1}{\ell^2(N^2-\omega^2)}$. It is worth noting that the local horizontal component of the earth's rotation vector has no effect on the dispersion relation Eq. (12). Once $V_n(y)$ is known, the solutions for u, w and p are easily found

$$u(y) = \frac{i}{k}\left[\left(\frac{N^2-\omega^2+\frac{2\Omega i\omega\ell}{k}}{N^2-\omega^2(1+\frac{\ell^2}{k^2})}\right)\frac{dV_n}{dy} - \frac{\beta\omega\ell^2 y}{k\left(N^2-\omega^2(1+\frac{\ell^2}{k^2})\right)}V_n\right], \tag{14}$$

$$w(y) = \frac{\omega}{k}\left[\frac{2\Omega - \frac{i\omega\ell}{k}}{\left(N^2-\omega^2(1+\frac{\ell^2}{k^2})\right)}\frac{dV_n}{dy} + \frac{i\ell\beta y}{\left(N^2-\omega^2(1+\frac{\ell^2}{k^2})\right)}V_n\right], \tag{15}$$

$$p(y) = \frac{i}{k}\left[\frac{\omega}{k}\left(\frac{N^2-\omega^2+4\Omega^2}{N^2-\omega^2(1+\frac{\ell^2}{k^2})}\right)\frac{dV_n}{dy} - \left(\frac{N^2-\omega^2-\frac{2\Omega i\omega\ell}{k}}{N^2-\omega^2(1+\frac{\ell^2}{k^2})}\right)\beta y V_n\right]. \tag{16}$$

The surface height is contoured in figure 1 for the $n = 1$ mode and is overlaid with the horizontal velocity field (the u', v' field). The zonal velocities for the modes $n = 1, 2, 3, 4$ are compared in figure 2. The direction of propagation of waves is consistent with [8] and [19]. Surfaces of constant phase for this system are given by

$$kx + \ell z + \frac{2\Omega\ell\beta\hat{y}^2/2}{\omega^2 - N^2 - 4\Omega^2} = constant. \tag{17}$$

It is clear that these constant phase surfaces are curved in the y, z plane, which differs from the planar phase surfaces in the x, y plane of the shallow water equations. This is due to the introduction of an imaginary component in the solutions from the local horizontal component of the Coriolis force, a result consistent with that of [22].

2.2 Kelvin waves

We consider the unique case in which the meridional component of the velocity vanishes - the analog of the equatorial Kelvin wave in the shallow water model. First, setting $v = 0$ in Eqs. (6) - (9), we proceed to eliminate u, w and p from these equations and solve for ω algebraically, obtaining

$$\omega^2 = \frac{N^2 k^2}{k^2 + \ell^2}, \tag{18}$$

which yields a non-trivial solution provided $\omega \neq 2\Omega ik/\ell$. The solutions $u(y)$ and $w(y)$ expressed as functions of $p(y)$ are

$$u(y) = \frac{i\ell\left(2\Omega - \frac{i\omega\ell}{k}\right)}{4\Omega^2 + \frac{N^2\ell^2}{k^2+\ell^2}} p(y), \qquad w(y) = -\frac{ik\left(2\Omega - \frac{i\omega\ell}{k}\right)}{4\Omega^2 + \frac{N^2\ell^2}{k^2+\ell^2}} p(y). \tag{19}$$

Finally, $p(y)$ is found by solving the following equation

$$\frac{dp(y)}{dy} + byp(y) = 0, \qquad \text{where} \qquad b = i\ell\beta\frac{\left(2\Omega - \frac{i\omega\ell}{k}\right)}{4\Omega^2 + \frac{N^2\ell^2}{k^2+\ell^2}}. \tag{20}$$

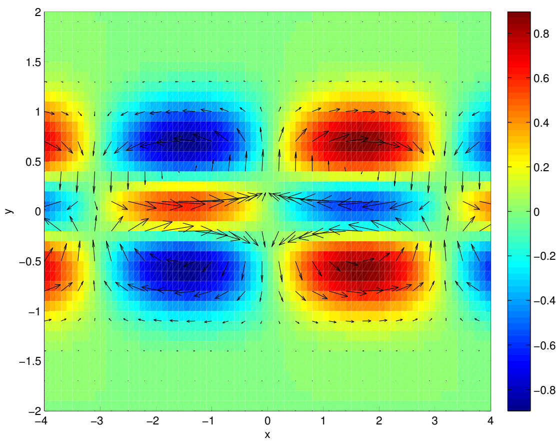
It is then clear that $p(y) = ae^{-by^2/2}$, where a is a complex constant amplitude. The p' is contoured in figure 3, with overlay of a purely zonal velocity field in arrows. Note that the meridional velocity is zero. Since we require that the waves decay meridionally north and south of the equator, the real part of b must be positive. That is,

$$Re\{b\} = \frac{\omega\ell^2\beta}{k(4\Omega^2 + \frac{N^2\ell^2}{k^2+\ell^2})} > 0. \tag{21}$$

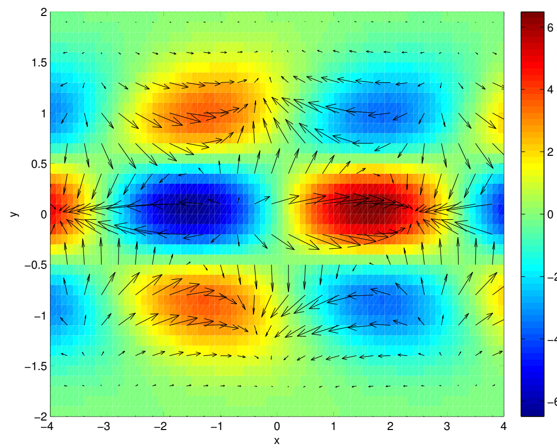
A consequence of this restriction is that ω is positive, namely

$$\omega = \frac{Nk}{\sqrt{k^2 + \ell^2}}. \tag{22}$$

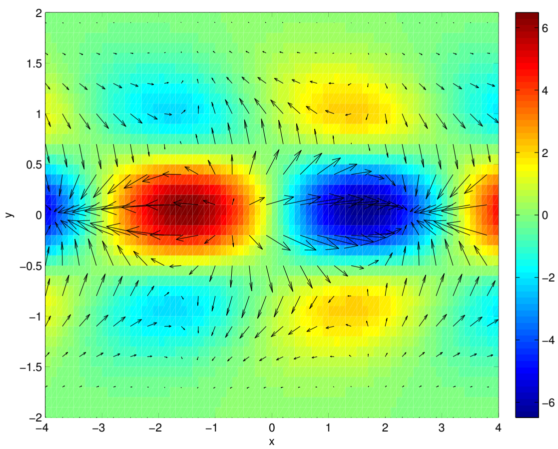
Defining the phase speed as $c_{p_x} = \omega/k$ and the group velocity as $\mathbf{c_{g_x}} = \partial\omega/\partial k$, we find that this Kelvin wave has a positive zonal phase speed and a positive group velocity, and



(a) A westwards propagating Rossby wave with small positive ω, $k = -1$.

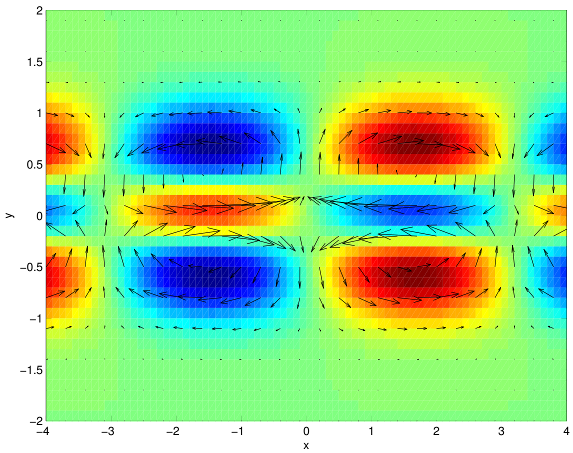


(b) An eastwards propagating inertio-gravity wave with large positive ω, $k = 1$.

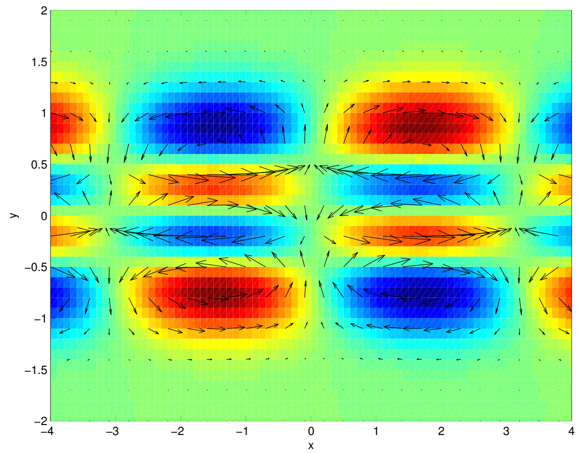


(c) A westwards propagating inertio-gravity wave with large positive ω, $k = -1$.

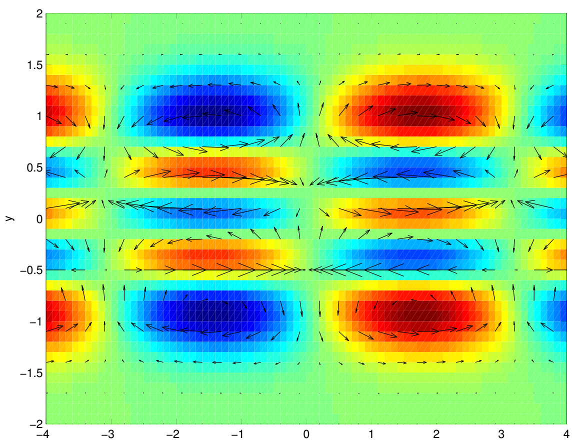
Figure 1: Pressure (surface height) in contours with overlay of the horizontal (u and v) velocity field in arrows, for $n = 1$ modes (Rossby and inertio-gravity modes) and $|\omega| < N$. The values for the constants are $\ell = 1$, $\beta = 1$, $N = 3$ and $2\Omega = 1$.



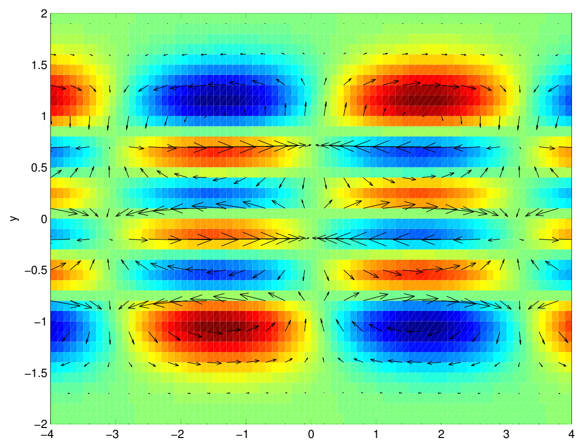
(a) $n = 1$



(b) $n = 2$



(c) $n = 3$



(d) $n = 4$

Figure 2: Pressure (surface height) in contours with overlay of the horizontal (u and v) velocity field in arrows, for a westwards propagating Rossby wave with increasing values of n, $|\omega| < N$ and $k = -1$. The values for the constants are $\ell = 1$, $\beta = 1$, $N = 3$ and $2\Omega = 1$.

hence propagates eastwards. However, the phase speed and group velocity are not equal, which means that the wave is dispersive; its shape is not conserved as the wave travels. By contrast, in the shallow water case, Kelvin waves of equivalent height H are non-dispersive; they all travel eastward with the phase speed $c = \sqrt{gH}$, which is equal to the group velocity. We deduce that the additional effects of vertical propagation and stratification, which are not present in the shallow water case, cause the Kelvin wave described by the dispersion relation in Eq. (22) to be dispersive. This result was also found by [12]. As in section 2.1, the local horizontal component of the Coriolis force has no effect on the dispersion relation for the Kelvin wave.

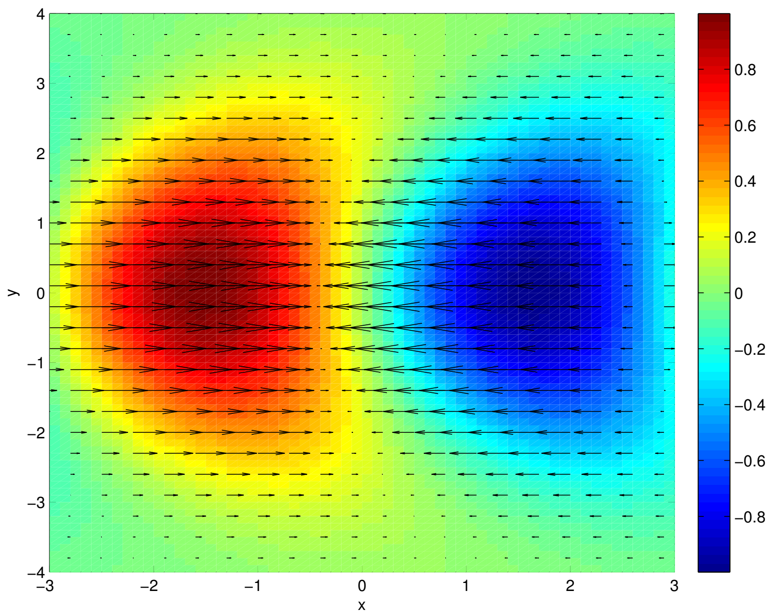


Figure 3: Pressure (surface height) in contours with overlay of the zonal velocity u field in arrows, for the "Kelvin"-like solution, for which $|\omega| < N$. The values for the constants are $k = 1$, $\ell = 1$, $\beta = 1$, $N = 3$ and $2\Omega = 1$.

2.3 Dispersion relation

The dispersion curves from Eqs. (12) and (22) are plotted in figure 4(a). The Kelvin wave is often denoted the $n = -1$ mode, although its dispersion relation comes from Eq. (22) rather than Eq. (12), as it corresponds to the unique case in which the meridional velocity is equal to zero. For the case when $n = 0$ in Eq. (12), known as the mixed Rossby-gravity mode or "Yanai" mode, two of the four solutions to the dispersion relation in Eq. (12) are $\omega = \pm\frac{Nk}{\sqrt{k^2+\ell^2}}$. In fact, these solutions are not permitted as in deriving the solution we assumed that $\omega^2(1 + \ell^2/k^2) - N^2 \neq 0$. There are two other roots of the dispersion relation when $n = 0$: one is positive and corresponds to an eastwards propagating equatorial inertio-gravity wave, which resembles a Kelvin mode for large zonal scales; the other is negative and corresponds to a westwards propagating wave, which resembles a Rossby wave for small zonal scales (see figure 4). The key differences between the Kelvin $n = -1$ and the eastwards propagating inertio-gravity wave $n = 0$ is that the Kelvin wave is centered on the equator and has no westwards-propagating component. In the shallow water case, an atmospheric Kelvin wave has an eastwards phase speed of 10-20 m s^{-1}, whereas an atmospheric inertio-gravity wave has a typical eastwards phase speed of 25-50 m s^{-1} [27].

An analog to each of the waves found in figure 4(a) exists in the hydrostatic shallow water equations [8; 19]. For the Kelvin mode, the shallow water dispersion relation is

$$\omega = ck, \tag{23}$$

where $c = \sqrt{gH}$ is the gravity phase speed. When the primitive system in Eqs. (6) - (9) is strongly stratified (i.e. $|\omega| \ll N$), the dispersion relation in Eq. (12) reduces to the shallow water dispersion relation for the higher order modes, namely,

$$\frac{\omega^2}{c^2} - k^2 - \frac{\beta k}{\omega} = (2n+1)\frac{\beta}{c}, \tag{24}$$

where $c^2 = N^2/\ell^2$ is the analog of the shallow water phase speed. It is well known (e.g. [8; 11]) that for the $n = 0$ mode the dispersion relation in Eq. (24) produces a root, $\omega = -ck$, which is spurious (this relation is assumed not to be true in the derivation of the solution). This root corresponds to a westwards propagating gravity wave; if it were valid, the solution for the meridional component of the velocity would no longer be equatorially-constrained. The key difference between the shallow water dispersion relations and those of Eqs. (12) and (22) is the presence of N^2 in the latter, which is the upper limit of the inertio-gravity, gravity and mixed Rossby-gravity mode frequencies. This is clearly illustrated in figures 4(a) and (b).

2.4 Evanescent waves

The results in the previous sections were obtained by assuming a traveling wave in the vertical direction, $\mathrm{e}^{i\ell z}$. If we instead assume that waves decay exponentially from the height $z = 0$ so that the vertical component of u', v', w' and p' is given by $\mathrm{e}^{-\ell z}$, and this time assume that $|\omega| > N$, we obtain the following solution also in terms of parabolic cylinder functions

$$u(y) = \frac{i}{k}\left[\left(\frac{\omega^2 - N^2 + \frac{2\Omega\omega\ell}{k}}{\omega^2\left(1 - \frac{\ell^2}{k^2}\right) - N^2}\right)\frac{dV_n}{dy} - \frac{\beta\omega l^2 y}{k\left(\omega^2\left(1 - \frac{\ell^2}{k^2}\right) - N^2\right)}V_n\right], \tag{25}$$

$$w(y) = \frac{\omega}{k}\left[-\frac{2\Omega + \frac{\omega\ell}{k}}{\left(\omega^2\left(1 - \frac{\ell^2}{k^2}\right) - N^2\right)}\frac{dV_n}{dy} + \frac{\beta\ell y}{\left(\omega^2\left(1 - \frac{\ell^2}{k^2}\right) - N^2\right)}V_n\right], \tag{26}$$

$$p(y) = \frac{i}{k}\left[\frac{\omega}{k}\left(\frac{\omega^2 - N^2 - 4\Omega^2}{\omega^2(1 - \frac{\ell^2}{k^2}) - N^2}\right)\frac{dV_n}{dy} - \left(\frac{\omega^2 - N^2 - \frac{2\Omega\omega\ell}{k}}{\omega^2(1 - \frac{\ell^2}{k^2}) - N^2}\right)\beta y V_n\right], \tag{27}$$

where

$$V_n(\hat{y}) = \frac{2^{-n/2}}{\sqrt{\omega^2 - N^2 - 4\Omega^2}}\mathrm{e}^{-\beta\hat{y}^2/2\alpha}H_n((\beta/\alpha)^{1/2}\hat{y})\mathrm{e}^{-\lambda\hat{y}^2/2}.$$

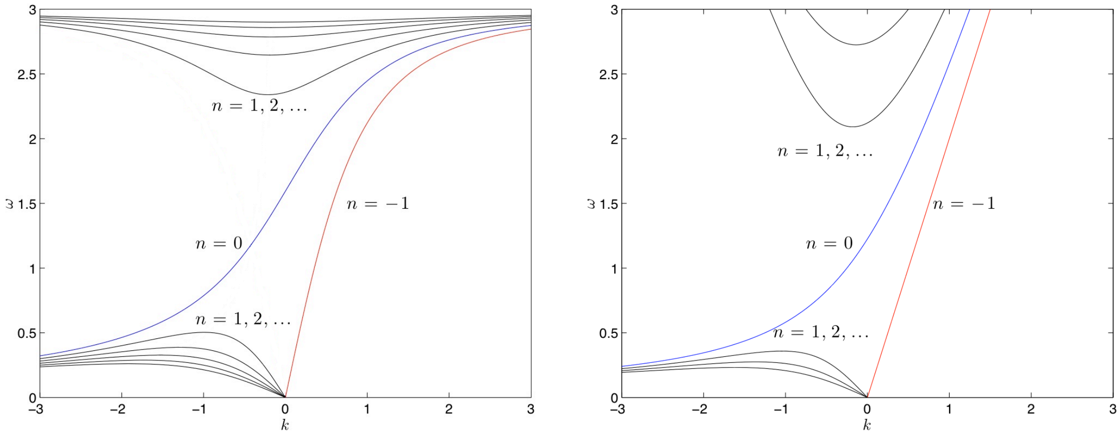


Figure 4: Dispersion curves for: (a) the non-hydrostatic primitive system; and (b) for the shallow water equations. In each figure, the red line is the Kelvin ($n = -1$) mode, refer to Eqs. (22) and (23), respectively, the blue lines are the mixed Rossby-gravity ($n = 0$) mode, refer to Eqs. (12) and (24), respectively, and the black lines are the Rossby (low-frequency) and inertio-gravity (higher-frequency) ($n = 1, 2, ...$) modes, refer to Eqs. (12) and (24), respectively. Values for the constants are $N = 3$, $\beta = 1$ and $c = 2$.

Here, y has been scaled such that $\hat{y} = y\sqrt{\omega^2 - N^2 + 4\Omega^2}$ and the constants λ and α are defined by

$$\lambda = \frac{\ell\beta}{\omega^2 - N^2 - 4\Omega^2}, \qquad \alpha^2 = \frac{1}{\ell^2(\omega^2 - N^2)}.$$

The dispersion curve for this system is

$$\omega^2\ell^2 - (\omega^2 - N^2)(k^2 - \frac{k\beta}{\omega}) - (2n+1)\beta\ell\sqrt{\omega^2 - N^2} = 0, \tag{28}$$

which is plotted in figure 5.

2.5 Evanescent Kelvin waves

Proceeding as in section 2.4 for the case in which the meridional component of the velocity vanishes, we find that

$$\omega^2 = \frac{N^2k^2}{k^2 - \ell^2}, \tag{29}$$

which is true provided $\omega \neq 2\Omega k/\ell$ (otherwise, as previously, we obtain the trivial solution for each of u', w' and p'). The solution for $p(y)$ is

$$p(y) = \mu e^{-\gamma y^2/2}, \quad \text{where,} \quad \gamma = -\frac{\beta\ell\left(2\Omega + \frac{\omega\ell}{k}\right)}{4\Omega^2 - \frac{N^2\ell^2}{k^2-\ell^2}}, \tag{30}$$

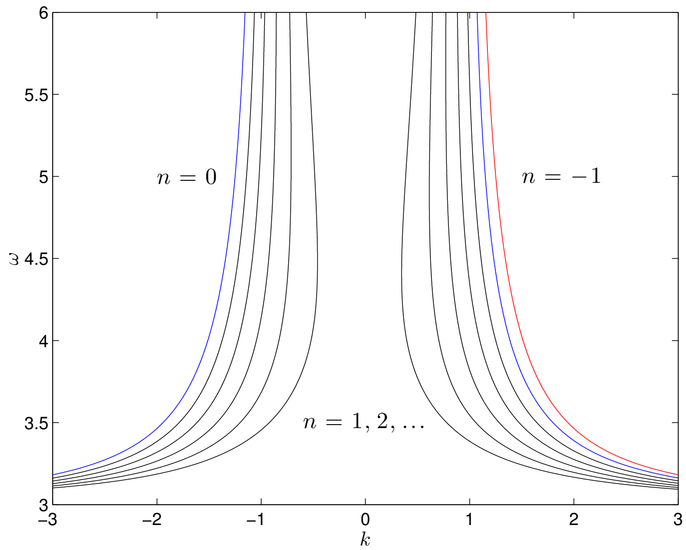


Figure 5: Dispersion curve for evanescent waves ($|\omega| > N$). The red line is the Kelvin mode, the blue lines are the $n = 0$ modes, and the black lines are the $n = 1, 2, ..$ modes.

and μ is a complex constant. Since we require that the waves decay meridionally north and south of the equator, we must have $\gamma < 0$. Then,

$$\omega = \frac{Nk}{\sqrt{k^2 - \ell^2}}, \tag{31}$$

which is plotted in figure 5, and we assume that $k > \ell > 0$, in which case the waves propagate eastwards and vertically upwards. The solutions u and w are

$$u(y) = -\frac{\ell\left(2\Omega + \frac{\omega\ell}{k}\right)}{4\Omega^2 - \frac{N^2\ell^2}{k^2-\ell^2}}p(y), \qquad w(y) = -\frac{ik\left(2\Omega + \frac{\omega\ell}{k}\right)}{4\Omega^2 - \frac{N^2\ell^2}{k^2-\ell^2}}p(y). \tag{32}$$

2.6 Remarks on the primitive equations

There are some clear distinctions between our results and those obtained from the hydrostatic shallow water equations on an equatorial β-plane: (i) the addition of the local horizontal component of the Coriolis force introduces an imaginary component in the solutions to the primitive equations, adds curvature to the y, z phase planes in the case when $v' \neq 0$, and adds a phase shift to the wave structure in the case when $v' = 0$; (ii) when a traveling wave is assumed in the vertical, non-hydrostatic effects modify the behavior of the inertio-gravity, gravity and mixed Rossby-gravity modes such that their frequencies are necessarily less than the Brunt-Väisälä frequency and the converse is true for evanescent waves in the vertical; and, (iii) the Kelvin mode no longer obeys the gravity phase speed $c = \sqrt{gH}$, but becomes dispersive under non-hydrostatic effects. This concludes investigation of the linearized primitive equations; we now turn to the SNH-QGE III model, comparing the linearized version of the model with the results from this section.

3 Quasi-geostrophic Model

We briefly outline the procedure used to derive the SNH-QGE III model analyzed in the following sections. Readers are directed to [13] for a more comprehensive derivation. First, consider the Boussinesq form of the dimensional Navier-Stokes equations in the following form:

$$D_t\mathbf{u}^* + 2\Omega\hat{\boldsymbol{\eta}} \times \mathbf{u}^* = -\frac{1}{\rho_r}\boldsymbol{\nabla} p^* + b^*\hat{\mathbf{z}}, \tag{33}$$

$$D_t b^* - \frac{g}{\rho_r} w^* \partial_z \overline{p} = 0, \tag{34}$$

$$\boldsymbol{\nabla} \cdot \mathbf{u}^* = 0. \tag{35}$$

Here, $D_t = \partial_t + \mathbf{u} \cdot \boldsymbol{\nabla}$ is the material derivative, $\mathbf{u}$ is the 3-dimensional velocity vector $\mathbf{u} = (u, v, w)$, and $b = -g\rho'/\rho_r$ is the buoyancy anomaly field of density perturbations, ρ', about the mean density field $\overline{\rho}$, where ρ_r is the reference density. The total planetary rotation vector is $2\Omega\hat{\boldsymbol{\eta}}$. We consider only motions at the equator and invoke the β-plane approximation with $\hat{\boldsymbol{\eta}} = \hat{\boldsymbol{\eta}}_0 - (y/R)\hat{\boldsymbol{\eta}}_1$, for planetary radius R, where $\hat{\boldsymbol{\eta}}_0 = (0, 1, 0)$ and $\hat{\boldsymbol{\eta}}_1 = (0, 0, -1)$. The asterisk notation in the equations above denotes a dimensional quantity. Let L be a characteristic length scale, and U a characteristic velocity and suppose that $T = L/U$ (we are interested in synoptic scale disturbances where typical values for L and U are $50-100$km and $0.4-1$m s^{-1}, respectively). We introduce the following non-dimensional numbers

$$\text{Rossby number: } Ro = \frac{U}{2\Omega L}, \qquad \text{Reynolds number: } Re = \frac{UL}{\nu},$$

$$\text{Euler number: } P = \frac{\delta p}{U^2\rho_r}, \qquad \text{Peclet number: } Pe = \frac{LU}{\kappa},$$

$$\text{Froude number: } Fr = \frac{U}{N_0 L}, \qquad \text{Buoyancy number: } \Gamma = \frac{BL}{U^2} = g\left|\frac{\delta\rho}{\rho_r}\right|\frac{L}{U^2},$$

$$\text{Buoyancy anomaly: } B = g\left|\frac{\delta\rho}{\rho_r}\right|, \qquad \text{Reference stratification: } N_0 = \sqrt{\frac{g}{H_\rho}},$$

$$\text{Reference density height: } H_\rho^{-1} = \frac{|(\partial_z\overline{\rho})_r|}{\rho_r},$$

where δp is a dynamic pressure scale, N_0 is a reference stratification, $\delta\rho$ is a density scale, $(\partial_z\overline{\rho})_r$ is a reference density gradient, ν is viscosity and κ is thermal diffusivity.

The system is non-dimensionalized and then scaled in the y-direction by introducing the modulation scale $y = A_Y Y$. The β-effect also operates on this scale, so $A_\beta y \sim A_\beta A_Y Y$ and $A_\beta A_Y = \mathcal{O}(Ro)$. We define $\boldsymbol{\nabla} \to \boldsymbol{\nabla}_\perp + A_Y^{-1} y\partial_Y$, where $\boldsymbol{\nabla}_\perp = \hat{\mathbf{x}}\partial_x + \hat{\mathbf{z}}\partial_z$. We are interested in flows with characteristic time scales greater than a day, and as such, consider small Rossby numbers ($Ro \sim 1/10$). Write $Ro \equiv \epsilon \ll 1$ and choose the spatial scales

$$A_Y = \epsilon^{-1}, \qquad A_\beta = \beta\epsilon^2, \qquad P = \epsilon^{-1}, \qquad \Gamma = 1,$$

assuming that $\beta \sim \mathcal{O}(1)$. Introducing asymptotic expansions of the form

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + ..., \tag{36}$$

$$v = v_0 + \epsilon v_1 + \epsilon^2 v_2 + ..., \tag{37}$$

$$w = w_0 + \epsilon w_1 + \epsilon^2 w_2 + ..., \tag{38}$$

$$p = p_0 + \epsilon p_1 + \epsilon^2 p_2 + ..., \tag{39}$$

$$b = b_0 + \epsilon b_1 + \epsilon^2 b_2 + ..., \tag{40}$$

the leading order set yields geostrophic balance and continuity on the x, z plane, given by

$$\hat{\boldsymbol{\eta}} \times \mathbf{u}_0 = -\boldsymbol{\nabla}_\perp p_0, \quad \text{and,} \quad \boldsymbol{\nabla}_\perp \mathbf{u} = 0, \tag{41}$$

respectively. The next order set is

$$D_t^0 \mathbf{u}_0 - \beta Y \hat{\boldsymbol{\eta}}_1 \times \mathbf{u}_0 + \hat{\boldsymbol{\eta}}_0 \times \mathbf{u}_1 = -\boldsymbol{\nabla}_\perp p_1 - \partial_Y p_0 \hat{\mathbf{y}} + b_0 \hat{\mathbf{z}}, \tag{42}$$

$$D_t^0 \left(b_0 - \frac{1}{\mathcal{F}^2} \overline{\rho}(z) \right) = 0, \tag{43}$$

$$\boldsymbol{\nabla}_\perp \cdot \mathbf{u}_1 + \partial_Y v_0 = 0. \tag{44}$$

Note that $D_t^0 = \partial_t + \mathbf{u} \cdot \boldsymbol{\nabla}_\perp$ and $\mathcal{F}^2 = \Gamma Fr^2$. For a closed system, we require that $\mathcal{F} = \mathcal{O}(1)$. Applying $\hat{\boldsymbol{\eta}}_0 \cdot$ and $\boldsymbol{\nabla} \times$ to Eq. (42), we find that

$$\hat{\boldsymbol{\eta}}_0 \cdot \boldsymbol{\nabla} p_1 = \hat{\boldsymbol{\eta}}_0 \cdot \left(-D_t^0 \mathbf{u}_0 + \beta Y \hat{\boldsymbol{\eta}}_1 \times \mathbf{u}_0 - \partial_Y p_0 \hat{\mathbf{y}} + b_0 \hat{\mathbf{z}} \right), \tag{45}$$

$$\hat{\boldsymbol{\eta}}_0 \cdot \boldsymbol{\nabla} \mathbf{u}_1 = D_t^0 \boldsymbol{\omega}_0 - \boldsymbol{\omega}_0 \cdot \boldsymbol{\nabla}_\perp \mathbf{u}_0 - \hat{\boldsymbol{\eta}}_0 \partial_Y v_0 - \boldsymbol{\nabla} \times \left(\beta Y \hat{\boldsymbol{\eta}}_1 \times \mathbf{u}_0 - \partial_Y p_o \hat{\mathbf{y}} + b_z \hat{\mathbf{z}} \right), \tag{46}$$

the latter being the equation for the vorticity $\boldsymbol{\omega}$. We have introduced the large spatial variable Y, which means that Eqs. (45) and (46) contain secular terms that grow with the small variable y. Hence, we require a solvability condition that ensures the terms in Eqs. (45) and (46) balance. In this case, the solvability conditions are obtained by averaging the equations over η and forcing the right-hand sides to equal 0. Then, projecting Eq. (46) onto $\hat{\boldsymbol{\eta}}_0$, we obtain the following closed system that can be written in terms of the geostrophic and ageostrophic streamfunctions, Ψ and Φ, respectively,

$$\mathbf{u_0} = -\boldsymbol{\nabla} \times (\Psi_0 \hat{\mathbf{y}} + \boldsymbol{\nabla} \times \Phi_0 \hat{\mathbf{y}}), \quad p_0 = \Psi_0, \tag{47}$$

$$D_t^0 \boldsymbol{\nabla}_\perp^2 \Psi_0 - (\partial_y + \beta y \partial_z) \boldsymbol{\nabla}_\perp^2 \Phi_0 = -\partial_x b_0, \tag{48}$$

$$D_t^0 \boldsymbol{\nabla}_\perp^2 \Phi_0 + (\partial_y + \beta y \partial_z) \Psi_0 = 0, \tag{49}$$

$$D_t^0 \left(b_0 - \frac{\overline{\rho}(z)}{\mathcal{F}^2} \right) = 0. \tag{50}$$

This is the SNH-QGE III model. The linearized version of this system is analyzed in the following section and compared to the linearized primitive equations of section 2. In particular, we are interested in whether the system describes the same equatorially constrained waves as the linearized primitive equations (the Kelvin, Rossby, mixed Rossby-gravity and inertio-gravity modes).

4 Investigation of the linearized quasi-geostrophic equations

The linearized reduced quasi-geostrophic model from Eqs. (48) - (50) can be written

$$\partial_t^0 \boldsymbol{\nabla}_\perp^2 \Psi - (\partial_y + \beta y \partial_z) \boldsymbol{\nabla}_\perp^2 \Phi = -\partial_x b, \tag{51}$$

$$\partial_t^0 \boldsymbol{\nabla}_\perp^2 \Phi + (\partial_y + \beta y \partial_z) \Psi = 0, \tag{52}$$

$$\partial_t^0 b - w \frac{\overline{\rho}_z(z)}{\mathcal{F}^2} = 0. \tag{53}$$

Eqs. (51) and (53) are combined by differentiating the former with respect to t. We write the Brunt-Väisälä frequency as $N^2 = -\overline{\rho}_z(z)/\mathcal{F}^2$ and assume that it is constant. Consider first the case when $v = \boldsymbol{\nabla}_\perp^2 \Phi = 0$ (from Eq. (47)). Ψ is found by solving $\partial_y \Psi + \beta y \partial_z \Psi = 0$. Assuming a solution of the form

$$\Psi(x, y, z, t) = \tilde{\Psi}(y) \mathrm{e}^{ikx + i\ell z - i\omega t}, \tag{54}$$

the y-dependent part of the solution is given by $\tilde{\Psi} = a\mathrm{e}^{-i\ell\beta y^2/2}$, where a is a complex constant. Note that this wave is not equatorially constrained; the solution Ψ does not decay to 0 as $y \to \infty$. The dispersion relation for this wave is similar to that in the primitive equations for the case when $|\omega| < N$, namely,

$$\omega = \pm \frac{Nk}{\sqrt{k^2 + \ell^2}}, \tag{55}$$

except that now there are two roots to the equation, representing eastwards and westwards traveling waves.

We next consider the higher order mode waves, where the meridional velocity is no longer zero. From Eq. (47), the zonal and vertical components of the velocity are $u = \partial_z \Psi$ and $w = -\partial_x \Psi$, so that Eqs. (51) - (53) can be written as one equation in terms of the variable Ψ, namely

$$\partial_{tt} \nabla_\perp^2 \Psi + N^2 \partial_{xx} \Psi + \partial_{yy} \Psi + \beta^2 y^2 \partial_{zz} \Psi + \beta \partial_z \Psi + 2\beta y \partial_{yz} \Psi = 0. \tag{56}$$

We again assume a solution of the form given in Eq. (54), which yields the following equation for Ψ

$$\frac{d^2 \tilde{\Psi}}{dy^2} + 2i\ell\beta y \frac{d\tilde{\Psi}}{dy} - \beta^2 \ell^2 y^2 \tilde{\Psi} + i\ell\beta \tilde{\Psi} + k^2 \left(\omega^2 \left[1 + \frac{\ell^2}{k^2} \right] - N^2 \right) \tilde{\Psi} = 0. \tag{57}$$

Using the standard transformation $\tilde{\Psi}(y) = \psi(y) \mathrm{e}^{-\lambda y^2/2}$, where $\lambda = i\ell\beta$, we obtain the simplified equation

$$\frac{d^2 \psi}{dy^2} + k^2 \left(\omega^2 \left[1 + \frac{\ell^2}{k^2} \right] - N^2 \right) \psi = 0. \tag{58}$$

The waveguide solution that existed in the primitive equations using an equivalent ansatz - traveling wave structures in the zonal and vertical directions - is not apparent in the reduced quasi-geostrophic model. That is, the solution to Eq. (56), for both the Kelvin mode and the higher order modes, is no longer given in terms of a parabolic cylinder function that decays exponentially north and south of the equator. What happens if we choose a different ansatz? In particular, how does the solution change if we assume a standing wave form in the vertical, rather than a traveling wave? We will run into difficulties if we simply use this ansatz in form of the system given in Eq. (56), hence to proceed, we rescale Eq. (56) in y and perform an asymptotic expansion about the small parameter β.

Recall that the y coordinate in Eq. (56) was scaled in section 3 such that $y = A_y^{-1}y^*$, where y^* was the unscaled meridional variable, and with $A_y = 1/Ro$. In what follows, we introduce a new scaling $y = \beta^{-1/2}Y$, where β is assumed to be small enough such that the distinguished limits in the asymptotic expansion of section 3 are unchanged. This yields the following rescaled equation for Ψ

$$(\partial_{tt}\nabla_\perp^2 + N^2\partial_{xx})\Psi + \beta\left(\partial_{YY} + Y^2\partial_{zz}\right)\Psi + \beta\left(\partial_z + 2Y\partial_{Yz}\right)\Psi = 0. \tag{59}$$

The slow time $\tau = \beta t$ is introduced along with the expansion

$$\Psi(x, Y, z, t, \tau) = \Psi_0 + \beta\Psi_1 + ... \tag{60}$$

Note that because we have both a fast and a slow time in the system now, we expect the dispersion relation to involve both a fast and a slow frequency, denoted ω_f and ω_s, respectively. This time we assume a standing wave form in the vertical

$$\Psi_0(x, Y, z, t, \tau) \propto \psi_0(Y, \tau)\mathrm{e}^{ikx - i\omega_f t}\sin(\ell z), \tag{61}$$

with the boundary conditions $w = 0$ at $z = 0, H$. At leading order, $\mathcal{O}(\beta^0)$, this expansion yields

$$\left[\partial_{tt}\nabla_\perp^2 + N^2\partial_{xx}\right]\Psi_0 = 0, \tag{62}$$

from which we derive an expression for the fast frequency,

$$\omega_f = \pm\frac{Nk}{\sqrt{k^2 + \ell^2}}. \tag{63}$$

This is the leading order frequency of the linearized system. At next order, $\mathcal{O}(\beta)$, we obtain

$$\left[\partial_{tt}\nabla_\perp^2 + N^2\partial_{xx}\right]\Psi_1 = -2\partial_{t\tau}\nabla_\perp^2\Psi_0 - \left(\partial_{YY} + Y^2\partial_{zz}\right)\Psi_0 - \left(\partial_z + 2Y\partial_{Yz}\right)\Psi_0. \tag{64}$$

The linear operator $\mathcal{L} = \partial_{tt}\nabla_\perp^2 + N^2\partial_{xx}$ is self-adjoint, and, by orthogonality of $\sin\ell z$ and $\cos\ell z$, the last two terms involving first derivatives with respect to z vanish from the solvability condition, leaving

$$\langle\Psi_0, \left[-2\partial_{t,\tau}\nabla_\perp^2 - (\partial_{YY} + Y^2\partial_{zz})\right]\Psi_0\rangle = 0. \tag{65}$$

Taking into account the ansatz from Eq. (61), and assuming that ψ_0 is separable in τ and Y such that $\psi_0(Y, \tau) = \tilde{\psi}_0(Y)\mathrm{e}^{i\omega_s\tau}$, we obtain the following expression for $\tilde{\psi}_0$

$$\frac{d^2\tilde{\psi}_0(Y)}{dY^2} - \ell^2Y^2\tilde{\psi}_0(Y) + 2\omega_f\omega_s(k^2 + \ell^2)\tilde{\psi}_0(Y) = 0. \tag{66}$$

Eq. (66) can be solved with parabolic cylinder functions, and is equatorially constrained when

$$2\omega_f\omega_s(k^2+\ell^2) = (2n+1)\ell, \qquad \text{for } n = 0, 1, 2, ..., \tag{67}$$

where ω_f is the fast frequency defined in Eq. (63). The dispersion relation for the reduced system can be written as $\omega = \omega_f + \beta\omega_s$, leading to the following expression for ω in terms of the horizontal and vertical wavelengths

$$\omega = \frac{Nk}{\sqrt{k^2+\ell^2}} + \frac{\beta(2n+1)\ell}{2Nk\sqrt{k^2+\ell^2}}. \tag{68}$$

How does Eq. (68) compare to the dispersion relations of the primitive equations, namely Eqs. (12) and (22)? Substituting $\omega = \omega_f + \beta\omega_s$ into Eq. (12), we expand in terms of the small parameter β. Then, the leading order expression is identical to Eq. (63) and at next order in β we obtain

$$2\omega_f\omega_s(k^2+\ell^2) = (2n+1\pm 1)\frac{N\ell^2}{\sqrt{k^2+\ell^2}}. \tag{69}$$

The main differences between Eq. (67) from the exact system and Eq. (69) from the reduced system are the presence of N, $\ell/\sqrt{k^2+\ell^2}$ and the $\pm$ term on the right-hand side of Eq. (69). The $\pm$ term indicates that the reduced system has filtered out modes that are even with respect to the equator ($n = 1, 3, 5, ...$). It is likely that the former two terms are a result of the scalings used to derive the quasi-geostrophic system from [13] (for example, that N is a result of the time scaling employed and that k is scaled such that $k = \mathcal{O}(1)$ in the reduced system corresponds to $k \ll 1$ in the primitive equations so that $\ell^2/\sqrt{k^2+\ell^2} \approx \ell$). In this case, and taking into account the limits of the reduced model, the dispersion relations in Eq. (12) and (68), and indeed the primitive and reduced systems, are then equivalent. We now turn to the fully nonlinear quasi-geostrophic model.

5 Investigation of the nonlinear quasi-geostrophic model

This section introduces two methods of analyzing the nonlinear quasi-geostrophic system in Eqs. (48) - (50): one method involves a similar scaling argument to that presented in section 4 for a constant Brunt-Väisälä frequency, while the other method is not restricted to small values of β and assumes a non-constant vertical stratification.

5.1 Constant Brunt-Väisälä frequency

Consider the nonlinear reduced equatorial system in Eqs. (48) - (50) from section 3. Following [2], we transform Eqs. (48) - (50) to a frame traveling with the wave speed c eastwards, by defining $\xi = x - ct$. We are interested in longwave solutions in the zonal direction that do not change the distinguished limits in the scaling arguments of section 3, namely $X = \epsilon\xi$, where ϵ is a small number. We also introduce the following scalings:

$$\Psi = \epsilon\Psi, \qquad \Phi = \epsilon^2\Phi, \qquad b = \epsilon b.$$

Proceeding as in section 4, y is rescaled by $y = \beta^{-1/2}Y$ and β by $\beta = \epsilon^4\hat{\beta}$. Eqs. (48) - (50) simplify to

$$\begin{aligned}
-\partial_X b &= -c(\epsilon^2\partial_X^2 + \partial_z^2)\partial_X\Psi - \epsilon^2(\partial_Y + \hat{\beta}Y\partial_z)(\epsilon^2\partial_X^2 + \partial_z^2)\Phi \\
&\quad + \epsilon\left[\partial_z\Psi(\epsilon^2\partial_X^2 + \partial_z^2)\partial_X\Psi - \partial_X\Psi(\epsilon^2\partial_X^2 + \partial_z^2)\partial_z\Psi\right], \quad (70)\\
0 &= -c(\epsilon^2\partial_X^2 + \partial_z^2)\partial_X\Phi + (\partial_Y + \hat{\beta}Y\partial_z)\Psi \\
&\quad + \epsilon\left[\partial_z\Psi(\epsilon^2\partial_X^2 + \partial_z^2)\partial_X\Phi - \partial_X\Psi(\epsilon^2\partial_X^2 + \partial_z^2)\partial_z\Phi\right], \quad (71)\\
0 &= -c\partial_X b + \epsilon\partial_z\Psi\partial_X b - \epsilon\partial_X\Psi\partial_z b - \partial_X\Psi N^2. \quad (72)
\end{aligned}$$

We make the assumption that the meridional velocity is of the same order as the zonal and vertical velocities and expand the variables Ψ, Φ, b and the phase speed c as follows

$$\Psi = \Psi_0 + \epsilon\Psi_1 + ..., \quad (73)$$

$$\Phi = \Phi_0 + \epsilon\Phi_1 + ..., \quad (74)$$

$$b = b_0 + \epsilon b_1 + ..., \quad (75)$$

$$c = c_0 + \epsilon c_1 + ... \quad (76)$$

The expansions in Eqs. (73) - (76) are substituted into Eqs. (70) - (72), yielding the leading order set, written as a system in terms of Ψ and Φ only

$$\frac{N^2}{c_0^2}\Psi_{0X} + \Psi_{0Xzz} = 0, \quad (77)$$

$$-c_0\Phi_{0Xzz} + (\partial_Y + \hat{\beta}Y\partial_z)\Psi_0 = 0, \quad (78)$$

which, when the boundary conditions $w = -\Psi_{0X} = 0$ on $z = 0, H$ are satisfied, yields

$$\Psi_0 = A_0(X,Y)\sin\left(\frac{\pi}{H}z\right). \quad (79)$$

This wave is a standing wave in the vertical, which results from the ansatz we have assumed here, which is analogous to that of the linear case in Eq. (61). Suppose that

$$\Phi_0 = B(X,Y)\sin\left(\frac{\pi}{H}z\right) + C(X,Y)\cos\left(\frac{\pi}{H}z\right). \quad (80)$$

Substituting this into Eq. (78), the amplitudes B and C can be expressed in terms of A_0, namely

$$C_X = -\frac{\hat{\beta}HY}{c_0\pi}A_0, \qquad B_X = -\frac{H^2}{c_0\pi^2}A_{0Y}. \quad (81)$$

We are interested in the meridional and zonal structure of the amplitude of Ψ, and particularly how it is modified by nonlinearities. To this end, we proceed to next order.

The first order equations written in terms of Ψ and Φ are

$$\begin{aligned}
c_0^2\Psi_{1Xzz} + N^2\Psi_{1X} &= c_0c_1\frac{\pi^2}{H^2}A_{0X}\sin\left(\frac{\pi}{H}z\right) + c_1\frac{N^2}{c_0}A_{0X}\sin\left(\frac{\pi}{H}z\right) \\
&\quad - A_{0X}D_z\sin\left(\frac{\pi}{H}z\right), \quad (82)\\
\Rightarrow (N^2 + c_0^2\partial_z^2)\Psi_{1X} &= 2c_0c_1\frac{\pi^2}{H^2}A_{0X}\sin\left(\frac{\pi}{H}z\right). \quad (83)
\end{aligned}$$

The linear operator $(N^2 + c_0^2\partial_z^2)$ is self-adjoint, which leads to the solvability condition

$$\left\langle \Psi_0, 2c_0c_1\frac{\pi^2}{H^2}A_{0X}\sin\left(\frac{\pi}{H}z\right)\right\rangle = 0. \tag{84}$$

We deduce that $c_1 = 0$ and the first order equations are simplified to

$$\frac{N^2}{c_0^2}\Psi_{1X} + c_0\Psi_{1Xzz} = 0, \tag{85}$$

$$c_0\Phi_{1Xzz} - (\partial_Y + \hat{\beta}Y\partial_z)\Psi_1 = \Psi_{0z}\Phi_{0Xzz} - \Psi_{0X}\Phi_{0zzz}. \tag{86}$$

The former equation is solved by imposing the boundary conditions $w = -\Psi_{0X} = 0$ on $z = 0, H$, yielding $\Psi_1 = A_1(X,Y)\sin\left(\frac{\pi}{H}z\right)$. Finally, Eq. (86) is solved for Φ_1, and integrating twice with respect to z, we find

$$\begin{aligned}\Phi_{1X} \quad = \quad & -\frac{\pi}{8c_0H}\left[(A_{0X}B - A_0B_X)\sin\left(\frac{2\pi}{H}z\right) + (A_{0X}C - A_0C_X)\cos\left(\frac{2\pi}{H}z\right)\right] \\ & -\frac{H^2}{c_0\pi^2}A_{1Y}\sin\left(\frac{\pi}{H}z\right) - \frac{H}{c_0\pi}A_1\beta Y\cos\left(\frac{\pi}{H}z\right) - \frac{\pi^3}{2c_0H^3}(A_{0X}C + A_0C_X)\left(\frac{z^2}{2} + Dz + E\right),\end{aligned}$$

where

$$-\frac{\pi^3}{2c_0H^3}(A_{0X}C + A_0C_X)E = \frac{\pi}{8c_0H}(A_{0X}C - A_0C_X) + \frac{H}{c_0\pi}A_1\beta Y, \tag{87}$$

$$\frac{\pi^3}{2c_0H^3}(A_{0X}C + A_0C_X)D = \frac{2}{c_0\pi}A_1\beta Y - \frac{\pi^3}{4c_0H^2}(A_{0X}C + A_0C_X). \tag{88}$$

We expect nonlinearities to appear at second order, so proceed with our expansions. At second order, there is cancellation of the nonlinear terms such that Ψ_2 can be written

$$\begin{aligned}c_0\Psi_{2Xzz} + \frac{N^2}{c_0}\Psi_{2X} \quad = \quad & \left(2c_2\frac{\pi^2}{H^2}A_{0X} - c_0A_{0XXX}\right)\sin\left(\frac{\pi}{H}z\right) + \frac{\pi^2}{H^2}\left(B_Y\sin\left(\frac{\pi}{H}z\right) + C_Y\cos\left(\frac{\pi}{H}z\right)\right) \\ & -\frac{\pi^3}{H^3}\left(-\hat{\beta}YB\cos\left(\frac{\pi}{H}z\right) + \hat{\beta}YC\sin\left(\frac{\pi}{H}z\right)\right).\end{aligned} \tag{89}$$

On applying the solvability condition $\langle \Psi_{0X}, RHS\rangle$ to Eq. (89) we find that

$$2c_2\frac{\pi^2}{H^2}A_{0X} - c_0A_{0XXX} + \frac{\pi^2}{H^2}B_Y - \frac{\pi^3}{H^3}\hat{\beta}YC = 0. \tag{90}$$

Eq. (81) defines C_X and B_X in terms of A_0. This is substituted into the above equation, and, after differentiating with respect to X, the following *linear* equation for the amplitude A_0 is obtained

$$2c_2\frac{\pi^2}{H^2}A_{0XX} - c_0A_{0XXXX} - \frac{1}{c_0}A_{0YY} + \frac{\pi^2}{c_0H^2}\hat{\beta}^2Y^2A_0 = 0. \tag{91}$$

By assuming A_0 is separable, i.e. $A_0(X,Y) = F(X)G(Y)$, we derive equations for $F(X)$ and $G(Y)$ that involve a separation constant μ. In a similar manner to that presented in preceding sections, the equation for $G(Y)$ can be solved in terms of equatorially-constrained parabolic cylinder functions only when the separation constant is defined as $\mu = \mu_n = \frac{\pi}{H}\hat{\beta}(2n+1)$, for the positive integer n. The equation for $F(X)$ is the linear Swift-Hohenberg equation. We assume that the solution $F(X)$ is periodic in X, such that

$$2c_2\frac{\pi^2}{H^2}\hat{k}^2 + c_0\hat{k}^4 + \frac{\mu_n}{c_0} = 0, \tag{92}$$

which is the analog of the linear dispersion relation from the preceding section 4, this time solving for the phase speed c_2 in terms of the zonal wavenumber k. This result matches that of the linear reduced theory, Eq. (67), when $\mu_n = \frac{\pi}{H}\hat{\beta}(2n+1)$. We proceed to solve Eq. (89) for Ψ_2. From the solvability condition $\langle \Psi_{0X}, RHS\rangle$ applied to Eq. (89), terms involving $\sin\left(\frac{\pi}{H}z\right)$ disappear due to orthogonality, and we obtain the following equation for Ψ_2

$$\Psi_{2Xzz} + \frac{N^2}{c_0^2}\Psi_{2X} = \frac{\pi^2}{c_0H^2}\left(C_Y + \hat{\beta}YB\frac{\pi}{H}\right)\cos\left(\frac{\pi}{H}z\right). \tag{93}$$

Once again taking into account the boundary conditions for Ψ on $z = 0, H$, the solution to Eq. (93) is $\Psi_2 = Wz\sin\left(\frac{\pi}{H}z\right)$, provided that on substitution into Eq. (93) the following relation between W and A holds

$$W_{XX} = -\frac{\hat{\beta}}{2c_0^2}(A_0 + 2YA_{0Y}). \tag{94}$$

Here we have also made use of Eq. (81). We were expecting that the second order solvability condition would introduce nonlinearity in our expression for the amplitude A_0. Instead, due to the scalings chosen, the nonlinearities at this order cancelled leaving the linear Swift-Hohenberg equation once the Y-dependent part had been accounted for by the appropriate parabolic cylinder function. It is therefore necessary to proceed to third order in ϵ to retrieve the nonlinear adjustment to the amplitude equation.

At third order, we find the following equation for Ψ

$$\begin{aligned}\frac{N^2}{c_0}\Psi_{3X} + c_0\Psi_{3Xzz} \quad = \quad & \left(2c_3\frac{\pi^2}{H^2}A_{0X} - c_0A_{1XXX} + 2c_2\frac{\pi^2}{H^2}A_{1X}\right)\sin\left(\frac{\pi}{H}z\right) \\ & - (\partial_Y + \hat{\beta}Y\partial_z)\Phi_{1zz} + \frac{\pi}{2H}(A_0A_{0XXX} - A_{0X}A_{0XX})\sin\left(\frac{2\pi}{H}z\right) \\ & + \frac{\pi^2}{H^2}\left[A_0W_X\left(1+\cos\left(\frac{2\pi}{H}z\right)\right) + A_{0X}W\left(1-\cos\left(\frac{2\pi}{H}z\right)\right)\right].\end{aligned}$$

The solvability condition yields

$$\begin{aligned}0 \quad = \quad & 2c_3\frac{\pi^2}{H^2}A_{0XX} - c_0A_{1XXXX} + 2c_2\frac{\pi^2}{H^2}A_{1XX} \\ & + \frac{2\pi^2}{c_0H^3}(A_{0X}C + A_0C_X)_Y + \frac{2\pi^2}{3c_0H^3}(A_{0X}C - A_0C_X)_Y - \frac{1}{c_0}A_{1YY} \\ & + \hat{\beta}Y\frac{4\pi^3}{3c_0H^4}(A_{0X}B - A_0B_X) + \frac{\pi^2}{c_0H^2}\hat{\beta}^2Y^2A_1 \\ & + \frac{8\epsilon\pi}{3H^2}(A_0W_X + 2A_{0X}W)_X.\end{aligned} \tag{95}$$

Writing A_0 and A_1 by $\tilde{A} = A_0 + \epsilon A_1 + ...$, and hence B and C from Eq. (81) in terms of $\tilde{A}$, we employ the transformation $\tilde{A} = \Theta_{XX}$ and combine Eqs. (91) and (95) to obtain the following equation for Θ

$$\begin{aligned}0 \;=\;& 2(c_2+\epsilon c_3)\frac{\pi^2}{H^2}\Theta_{XXXX} - c_0\Theta_{XXXXXX} - \frac{1}{c_0}\Theta_{XXYY} + \frac{\pi^2}{c_0H^2}\hat{\beta}^2Y^2\Theta_{XX}\\ &- \frac{4\epsilon\pi\hat{\beta}}{3c_0^2H^2}(2Y\Theta_{XXX}\Theta_X + Y\Theta_{XX}^2)_Y\\ &- \frac{4\epsilon\pi\hat{\beta}}{3c_0^2H^2}(Y\Theta_{XXX}\Theta_{XY} - Y\Theta_{XX}\Theta_{XXY})\\ &- \frac{4\epsilon\pi\hat{\beta}}{3c_0^2H^2}(\Theta_{XX}[\Theta_X + 2Y\Theta_{XY}] + 2\Theta_{XXX}[\Theta + 2Y\Theta_Y])_X. \qquad (96)\end{aligned}$$

Here the constant c_3 is the nonlinear correction to the phase speed. The constants c_0 and c_2 are defined as

$$c_0 = \frac{NH}{\pi}, \qquad c_2 = -\frac{NH^3}{2\pi^3}\hat{k}^2 - \frac{\hat{\beta}(2n+1)}{2N\hat{k}^2}.$$

Note the appearance of the small parameter ϵ in Eq. (96), which implies that the nonlinear terms are small compared to the linear terms from the leading order expansion.

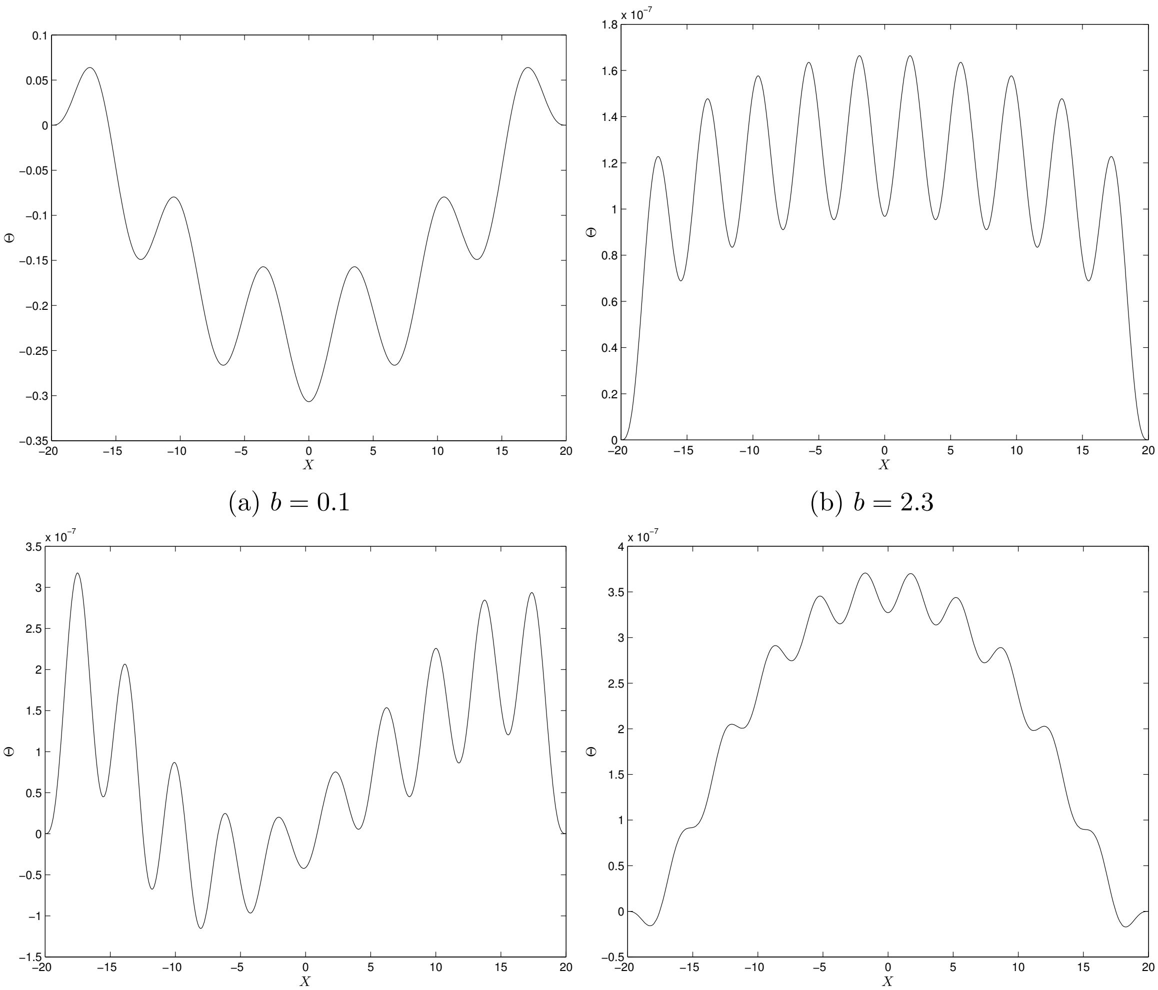
Suppose Θ is separable in X and Y, namely $\Theta(X,Y) = F(X)G(Y)$. Then, from the homogeneous part of Eq. (96) (the first line), we find equations for $F(X)$ and $G(Y)$ that involve a separation constant γ. As previously, provided that $\gamma = \gamma_n = (2n+1)\pi\beta/H$, an equatorially-constrained solution for $G(Y)$ in terms of parabolic cylinder functions exists, and is given by

$$G(Y) = G_n(Y) = 2^{-n/2}\mathrm{e}^{-\pi\hat{\beta}Y^2/2H}H_n\left(\sqrt{\frac{\pi\hat{\beta}}{H}}Y\right), \quad \text{for } n = 0,1,2,..., \qquad (97)$$

where H_n is the Hermite polynomial. We wish to remove the Y-dependence of Eq. (96) to investigate the effect of nonlinearities on the zonal wave structure. Consider the simplest choice of function for G_n, that is, $G_n = G_0 = \mathrm{e}^{-\pi\hat{\beta}y^2/2H}$. Multiplying each term in Eq. (96) by G_0 and integrating with respect to Y from $-\infty$ to ∞ yields

$$\theta_{\eta\eta\eta\eta\eta\eta} + b\theta_{\eta\eta\eta\eta} - \theta_{\eta\eta} + 4\theta_\eta\theta_{\eta\eta\eta} + 3\theta_{\eta\eta}^2 + 2\theta_{\eta\eta\eta\eta}\theta = 0, \qquad (98)$$

which is a new equation that is similar to the conserved Swift-Hohenberg (SH) equation [16; 25]. Here, $\theta(\eta)$ and η are the rescaled $F(X)$ and X and $b = -2(c_2+\epsilon c_3)\frac{\pi^{3/2}\alpha}{H^2}$ is a positive constant. Eq. (98) has the symmetry $\eta \to -\eta$, $\theta \to \theta$. We are particularly interested in finding localized solutions to Eq. (98), which have been found in the conserved SH equation (e.g. [20]). Some solutions to Eq. (98) obtained using Neumann boundary conditions are illustrated in figure 6. It is important to note that these are not localized solutions as they depend on the boundary conditions even when the domain becomes large. A future work will investigate possible localized solutions to Eq. (98).



(a) $b = 0.1$ (b) $b = 2.3$

(c) $b = 2.5$ (d) $b = 2.7$

Figure 6: Some solutions to the amplitude equation, Eq. (98), for different values of b.

If we were to choose a parabolic cylinder function that was even with respect to the equator, i.e G_n for $n = 1, 3, 5, ...$, rather than an odd mode, such as the G_0 chosen above, the nonlinearities would cancel. As in [2], this does not mean that the nonlinear terms are identically zero; rather, choosing a parabolic cylinder function that is even with respect to the equator forces a symmetry that leads to cancellation of the nonlinear terms. It is likely that an alternative scaling for the reduced system exists in which the even modes are retained.

So far we have assumed that the Brunt-Väisälä frequency N is constant and have introduced the scalings such that Eq. (98) applies only in the small β limit. In the following section we return to Eqs. (48) - (50), this time assuming that that the Brunt-Väisälä frequency varies with height, which allows us to derive a nonlinear equation for the amplitude of the streamfunction that does not require the imposition of a small β limit.

5.2 Non-constant Brunt-Väisälä frequency

Consider Eqs. (48) - (50) once more, again transforming to the moving frame $\xi = x - ct$ and assuming long waves in the zonal direction $X = \epsilon\xi$. This time we introduce the following scalings

$$\Psi = \epsilon^2\Psi, \qquad \Phi = \epsilon^3\Phi, \qquad b = \epsilon^2 b, \tag{99}$$

where $y = \beta^{-1/2}Y$ and $\beta^{1/2} = \hat{\beta}^{1/2}\epsilon^2$. The physical interpretation of these scalings is that the meridional current is small compared with the zonal and vertical equatorial currents. Eqs. (48) - (50) become

$$\begin{aligned} -\partial_X b &= -c(\epsilon^2\partial_X^2 + \partial_z^2)\partial_X\Psi - \epsilon^2\hat{\beta}^{1/2}(\partial_Y + Y\partial_z)(\epsilon^2\partial_X^2 + \partial_z^2)\Phi \\ &\quad + \epsilon^2\partial_z\Psi(\epsilon^2\partial_X^2 + \partial_z^2)\partial_X\Psi - \epsilon^2\partial_X\Psi(\epsilon^2\partial_X^2 + \partial_z^2)\partial_z\Psi, && (100)\\ 0 &= -c(\epsilon^2\partial_X^2 + \partial_z^2)\partial_X\Phi + \hat{\beta}^{1/2}(\partial_Y + Y\partial_z)\Psi \\ &\quad + \epsilon^2\partial_z\Psi(\epsilon^2\partial_X^2 + \partial_z^2)\partial_X\Phi - \epsilon^2\partial_X\Psi(\epsilon^2\partial_X^2 + \partial_z^2)\partial_z\Phi, && (101)\\ 0 &= -c\partial_X b + \epsilon^2\partial_z\Psi\partial_X b - \epsilon^2\partial_X\Psi\partial_z b - \partial_X\Psi N^2. && (102) \end{aligned}$$

We assume the following expansions about the small parameter ϵ

$$\Psi = \Psi_0 + \epsilon\Psi_1 + ..., \tag{103}$$

$$\Phi = \Phi_0 + \epsilon\Phi_1 + ..., \tag{104}$$

$$b = b_0 + \epsilon b_1 + ..., \tag{105}$$

$$c = c_0 + \epsilon c_1 + ..., \tag{106}$$

and obtain the leading order set in terms of Ψ and Φ

$$\frac{N^2}{c_0}\Psi_{0X} + c_0\Psi_{0Xzz} = 0, \tag{107}$$

$$-c_0\Phi_{0Xzz} + \hat{\beta}^{1/2}(\partial_Y + Y\partial_z)\Psi_0 = 0. \tag{108}$$

Suppose that N is a function of z, which takes the form $N(z) = N_0\mathrm{e}^{-z}$, for some constant N_0. Assume also that $\Psi_0(X, Y, z)$ is separable, writing $\Psi_0 = A_0(X, Y)J(z)$, and let $z = -\ln t$.

Then, the solution to J is simply the zeroth mode Bessel function of the first kind, J_0, defined by the equation

$$\frac{1}{t}\frac{d}{dt}\left(t\frac{dJ_0}{dt}\right) = -\frac{N_0^2}{c_0^2}J_0. \tag{109}$$

We proceed to second order in ϵ and obtain the following set of equations

$$\begin{aligned}
-b_{2X} + c_0\Psi_{2Xzz} &= -c_0\Psi_{0XXX} - c_2\Psi_{0Xzz} - \hat{\beta}^{1/2}(\partial_Y + Y\partial_z)\Phi_{0zz} \\
&\quad + \Psi_{0z}\Psi_{0Xzz} - \Psi_{0X}\Psi_{0zzz}, \qquad (110)\\
c_0\Phi_{2Xzz} - \hat{\beta}^{1/2}(\partial_Y + Y\partial_z)\Psi_2 &= -c_0\Phi_{0XXX} - c_2\Phi_{0Xzz} + \Psi_{0z}\Phi_{0Xzz} - \Psi_{0X}\Phi_{0zzz}, \qquad (111)\\
-b_{2X} &= \frac{N^2}{c_0}\Psi_{2X} + \frac{c_2}{c_0}b_{0X} - \frac{1}{c_0}\Psi_{0z}b_{0X} + \frac{1}{c_0}\Psi_{0X}b_{0z}. \qquad (112)
\end{aligned}$$

Substituting Eq. (112) into Eq. (110) and using Eq. (108) we obtain the solvability condition

$$\begin{aligned}
0 &= -c_0^2B_{0XXXX}\int_0^\infty J_0^2(z)dz + \frac{2c_2N_0^2}{c_0}B_{0XX}\int_0^\infty \mathrm{e}^{-2z}J_0^2(z)dz - \frac{4N_0^2}{c_0}B_{0XX}B_{0X}\int_0^\infty \mathrm{e}^{-2z}J_0^3(z)dz \\
&\quad - \hat{\beta}Y^2B_0\int_0^\infty \frac{d^2J_0(z)}{dz^2}J_0(z)dz - \hat{\beta}(1+2Y\partial_Y)B_0\int_0^\infty \frac{dJ_0(z)}{dz}J_0(z)dz \\
&\quad - \hat{\beta}B_{0YY}\int_0^\infty J_0^2(z)dz, \qquad (113)
\end{aligned}$$

where $B_0 = \int A_0 dX$ (note that the operator $[N^2 + c_0^2\partial_z^2]$ is self-adjoint and we assume that $B_{0X} \neq 0$). Again using the transformation $z = -\ln t$, this becomes

$$\begin{aligned}
0 &= -c_0^2B_{0XXXX}\int_0^1 \frac{1}{t}J_0^2(\ln t)dt + \frac{2c_2N_0^2}{c_0}B_{0XX}\int_0^1 tJ_0^2(\ln t)dt - \frac{4N_0^2}{c_0}B_{0XX}B_{0X}\int_0^1 tJ_0^4(\ln t)dt \\
&\quad - \frac{\hat{\beta}}{2}Y^2B_0\int_0^1 \frac{1}{t}\left[J_0(\ln t)J_2(\ln t) - J_0(\ln t)^2\right]dt - \hat{\beta}(1+2Y\partial_Y)B_0\int_0^1 \frac{1}{t}J_0(\ln t)J_1(\ln t)dt \\
&\quad - \hat{\beta}B_{0YY}\int_0^1 \frac{1}{t}J_0^2(\ln t)dt. \qquad (114)
\end{aligned}$$

In this form the integrals involving the factor $1/t$ are infinite in the interval $t = [0, 1]$. Hence, we restrict the interval of integration further such that we integrate from $t = b$ to $t = 1$, where $b = \mathrm{e}^{-j_{0,1}}$ and $j_{0,1}$ is the first root of the Bessel function of the first kind $J_0(z)$. This is equivalent to integrating from $z = 0$ to $z = j_{0,1}$. Proceeding in this manner, we obtain

$$\begin{aligned}
0 &= -c_0^2B_{0XXXX}\ {}_2F_3\left(\frac{1}{2},\frac{1}{2};1,1,\frac{3}{2};-(j_{0,1})^2\right)j_{0,1} + \frac{2c_2N_0^2r}{c_0}B_{0XX} - \frac{4N_0^2s}{c_0}B_{0XX}B_{0X} \\
&\quad + \frac{\hat{\beta}}{2}Y^2B_0\left[{}_2F_3\left(\frac{1}{2},\frac{1}{2};1,1,\frac{3}{2};-(j_{0,1})^2\right)j_{0,1} - \frac{1}{6}{}_2F_3\left(\frac{3}{2},\frac{3}{2};1,\frac{5}{2},3;-(j_{0,1})^2\right)(j_{0,1})^3\right] \\
&\quad - \hat{\beta}B_{0YY}\ {}_2F_3\left(\frac{1}{2},\frac{1}{2};1,1,\frac{3}{2};-(j_{0,1})^2\right)j_{0,1} + \frac{\hat{\beta}}{2}(1+2Y\partial_Y)B_0. \qquad (115)
\end{aligned}$$

Here, ${}_2F_3$ is the generalized hypergeometric function and $r \approx 0.4171$, $s \approx 0.3756$. We write Eq. (115) more compactly as

$$-a_1B_{0XXXX} + a_2B_{0XX} - a_3B_{0XX}B_{0X} + a_4Y^2B_0 - a_5B_{0YY} + a_6(1+2Y\partial_Y)B_0 = 0, \tag{116}$$

where the constants a_1 to a_6 are positive. Then, by introducing the scalings $S = f_1 B_0$, $\xi = f_2 X$ and $\eta = f_3 Y$, we obtain the simpler equation

$$S_{\xi\xi\xi\xi} - S_{\xi\xi} + S_{\xi\xi}S_{\xi} + \alpha_1 S_{\eta\eta} - (\eta^2 + \alpha_2 + 2\alpha_2\eta\partial_\eta)S = 0. \tag{117}$$

The constants $\alpha_1 = a_5 f_2^4/a_1 f_3^2$ and $\alpha_2 = a_6 f_2^4/a_1$ are functions of $\hat{\beta}$, N_0 and the phase speeds c_0 and c_2. Eq. (117) is a generalization of the Kadomtsev-Petviashvili (KP) equation [14], with the additional terms $2\eta S_\eta$, $\eta^2 S$ and $S_{\eta\eta}$. As with the SH equation, the KP equation permits localized solutions (e.g. [15; 18]). It is possible to remove the η dependence in Eq. (117) such that it is an ODE in ξ only. We assume that S is separable, e.g. $S(\xi, \eta) = E(\xi)D(\eta)$, where $E(\xi)$ and $D(\eta)$ are related by a separation constant γ, and find that the solution to $D(\eta)$ from the homogeneous part of Eq. (117) (i.e. ignoring the nonlinear term) can be expressed in terms of parabolic cylinder functions. When $\gamma = \gamma_n = (2n+1)\alpha_2$ for the nonnegative integer n, then D is equatorially constrained. Choosing the lowest order mode, D_0, we multiply Eq. (117) by D_0 and integrate from $-\infty$ to ∞ to remove the η dependence. Finally we rescale $E(\xi)$ to obtain the following equation

$$E_{\xi\xi\xi\xi} - E_{\xi\xi} + E_{\xi\xi}E_{\xi} + \kappa E = 0. \tag{118}$$

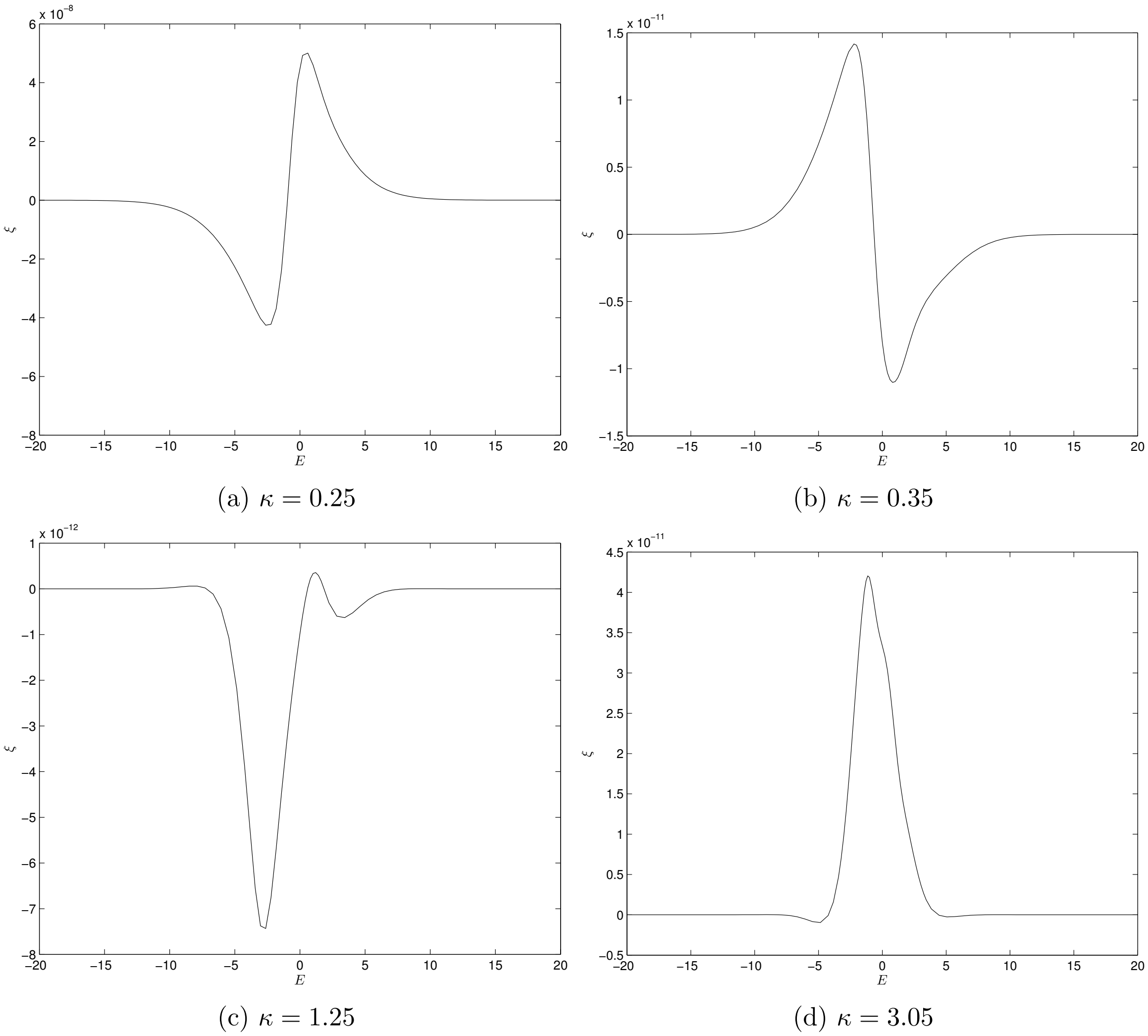
Like Eq. (98), Eq. (118) contains only the one parameter $\kappa = \alpha_2(2-\sqrt{2}) + \alpha_1\alpha_2(\sqrt{2} - 1) - \sqrt{2}\alpha_1/\alpha_2(\alpha_1 - 1)$. Some solutions to Eq. (118) using Neumann boundary conditions are illustrated in figure 7. As with Eq. (98), these are not true localized solutions as they depend on the boundary conditions even for a large domain.

In this section we have undertaken a preliminary investigation of the nonlinear quasi-geostrophic model. Two alternative scalings have been presented that result in two different amplitude equations when the stratification is assumed to be constant and a function of height, respectively. The benefit of the latter scaling is that it does not restrict the resulting equation to the small β limit. As for Eq. (98), a future work will further investigate localized solutions to Eq. (118).

6 Conclusion

The aim of this work was to further that of Boyd in understanding the dynamics and evolution of nonlinear equatorial waves in the context of a reduced, quasi-geostrophic model. The model was derived based on the assumption that the Rossby number is small at the equator, which is perfectly valid when both the vertical and horizontal components of the Earth's rotation are taken into account. Non-hydrostatic effects and the vertical component of momentum were also included in the model. The derivation assumed that the meridional length scale is large compared to the zonal and vertical length scales, such that the quasi-geostrophic balance in this model was between the u and w components of the momentum.

In the first section, the results from the non-hydrostatic, linearized primitive equations were contrasted with those of the shallow water equations. The horizontal component of the Earth's rotation added curvature to the y, z phase planes and introduced an imaginary component into the solutions. Furthermore, non-hydrostatic effects modified the dispersion relation for the inertio-gravity, Kelvin and mixed Rossby-gravity modes, such that their



(a) $\kappa = 0.25$ (b) $\kappa = 0.35$

(c) $\kappa = 1.25$ (d) $\kappa = 3.05$

Figure 7: Some solutions to the amplitude equation, Eq. (118), for different values of κ.

frequencies were necessarily less than or greater than the Brunt-Väisälä frequency in the case of vertical traveling and standing waves, respectively.

In contrast with the linearized primitive equations, when a vertical traveling wave solution was assumed in the linearized quasi-geostrophic model, the system was not equatorially-constrained. Rather, a vertical standing wave with rigid lid boundary conditions was required to produce an equatorially-constrained wave. This result held in the small β limit, under which the even modes with respect to the equator were filtered out. It is likely that an alternative scaling of the linearized quasi-geostrophic model that includes the even modes can be found.

Multiple scalings of the nonlinear quasi-geostrophic model were undertaken. The first involved a small β limit, and under the assumption of constant stratification, equatorial waves behaved according to a generalized Swift-Hohenberg equation. Under an alternative scaling that did not require β to be small, and with the assumption that the stratification depended on height, equatorial waves behaved according to a generalized Kadomtsev-Petviashvili equation. Solitary wave solutions to the conserved Swift-Hohenberg equation and the Kadomtsev-Petviashvili equation have been found in previous studies and are a motivation for further work on the equations presented here.

This work is a first step in describing the effects of nonlinearity on equatorial waves in a context apart from the shallow water equations. It is acknowledged that many questions remain to be addressed with respect to the quasi-geostrophic model: for example, what rescaling will yield the even modes, and what effect does changing the vertical dependence of the Brunt-Väisälä frequency have? One clear benefit of this work on the quasi-geostrophic model is that it hints at appropriate methods for analyzing the nonlinear primitive equations, which will be investigated in a subsequent work.

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References

[1] R. Blandford, *Mixed gravity-rossby waves in the ocean*, Deep-Sea Research, 13 (1966), pp. 941–961.

[2] J. P. Boyd, *Equatorial solitary waves. Part I: Rossby solitons*, Journal of Physical Oceanography, 10 (1980), pp. 1699–1717.

[3] ——, *The nonlinear equatorial Kelvin wave*, Journal of Physical Oceanography, 10 (1980), pp. 1–11.

[4] ——, *Equatorial solitary waves. part 2: envelope solitons*, Journal of Physical Oceanography, 13 (1983), pp. 428–449.

[5] ——, *Equatorial solitary waves. part 3: westward-traveling modons*, Journal of Physical Oceanography, 15 (1985), pp. 46–54.

[6] J. G. Charney, *On the scale of atmospheric motions*, Geofys. Publ., 17 (1948), pp. 3–17.

[7] ——, *Geostrophic turbulence*, Journal of the Atmospheric Sciences, 28 (1971), pp. 1087–1095.

[8] A. E. Gill, *Atmosphere-Ocean Dynamics*, Academic Press, 1982.

[9] R. H. J. Grimshaw, *A note on the β-plane approximation*, Tellus, 27 (1975), pp. 351–356.

[10] M. Hayashi and H. Itoh, *The importance of the nontraditional Coriolis terms in large-scale motions in the tropics forced by prescribed cumulus heating*, Journal of the Atmospheric Sciences, 69 (2012), pp. 2699–2716.

[11] J. R. Holton, *An Introduction to Dynamic Meteorology*, Academic Press, New York, 1979.

[12] J. R. Holton, J. H. Beres, and X. Zhou, *On the vertical scale of gravity waves excited by localized thermal forcing*, Journal of the Atmospheric Sciences, 59 (2002).

[13] K. Julien, E. Knobloch, R. Milliff, and J. Werne, *Generalized quasi-geostrophy for spatially anisotropic rotationally constrained flows*, Journal of Fluid Mechanics, 555 (2006), pp. 233–274.

[14] B. B. Kadomtsev and V. I. Petviashvili, *On the stability of solitary waves in weakly dispersing media*, Soviet Physics - Doklady, 15 (1970), pp. 539–541.

[15] Y. Liu and X.-P. Wang, *Nonlinear stability of solitary waves of a generalized Kadomtsev-Petviashvili equation*, Communications in Mathematical Physics, 183 (1997), pp. 253–266.

[16] D. J. B. Lloyd, B. Sandstede, D. Avitabile, and A. R. Champneys, *Localized hexagon patterns of the planar Swift-Hohenberg equation*, SIAM Journal of Applied Dynamical Systems, 7 (2008), pp. 1049–1100.

[17] R. A. Madden and P. R. Julian, *Detection of a 40-50 day oscillation in the zonal wind in the tropical Pacific*, Journal of the Atmospheric Sciences, 28 (1971), pp. 702–708.

[18] S. V. Manakov, V. E. Zakharov, L. A. Bordag, A. R. Its, and V. B. Matveev, *Two-dimensional solitons of the Kadomtsev-Petviashvili equation and their interaction*, Physics Letters, 63A (1977), pp. 205–206.

[19] J. Matsuno, *Quasi-geostrophic motions in the equatorial area*, Journal of the Meteorological Society of Japan, 44 (1966), pp. 25–43.

[20] P. C. Matthews and S. M. Cox, *Pattern fomation with a conservation law*, Nonlinearity, 13 (2000), pp. 1293–1320.

[21] D. W. Moore, *Planetary-gravity waves in an equatorial ocean*, PhD thesis, Harvard University, 1968.

[22] ——, *Equatorial waves and the effect of the horizontal component of the earth's rotation.* June 1993.

[23] D. W. Moore and S. G. H. Philander, *Modeling of the tropical oceanic circulation*, in The Sea, E. Goldberg, ed., vol. 6, Wiley, 1977, ch. 8, pp. 319–347, 359–361.

[24] J. D. Neelin, D. S. Battisti, A. C. Hirst, F.-F. Jin, Y. Wakata, T. Yamagata, and S. E. Zebiak, *ENSO theory*, Journal of Geophysical Research, 103 (1998), pp. 14,261–14,290.

[25] J. B. Swift and P. C. Hohenberg, *Hydrodynamic fluctuations at the convective instability*, Physical Review A, 15 (1977), pp. 319–328.

[26] G. Veronis, *Comments on Phillips' proposed simplification of the equations of motion for a shallow rotating atmosphere*, Journal of the Atmospheric Sciences, 25 (1968), pp. 1154–1155.

[27] M. Wheeler and G. N. Kiladis, *Convectively coupled equatorial waves: analysis of clouds and temperature in the wavenumber-frequency domain*, Journal of the Atmospheric Sciences, 56 (1999), pp. 374–399.

[28] A. A. White and R. A. Bromley, *Dynamically consistent, quasi-hydrostatic equations for global models with a complete representation of the coriolis force*, Quarterly Journal of the Royal Meteorological Society, 121 (1995), pp. 399–418.

[29] C. Wunsch and A. E. Gill, *Observations of equatorially trapped waves in Pacific sea level variations*, Deep-Sea Research, 23 (1975), pp. 371–390.

[30] C. Zhang, *Madden-Julian oscillation*, in Reviews of Geophysics, vol. 43, American Geophysical Union, 2005, pp. 1–36.