Lecture 6 - Climate Variability and its Null Hypothesis

Henk A. Dijkstra; notes by Yana Bebieva & Giovanni Fantuzzi

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1 Introduction

The Earth's system is very non-linear and complicated. For instance, if a periodic forcing, as an input signal, is coming into the Earth's system (e.g. variability in a solar constant), then very complicated signal is coming out and we want to understand what process can explain this transformation. The ice core oxygen isotope records, that is usually linked to the local temperature, (Figure 1) shows various oscillation and variation in temperature of $5°C$. The other example of a noisy signal is El Ni$\tilde{n}$o/Southern Oscillation. Temperature anomaly with respect to the mean in the equatorial Pacific is shown in Figure 2.

Now, we are going to analyze these two cases by choosing the special time and spatial scale of phenomenon. For example, for El Ni$\tilde{n}$o it would be Pacific Basin and few years (inter-annual) correspondingly. And we try to understand that phenomenon using deterministic type of model and all unresolved processes we consider as a noise. In this sense stochastic dynamical system is obtained (Figure 3. We start with the simple example where underlying dynamical system is linear.

2 The Null Hypothesis

Consider stochastic climate model developed by Hasselmann in 1976 (Figure 4). It is a layer of the ocean (mixed layer) that has the heat flux coming from the atmosphere (Q_{oa}) and we are interested how mixed layer temperature (T) evolves in time. Write down the heat balance

$$\rho C_p \frac{\partial T}{\partial t} = \lambda \frac{\partial^2 T}{\partial z^2}$$

where C_p is the specific heat of seawater, ρ is density, λ is mixing coefficient, t is time, and z is vertical coordinate. Boundary conditions are

$$z = 0 : \lambda \frac{\partial T}{\partial z} = Q_{oa}$$
$$z = -h : \frac{\partial T}{\partial z} = 0,$$

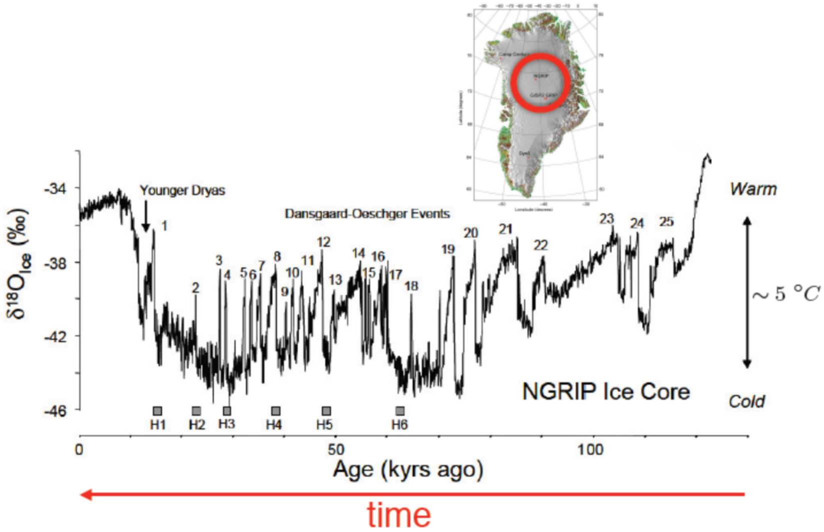

Figure 1: Ice core oxygen isotope record (ratio of ^{18}O to ^{16}O) taken from the NGRIP ice core at Greenland.

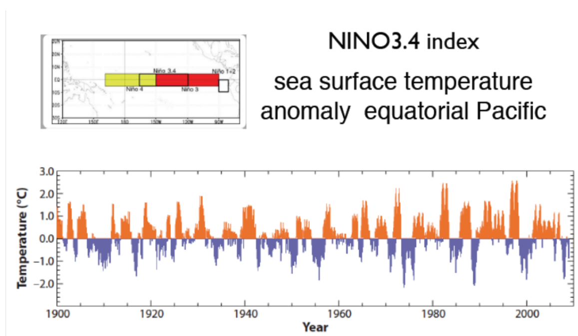

Figure 2: Temperature anomaly with respect to the mean in the equatorial Pacific in the Nino 3.4 region.

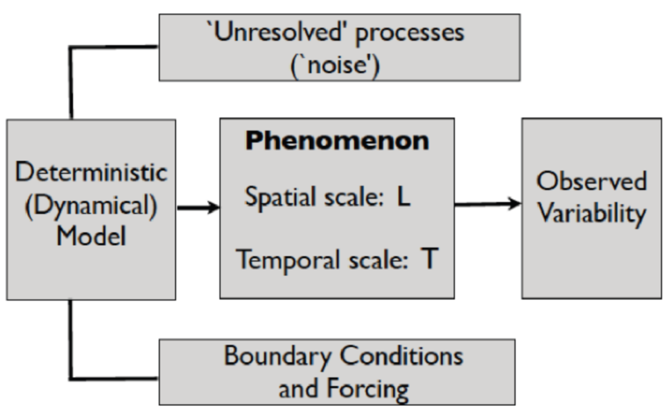

Figure 3: Stochastic dynamical systems approach scheme.

where h is depth of the layer. Now, we define depth averaged temperature

$$\overline{T} = \frac{1}{h}\int_{-h}^{0} T dz$$

This implies

$$\rho C_p h \frac{\partial \overline{T}}{\partial t} = Q_{oa} = \alpha \left(T_a - \overline{T}\right),$$

where T_a is atmospheric temperature.

We eventually get the equation

$$\begin{aligned}
\frac{\partial \overline{T}}{\partial t} &= \frac{\alpha}{\rho C_p h}\left(T_a - \overline{T}\right) \\
\gamma &= \frac{\alpha}{\rho C_p h} \\
\frac{\partial \overline{T}}{\partial t} &= -\gamma \overline{T} + \gamma T_a.
\end{aligned}$$

The time scale $1/\gamma$ is roughly 100 days. The question is how to represent γT_a. We look at a Fast-Slow systems with fast variable x and slow variable y.

$$\begin{aligned}
\frac{dx}{dt} &= f(x, y) \\
\frac{dy}{dt} &= g(x, y).
\end{aligned}$$

Looking at data measurements, one finds that variations of atmospheric temperature are much faster than ocean temperature fluctuations. Let y_0 be an initial condition, $\Delta y = y - y_0$, $t \ll \tau_y$ and take ensemble average. Assuming ergodicity

$$\begin{aligned}
\langle \frac{\partial y}{\partial t} \rangle &= \frac{d}{dt}\langle y \rangle = \langle g(x, y) \rangle = \frac{d\Delta y}{dt} \\
\langle \Delta y \rangle &= \langle g(x, y) \rangle t \\
\tilde{y} &= \Delta y - \langle \Delta y \rangle \\
\frac{d\tilde{y}}{dt} &= g(x, y) - \langle g(x, y) \rangle = \tilde{g}(x, y) \\
\langle \tilde{g}(x, y) \rangle &= 0.
\end{aligned}$$

If x is stationary, so is $\tilde{g}$. Then, $\tilde{g}$ looks like white noise and the model is

$$\frac{d\tilde{y}}{dt} = \sigma \xi(t).$$

In total we have

$$\frac{d\overline{T}}{dt} = -\gamma \overline{T} + \sigma \xi(t),$$

or

$$d\overline{T}_t = -\gamma \overline{T}_t dt + \sigma dW_t.$$

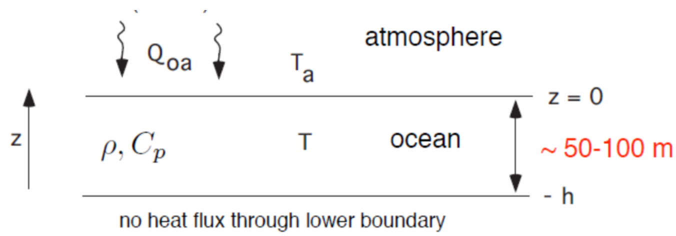

Figure 4: Scheme of Hasselmann's stochastic climate model. See text for definition of all parameters.

3 Analysis of the SDE

Let $X_t = \overline{T}_t$ be a stochastic process that satisfies the equation

$$dX_t = -\gamma X_t dt + \sigma dW_t$$
$$X_t = X_0 - \gamma \int_0^t X_s ds + \int_0^t \sigma dW_s.$$

How do we define a stochastic integral $\int_0^t \sigma dW_s$? The Itô stochastic integral definition is

$$\int_0^T h(t) dW_t = \lim_{N \to \infty} \sum_{j=0}^{N-1} h(t_j) \left(W(t_{j+1}) - W(t_j) \right)$$

and the Stratonovich stochastic integral is

$$\int_0^T h(t) \circ dW_t = \lim_{N \to \infty} \sum_{j=0}^{N-1} h\left(\frac{t_j + t_{j+1}}{2} \right) \left(W(t_{j+1}) - W(t_j) \right).$$

Where the limits exist in the L_2 norm (mean square) sense. So find 1) $\int_0^\tau dW_t = W_\tau$ (apply definition and the fact that $W_0 = 0$) and 2) $\int_0^\tau W_t dW_t$. Start with Taylor expansion using $(dW_t)^2 = dt$

$$h(W_t + dW_t) - h(W_t) = h'(W_t) dW_t + \frac{1}{2} h''(W_t)(dW_t)^2 + ...$$
$$= h'(W_t) dW_t + \frac{1}{2} h''(W_t) dt + ...$$

Integrate both sides

$$h(W_T) - h(W_0) = \int_0^T h'(W_t) dW_t + \int_0^T \frac{1}{2} h''(W_t) dt.$$

We obtained the Itô's lemma.

Now, choosing $h(W_t) = W_t^2$ (consequently $h' = 2W_t$ and $h'' = 2$) we find using the Itô's lemma

$$W_T^2 - W_0^2 = \int_0^T 2W_t dW_t + \frac{1}{2} \int_0^T 2dt.$$

Again using that $W_0 = 0$ we obtain

$$\int_0^T W_t dW_t = \frac{1}{2}\left(W_T^2 - T\right).$$

Thus, any linear 1-D stochastic differential equation can be solved analytically via the Itô lemma.

$$\begin{aligned} dX_t &= A_t dt + B_t dW_t \\ f &= f(x,t), f_1(x,t) = \frac{\partial f}{\partial t}, f_2(x,t) = \frac{\partial f}{\partial x}, f_{22}(x,t) = \frac{\partial^2 f}{\partial x^2}. \end{aligned}$$

Now, Taylor expand

$$f(t+dt, x_t + dX_t) - f(t, X_t) = f_1 dt + f_2 dX_t + \frac{1}{2}(f_{11}(dt)^2 + 2f_{12}dtdX_t + f_{22}(dX_t)^2) + ...$$

Then plug SDE into this and find

$$f(t, X_t) - f(0, X_0) = \int_0^t \left(f_1 + f_2 A_s + \frac{1}{2} f_{22} B_s^2\right) ds + \int_0^t f_2 B_s dW_s.$$

For our SDE model, choose

$$\begin{aligned} f(x,t) &= e^{\gamma t} x \\ f_1(x,t) &= \gamma e^{\gamma t} x \\ f_2(x,t) &= e^{\gamma t} \\ f_{22}(x,t) &= 0. \end{aligned}$$

Using this with our stochastic ODE and applying the Itô lemma

$$e^{\gamma t} X_t - X_0 = \int_0^t \left(\gamma e^{\gamma s} X_s - \gamma e^{\gamma s} X_s\right) ds + \sigma \int_0^t e^{\gamma s} dW_s.$$

And we have solved the SDE exactly as

$$X_t = e^{-\gamma t}\left[X_0 + \int_0^t \sigma e^{\gamma s} dW_s\right].$$

4 Numerical Solution of SDEs

One often requires a numerical solution of the SDE (interpreted in Itō's sense)

$$dX_t = f(t, X_t)dt + g(t, X_t)dW_t \tag{1}$$

over the time interval $[0, T]$. A basic approach to this problem is to consider discrete time instants $t_n = n\Delta t$, for a given time-step Δt, and approximate (1) with the *Euler-Maruyama scheme*

$$\tilde{X}_{n+1} = \tilde{X}_n + f(t_n, \tilde{X}_n)\Delta t + g(t_n, \tilde{X}_n)\Delta W_{n+1}. \tag{2}$$

Here, we have denoted $\tilde{X}_n = \tilde{X}(t_n)$ the approximate solution of (1) and $\Delta W_{n+1} = W_{n+1} - W_n$ is the jump of a Wiener process over the time interval Δt. The jumps ΔW are independent, Gaussian random variables $\Delta W \sim \mathcal{N}(0, \Delta t)$ and can therefore be generated at each iteration with an appropriate random number generator (e.g. `DW = sqrt(Dt)*randn(1)` in MATLAB). We also remark that Itō's interpretation of an SDE consists of the continuous-time limit of the Euler-Maruyama scheme.

Of course, more sophisticated schemes can be derived. An example is the *Milstein scheme*, which approximates the solution $X(t)$ of (1) as

$$\tilde{X}_{n+1} = \tilde{X}_n + f(t_n, \tilde{X}_n)\Delta t + g(t_n, \tilde{X}_n)\Delta W_{n+1} + \frac{1}{2}\left[(\Delta W_{n+1})^2 - \Delta t\right]. \tag{3}$$

The main difference between the various numerical schemes regards their convergence properties in the limit $\Delta t \to 0$. In the context of numerical schemes for SDEs, there are two notions of convergence. The first, known as *weak convergence*, considers convergence of the expectations of the approximation error $|X(t_n) - \tilde{X}_n|$; that is, a numerical scheme is weakly convergent if

$$\mathbb{E}\left(|X(t_n) - \tilde{X}_n|\right) \leq c\Delta t^{\eta}. \tag{4}$$

Here, c and η are constants (dependent on the type of scheme and the SDE considered); η is the convergence rate of the scheme.

Similarly, we say that a numerical scheme is *strongly convergent* if the expectations $\mathbb{E}[X(t_n)]$ and $\mathbb{E}(\tilde{X}_n)$ converge, i.e.

$$|\mathbb{E}[X(t_n)] - \mathbb{E}(\tilde{X}_n)| \leq c\Delta t^{\eta}. \tag{5}$$

Example 1 (Ornstein-Uhlenbeck Process). Consider the Ornstein-Uhlenbeck process $dX_t = -\gamma X_t dt + \sigma dW_t$, discretised with the Euler-Maruyama (EM) scheme as

$$X_{n+1} = X_n - \gamma X_n \Delta t + \sigma \Delta W \quad = (1 - \gamma\Delta t)X_n + \sigma\Delta W. \tag{6}$$

We know that the stationary state solution $X^{\text{stat}}(t)$ is a Gaussian random variable with zero mean ($\mathbb{E}[X(t)] = 0$). In order to check the strong convergence of the EM scheme, we can take the expectation of (6), obtaining

$$\mathbb{E}(X_{n+1}) = (1 - \gamma\Delta t)\mathbb{E}(X_n) = (1 - \gamma\Delta t)^{n+1} X_0 \tag{7}$$

As $n \to \infty$, the expectation $\mathbb{E}(X_{n+1})$ converges to the analytic stationary result $\mathbb{E}[X(t)] = 0$ only if

$$|1 - \gamma\Delta t| \leq 1 \Rightarrow \Delta t \leq \frac{2}{\gamma}, \tag{8}$$

that is the EM scheme converges strongly if $\Delta t \leq \frac{2}{\gamma}$.

5 Applicability of the Hasselmann's Model to the SST Anomaly

Hasselmann's linear model for the SST anomaly, derived in Section 2, resulted in the SDE

$$\frac{d\bar{T}}{dt} = -\gamma\bar{T} + \sigma\xi(t) \tag{9}$$

which is the Ornstein-Uhlenbeck process. Thus, according to Hasselmann's model, the transition density ρ of $\bar{T}$ satisfies the Fokker-Planck equation

$$\frac{\partial \rho}{\partial t} = \frac{\partial(\gamma x \rho)}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 \rho}{\partial x^2}, \tag{10}$$

and there exists a stationary distribution

$$\rho^{\text{stat}}(x) = \sqrt{\frac{\gamma}{\pi\sigma^2}}\, e^{-\frac{\gamma x^2}{\sigma^2}}. \tag{11}$$

Moreover, we recall that the stationary autocorrelation and the spectrum of the Ornstein-Uhlenbeck solution are, respectively,

$$\mathbb{E}(X_s X_t) = \frac{\sigma^2}{\gamma}\, e^{-|t-s|}, \tag{12}$$

$$S(\omega) = \frac{\sigma^2}{\gamma^2 + \omega^2}. \tag{13}$$

The main question to be answered at this point is how this model can be tuned using experimental observations of the SST anomaly (say, $\{\bar{T}_0^{\text{ex}},\ \bar{T}_1^{\text{ex}},\ \dots\ ,\ \bar{T}_N^{\text{ex}}\}$), so that any predictions based on Hasselmann's model can be trusted. In fact, we would like the analytic solution of the continuous time Ornstein-Uhlenbeck process, i.e.

$$\bar{T}(t) = e^{-\gamma t}\bar{T}_0 + \sigma e^{-\gamma t}\int_0^t e^{\gamma s} dW_s, \tag{14}$$

to reproduce the statistics of the time-series of measurements $\bar{T}_0^{\text{ex}},\ \bar{T}_1^{\text{ex}},\ \dots\ ,\ \bar{T}_N^{\text{ex}}$. To this purpose, we discretise (14) by looking at the time instants t_n and t_{n+1}, i.e. we consider

$$\bar{T}_{n+1} = e^{-\gamma t_{n+1}}\bar{T}_0 + \sigma e^{-\gamma t_{n+1}}\int_0^{t_{n+1}} e^{\gamma s} dW_s, \tag{15a}$$

$$\bar{T}_n = e^{-\gamma t_n}\bar{T}_0 + \sigma e^{-\gamma t_n}\int_0^{t_n} e^{\gamma s} dW_s. \tag{15b}$$

We can multiply (15b) by $e^{-\gamma(t_{n+1}-t_n)} = e^{-\gamma\Delta t}$ and subtract it from (15a) to obtain

$$\bar{T}_{n+1} = e^{-\gamma\Delta t}\bar{T}_n + \sigma e^{-\gamma t_{n+1}}\int_{t_n}^{t_{n+1}} e^{\gamma s} dW_s. \tag{16}$$

When Δt is small, we can approximate the integral in the last expression by $c\Delta W_{n+1}$, where c is a suitable constant. So, the last term can be approximated with a Gaussian random variable Z_n with zero mean and variance $\tilde{\sigma}^2$ (chosen appropriately). Then, we may approximate the solution of the Ornstein-Uhlenbeck process by the discrete stochastic process

$$\bar{T}_{n+1} = \alpha\bar{T}_n + Z_{n+1}, \tag{17}$$

where $\alpha = e^{-\gamma\Delta t} \in (0,1)$. This type of discrete process is known as the *AR(1) process* (where AR stands for auto-regressive), or as a *red noise process*. We can now compute the spectrum of the AR(1) process and compare it to the spectrum obtained from the experimental data set $\{\bar{T}_0^{\text{ex}}, \bar{T}_1^{\text{ex}}, \dots , \bar{T}_N^{\text{ex}}\}$ to compute appropriate values of α and $\tilde{\sigma}$. To this purpose, we first note that

$$\begin{aligned} \bar{T}_k &= \alpha \bar{T}_{k-1} + Z_k \\ &= \alpha(\alpha \bar{T}_{k-2} + Z_{k-1}) + Z_k \\ &\vdots \\ &= \alpha^k Z_0 + \alpha^{k-1} Z_1 + \dots + \alpha Z_{k-1} + Z_k \end{aligned} \tag{18}$$

so that

$$\begin{aligned} c_0 \stackrel{\text{def}}{=} \mathbb{E}(\bar{T}_k^2) &= \left[1 + \alpha^2 + \left(\alpha^2\right)^2 + \dots + \left(\alpha^2\right)^k\right] \tilde{\sigma}^2 \\ &\stackrel{k\to\infty}{=} \frac{\tilde{\sigma}^2}{1-\alpha^2} \end{aligned} \tag{19}$$

Note that we have used the relation $\mathbb{E}(Z_n Z_m) = \tilde{\sigma}^2 \delta_{nm}$ and the fact that the geometric series converges since $0 < \alpha < 1$. Thus, recalling that the random variable Z_n is independent of any past realisation $\bar{T}_m$, the correlation of the discrete process becomes

$$\begin{aligned} c_k \stackrel{\text{def}}{=} \mathbb{E}(\bar{T}_i \bar{T}_{k+i}) &= \frac{1}{N} \sum_{i=0}^{N} \bar{T}_i \bar{T}_{k+i} \\ &= \frac{1}{N} \sum_{i=0}^{N} \bar{T}_i \left(\alpha \bar{T}_{k-1+i} + Z_{k+i}\right) \\ &= \frac{\alpha}{N} \sum_{i=0}^{N} \bar{T}_i \bar{T}_{k-1+i} \\ &= \alpha\, c_{k-1} \\ &= \alpha^k\, c_0 \end{aligned} \tag{20}$$

so as $k \to \infty$ (i.e. the discrete time series becomes infinitely long) we obtain

$$c_k = \frac{\alpha^k\, \tilde{\sigma}^2}{1-\alpha^2} = \frac{\tilde{\sigma}^2}{1-\alpha^2}\, e^{-\gamma t_k}, \tag{21}$$

where we have used $\alpha = e^{-\gamma \Delta t}$ (constant for a given Δt) and that, for uniform time-steps, $k\Delta t = t_k$. Taking the continuous-time version $c(t)$ of (21), we can compute the spectrum of the AR(1) process as

$$\begin{aligned} S(\omega) &= \int_{-\infty}^{\infty} c(t) e^{-i\omega t} dt \\ &= \int_{-\infty}^{\infty} \frac{\tilde{\sigma}^2}{1-\alpha^2}\, e^{-\gamma |t|} e^{-i\omega t} dt \\ &= \frac{2\gamma}{1-\alpha^2} \frac{\tilde{\sigma}^2}{\gamma^2 + \omega^2} \end{aligned} \tag{22}$$

The absolute value was introduced to maintain the negativity of the argument of the exponential term in $c(t)$. Note that, up to a normalisation constant, this spectrum is the same as for the continuous-time Ornstein-Uhlenbeck process. Thus, one can tune the AR(1) to fit the measured data in the following way: compute the spectrum of the discrete measurements $\{\bar{T}_0^{\mathrm{ex}},\ \bar{T}_1^{\mathrm{ex}},\ \ldots\ ,\ \bar{T}_N^{\mathrm{ex}}\}$, then fit appropriate values of α and $\tilde{\sigma}$ in (22). Then, the corresponding AR(1) process reproduces the measured statistics, and can be used to estimate the statistical properties of the SST anomaly according to Hasselman's model.