Lecture 10

Unification of Variational Principles for Turbulent Shear Flows: the Mean-fluctuation Formulation of Howard-Busse and the Background Method of Doering-Constantin

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1 Introduction

An upper bound on the energy dissipation rate in turbulent shear flows can be found either using Howard-Busse's mean-fluctuation method [1] or Doering-Constantin's background method[2]. Howard-Busse's method grew out of ideas put forward by Malkus [3], whereas the Doering-Constantin approach is based upon a mathematical device invented by Hopf [4]. Although the methods have very different origins and look unrelated, we show in this lecture that they are in fact intimately connected. They both seek to make stationary the same functional. However, the Howard-Busse method seeks to estimate this stationary (saddle) point from below as a maximization problem, whereas the Doering-Constantin method estimates this part from above as part of a minimization problem. We show this explicitly for the canonical problem of plane Couette flow.

2 Couette Shear Flow

We consider a homogeneous incompressible fluid with viscosity χ between two parallel, infinite plates at $z = \pm\frac{1}{2}d$, which are sliding across each other with relative velocity V_0 in the $\underline{\hat{i}}$ direction. $\underline{\hat{i}}$ is the unit vector. The non-dimensionalized governing equations are:

$$\frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \underline{\nabla V} + \underline{\nabla} p = \nabla^2 \underline{V} \tag{1}$$

$$\underline{\nabla} \cdot \underline{V} = 0$$

with the boundary condition $\underline{V} = \mp\frac{1}{2}Re\underline{\hat{i}}$ at $z = \pm\frac{1}{2}$, where Reynolds number $Re = \frac{V_0 d}{\chi}$. We will seek upper bounds on the momentum transport which equals the viscous dissipation rate $\left\langle |\nabla \underline{V}|^2 \right\rangle$. Where
$\left\langle |\nabla \underline{V}|^2 \right\rangle := \lim_{L\to\infty} \frac{1}{4L^2} \int_{-L}^{L} dx \int_{-L}^{L} dy \int_{-\frac{1}{2}}^{\frac{1}{2}} dz \, |\nabla \underline{V}|^2$.

2.1 Howard-Busse Method

The Howard-Busse variational formulation is based on a mean-fluctuation decomposition of the velocity field $\underline{V}(\underline{x}, t) = U(z)\,\hat{i} + \underline{v}(\underline{x}, t)$, and consists of solving the variational problem:

$$\min(R_e) = \frac{\left\langle |\nabla \underline{v}|^2 \right\rangle}{\langle v_1 v_3 \rangle} + \frac{\mu \left\langle \left(\overline{v_1 v_3} - \langle v_1 v_3 \rangle\right)^2 \right\rangle}{\langle v_1 v_3 \rangle^2}$$

under the constraints $\nabla\cdot\underline{\upsilon}=0, \underline{\upsilon}(x,y,\pm\frac{1}{2})=0, \langle\upsilon_1\upsilon_3\rangle=1$. where $\overline{(\bullet)} := \lim_{L\to\infty}\frac{1}{4L^2}\int_{-L}^{L}dx\int_{-L}^{L}dy\,(\bullet)$. This problem can be eqivalently formulated as the following. Substitute $\underline{V}(\underline{x},t)$ into equation (1) and subtract the horizontal average, we obtain the power balance:

$$D-Re^2=\left\langle|\nabla\underline{\upsilon}|^2\right\rangle+\left\langle\left(\overline{\upsilon_1\upsilon_3}-\langle\upsilon_1\upsilon_3\rangle\right)^2\right\rangle=Re\,\langle\upsilon_1\upsilon_3\rangle \tag{2}$$

where D is the statistically averaged viscous dissipation. We maximize $Re\,\langle\upsilon_1\upsilon_3\rangle$ under the constraints of (2), the continuity equation, and the boundary conditions by considering the Lagrangian

$$L=Re\,\langle\upsilon_1\upsilon_3\rangle+\Lambda\left\{\left\langle|\nabla\underline{\upsilon}|^2\right\rangle+\left\langle\left(\overline{\upsilon_1\upsilon_3}-\langle\upsilon_1\upsilon_3\rangle\right)^2\right\rangle-Re\,\langle\upsilon_1\upsilon_3\rangle\right\}-\langle 2p\,(\underline{x})\,\nabla\cdot\underline{\upsilon}\rangle$$

where Λ and $p\,(\underline{x})$ are Lagrange multipliers. The Euler-Lagrange equation for the velocity field is

$$\left[\left(\overline{\upsilon_1\upsilon_3}-\langle\upsilon_1\upsilon_3\rangle\right)+Re\left(\frac{1-\Lambda}{2\Lambda}\right)\right]\begin{bmatrix}\upsilon_3\\0\\\upsilon_1\end{bmatrix}+\nabla p=\nabla^2\underline{\upsilon} \tag{3}$$

Eliminating Λ by using $\langle\underline{\upsilon}\cdot(3)\rangle$ and the constraint of equation (2), gives the optimization problem:

$$\left[\left(\overline{\upsilon_1\upsilon_3}-\langle\upsilon_1\upsilon_3\rangle\right)-\tfrac{1}{2}Re-\tfrac{1}{2}\frac{\left\langle\left(\overline{\upsilon_1\upsilon_3}-\langle\upsilon_1\upsilon_3\rangle\right)^2\right\rangle}{\langle\upsilon_1\upsilon_3\rangle}\right]\begin{bmatrix}\upsilon_3\\0\\\upsilon_1\end{bmatrix}+\nabla p=\nabla^2\underline{\upsilon}$$
$$\nabla\cdot\underline{\upsilon}=0,\ \underline{\upsilon}\left(x,y,\pm\tfrac{1}{2}\right)=0$$

from which the upper bound $D=Re\,\langle\upsilon_1\upsilon_3\rangle+Re^2$ follows.

2.2 Doering-Constantin Method

The Doering-Constantin method decomposes the velocity into "background" and "fluctuation" fields $\underline{V}\,(\underline{x},t)=\phi\,(z)\,\hat{\underline{i}}+\underline{\nu}\,(\underline{x},t)$. The background flow $\phi\,(z)$ satisfies the boundary condition $\phi\left(\pm\frac{1}{2}\right)=\mp\frac{1}{2}Re$ so that the fluctuation field satisfies homogeneous boundary conditions.

Putting $\underline{V}\,(\underline{x},t)=\phi\,(z)\,\hat{\underline{i}}+\underline{\nu}\,(\underline{x},t)$ into $\partial\underline{V}\big/\partial t+\underline{V}\cdot\underline{\nabla V}+\underline{\nabla p}=\nabla^2\underline{V}$, we obtain

$$\frac{\partial\underline{\nu}}{\partial t}+\underline{\nu}\cdot\underline{\nabla\nu}+\phi'\nu_3\hat{\underline{i}}+\phi\frac{\partial\underline{\nu}}{\partial x}+\underline{\nabla p}=\nabla^2\underline{\nu}+\phi''\hat{\underline{i}}$$

where $\phi':=d\phi\big/dz$ and $\phi'':=d^2\phi\big/dz^2$. Performing $\langle\underline{\nu}\cdot(4)\rangle$, we obtain

$$\frac{\partial}{\partial t}\left\langle\frac{1}{2}\underline{\nu}^2\right\rangle=\left\langle\phi''\nu_1\right\rangle-\left\langle|\nabla\underline{\nu}|^2\right\rangle-\left\langle\phi'\nu_1\nu_3\right\rangle\cdot \tag{4}$$

We also have the identity

$$\left\langle|\nabla\underline{V}|^2\right\rangle=\left\langle\phi'2\right\rangle-2\left\langle\phi''\nu_1\right\rangle+\left\langle|\nabla\underline{\nu}|^2\right\rangle \tag{5}$$

Performing $a\cdot(5)+(6)$ where a is some scalar gives

$$\left\langle |\nabla \underline{V}|^2 \right\rangle + a\frac{\partial}{\partial t}\left\langle \frac{1}{2}\underline{\nu}^2 \right\rangle = \left\langle \phi'^2 \right\rangle - G\left(\phi, \underline{\nu}; a\right)$$

where $G\left(\phi, \underline{\nu}; a\right) = \left\langle (a-1)\left|\nabla \underline{\nu}\right|^2 + a\phi' \nu_1 \nu_3 - (a-2)\phi'' \nu_1 \right\rangle$.
Taking long time averages leads to

$$D = \lim_{T\to\infty} \frac{1}{T}\int_0^T \left\langle |\nabla \underline{V}|^2 \right\rangle dt = \left\langle \phi'^2 \right\rangle - \lim_{T\to\infty} \frac{1}{T}\int_0^T G\left(\phi, \underline{\nu}; a\right) dt$$

If ϕ and a are such that $\inf_{\underline{\nu}} G\left(\phi, \underline{\nu}; a\right) > -\infty$ then there exists the bound

$$D \leqslant \left\langle \phi'^2 \right\rangle - \inf_{\underline{\nu}} G\left(\phi, \underline{\nu}; a\right).$$

The Doering-Constantin problem is to minimize the background dissipation $\left\langle \phi' 2 \right\rangle$ subject to the spectral constraint $\inf G > -\infty$ over all possible fluctuation fields $\underline{\nu}\left(\underline{x}\right)$. Solving the Euler-Lagrange equation gives the stationary value of

$$G\left(\phi, \underline{\nu}^*; a\right) = -\frac{(a-2)^2}{4(a-1)}\left[\left\langle \phi'^2 \right\rangle - Re^2\right].$$

This will be an infimum if and only if the dominant quadratic terms are positive definite, i.e.

$$H\left(\phi, \underline{\nu}, a\right) := (a-1)\left\langle |\nabla \underline{\nu}|^2 \right\rangle + a\left\langle \phi' \nu_1 \nu_3 \right\rangle \geq 0$$

for all allowable $\underline{\nu}$. This is called the spectral constraint.
The optimization problem is then to minimize the bound

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \left\langle |\nabla \underline{\nu}|^2 \right\rangle dt \leq \mathcal{D} := \left\langle \phi'^2 \right\rangle - \inf_{\underline{\nu}} G = \frac{a^2}{4\left(a-1\right)}\left\langle \left(\phi' + Re\right)^2 \right\rangle + Re^2$$

subject to the spectral constraint.

3 Unification

The Howard-Busse and Doering-Constantin methods can be unified by defining the following functional

$$D(\underline{V}, \underline{\nu}, a) := \lim_{T\to\infty} \frac{1}{T}\int_0^T \left\langle |\nabla \underline{V}|^2 \right\rangle - \left\langle a\underline{\nu} \cdot \left(\frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \underline{\nabla V} + \underline{\nabla} p - \nabla^2 \underline{V}\right)\right\rangle dt$$

where $\underline{V} = \phi\left(z\right)\hat{\underline{i}} + \underline{\nu}\left(\underline{x}, t\right) = \left[\phi\left(z\right) + \overline{\nu_1}\left(z\right)\right]\hat{\underline{i}} + \underline{\upsilon}(\underline{x}, t)$. Substitute $\underline{V} = \phi\left(z\right)\hat{\underline{i}} + \underline{\nu}\left(\underline{x}, t\right)$ into D, we obtain

$$
\begin{aligned}
D\left(\underline{V},\phi,a\right) &= \lim_{T\to\infty}\tfrac{1}{T}\int_0^T\left\langle\left|\nabla\underline{V}\right|^2\right\rangle-\left\langle a\left(\underline{V}-\phi\left(z\right)\hat{\underline{i}}\right)\cdot\left(\tfrac{\partial\underline{V}}{\partial t}+\underline{V}\cdot\underline{\nabla V}+\underline{\nabla}p-\nabla^2\underline{V}\right)\right\rangle dt\\
&= \lim_{T\to\infty}\tfrac{1}{T}\int_0^T\left\langle\left|\nabla\underline{V}\right|^2\right\rangle-a\left\langle\underline{V}\cdot\left(\tfrac{\partial\underline{V}}{\partial t}+\underline{V}\cdot\underline{\nabla V}+\underline{\nabla}p-\nabla^2\underline{V}\right)\right\rangle\\
&\quad +a\left\langle\phi\left(z\right)\hat{\underline{i}}\cdot\left(\tfrac{\partial\underline{V}}{\partial t}+\underline{V}\cdot\underline{\nabla V}+\underline{\nabla}p-\nabla^2\underline{V}\right)\right\rangle dt
\end{aligned}
$$

This makes it clear that a is acting as a Lagrange multiplier which imposes the total power balance and that $a\phi\left(z\right)$ is the Lagrange multiplier which imposes the mean momentum balance. Also

$$
D\left(\underline{\nu},\phi,a\right)=\left\langle\phi^{'2}\right\rangle-\lim_{T\to\infty}\frac{1}{T}\int_0^T\left\langle\left(a-1\right)\left|\nabla\underline{\nu}\right|^2+a\nu_1\nu_3\phi^{'}-\left(a-2\right)\phi^{''}\nu_1\right\rangle dt
$$

and finally

$$
D\left(\underline{\upsilon},\overline{\nu}_1,\phi,a\right)=\left\langle\phi^{'2}\right\rangle-\lim_{T\to\infty}\frac{1}{T}\int_0^T\left\langle\left(a-1\right)\left|\nabla\underline{\upsilon}\right|^2+a\upsilon_1\upsilon_3\phi^{'}+(a-1)\bar{\nu}_1^{'2}-(a-2)\phi^{''}\overline{\nu}_1\right\rangle dt.
$$

The full variational problem is to solve the Euler-Lagrange equations $\frac{\delta D}{\delta\phi}=0$; $\frac{\delta D}{\delta\underline{\upsilon}}=\underline{0}$; $\frac{\delta D}{\delta\bar{\nu}_1}=\underline{0}$; $\frac{\delta D}{\delta a}=0$. The Howard-Busse and Doering-Constantin methods consider complementary subsets of these equations.

3.1 Howard-Busse Problem

The Howard-Busse method solves $\frac{\delta D}{\delta\phi}=0$, $\frac{\delta D}{\delta\bar{\nu}_1}=0$ and $\frac{\delta D}{\delta a}=0$ leaving a maximization problem in $\underline{\upsilon}$.

$$
\begin{aligned}
\tfrac{\delta D}{\delta\phi}=0 \qquad & \Rightarrow -2\phi^{''}+a\overline{\upsilon_1\upsilon_3}^{'}+\left(a-2\right)\bar{\nu}_1^{''}=0\\
& \Rightarrow \phi^{'}+Re=\tfrac{1}{2}a\left(\overline{\upsilon_1\upsilon_3}-\left\langle\upsilon_1\upsilon_3\right\rangle\right)+\tfrac{1}{2}\left(a-2\right)\overline{\nu}_1^{'}
\end{aligned}
$$

$$
\frac{\delta D}{\delta\overline{\nu}_1}=0\Rightarrow 2\left(a-1\right)\overline{\nu}_1^{''}+\left(a-2\right)\phi^{''}=0\Rightarrow\overline{\nu}_1=-\frac{\left(a-2\right)}{2\left(a-1\right)}\left(\phi+Rez\right)
$$

Substituting these results into $D\left(\underline{\upsilon},\bar{\nu}_1,\phi,a\right)$, we obtain

$$
D\left(\underline{\upsilon},a\right)=Re^2+Re\left\langle\upsilon_1\upsilon_3\right\rangle+\left(a-1\right)\left\{Re\left\langle\upsilon_1\upsilon_3\right\rangle-\left\langle\left|\nabla\underline{\upsilon}\right|^2\right\rangle-\left\langle\left(\overline{\upsilon_1\upsilon_3}-\left\langle\upsilon_1\upsilon_3\right\rangle\right)^2\right\rangle\right\}
$$

This is equivalent to the problem of finding the maximum of $\left(Re^2+Re\left\langle\upsilon_1\upsilon_3\right\rangle\right)$subject to the power constraint $Re\left\langle uw\right\rangle=\left\langle\left|\nabla\underline{\upsilon}\right|^2\right\rangle+\left\langle\left(\overline{\upsilon_1\upsilon_3}-\left\langle\upsilon_1\upsilon_3\right\rangle\right)^2\right\rangle$ with $\left(a-1\right)$ being the Lagrange multiplier and $\nabla\cdot\underline{\upsilon}=0$.

3.2 Doering-Constantin problem

The Doering-Constantin method solves $\frac{\delta D}{\delta\underline{\upsilon}}=\underline{0}$ and $\frac{\delta D}{\delta\overline{\nu}_1}=0$ leaving a minimization problem for ϕ, a.

$\frac{\delta D}{\delta\overline{\nu}_1}=0\Rightarrow\overline{\nu}_1=-\frac{(a-2)}{2(a-1)}\left(\phi+Rez\right)$ as before. Now

$$\frac{\delta D}{\delta \underline{v}} = \underline{0} \Rightarrow 2(a-1)\nabla^2 \underline{v} - a\phi^{'} \begin{bmatrix} v_3 \\ 0 \\ v_1 \end{bmatrix} + \underline{\nabla p} = \underline{0}$$

Taking $\langle \underline{v} \cdot () \rangle$ of this expression gwes

$$\left\langle \underline{v} \cdot \frac{\delta D}{\delta \underline{v}} \right\rangle = 0 = \left\langle (a-1) \left| \nabla \underline{v} \right|^2 + a\phi^{'} v_1 v_3 \right\rangle$$

So, $D\left(\underline{v}, \phi, a\right) = \frac{a^2}{4(a-1)} \left\langle \left(\phi^{'} + Re\right)^2 \right\rangle + Re^2 - \left\langle (a-1) \left| \nabla \underline{v} \right|^2 + a\phi^{'} v_1 v_3 \right\rangle$ is equivalent to

$$D(\phi, a) = \frac{a^2}{4(a-1)} \left\langle \left(\phi^{'} + Re\right)^2 \right\rangle + Re^2$$

provided ϕ and a satisfy the spectral constraint which ensures overestimation of the highest saddle point of D. This highest saddle point bounds the energy dissipation (see [5] for details).

4 Discussion

In this lecture, we have made a direct link between the Howard-Busse and Doering-Constantin variational methods for upper bounding turbulent transport in plane Couette shear flow. Similar arguments can be applied to turbulent heat transport for convection as well [6]. Both methods revolve around the same underlying functional. The Howard-Busse method seeks to find the highest saddle point of this functional by maximizing from below, while the Doering-Constantin method seeks to minimize from above. The consequence is that the ideal upper bounds derived from each method should coincide at the highest saddle point. Historically, this is seen in the results obtained in each approach. The original bound produced by Doering and Constantin [6] in 1992 was $\frac{1}{8\sqrt{2}} \approx \frac{1}{11.3}$ as opposed to Busse's estimate [1] of $\simeq \frac{1}{99.7}$. Nicodemus et al. [8] improved the Doering-Constantin result down to $\sim \frac{1}{92.0}$ in 1998. Recently Plasting & Kerswell [9] have solved the full problem to find the asymptotic result that $D \leq 0.008553$ in units of $\frac{V_0^3}{d}$ as $Re \to \infty$.

References

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