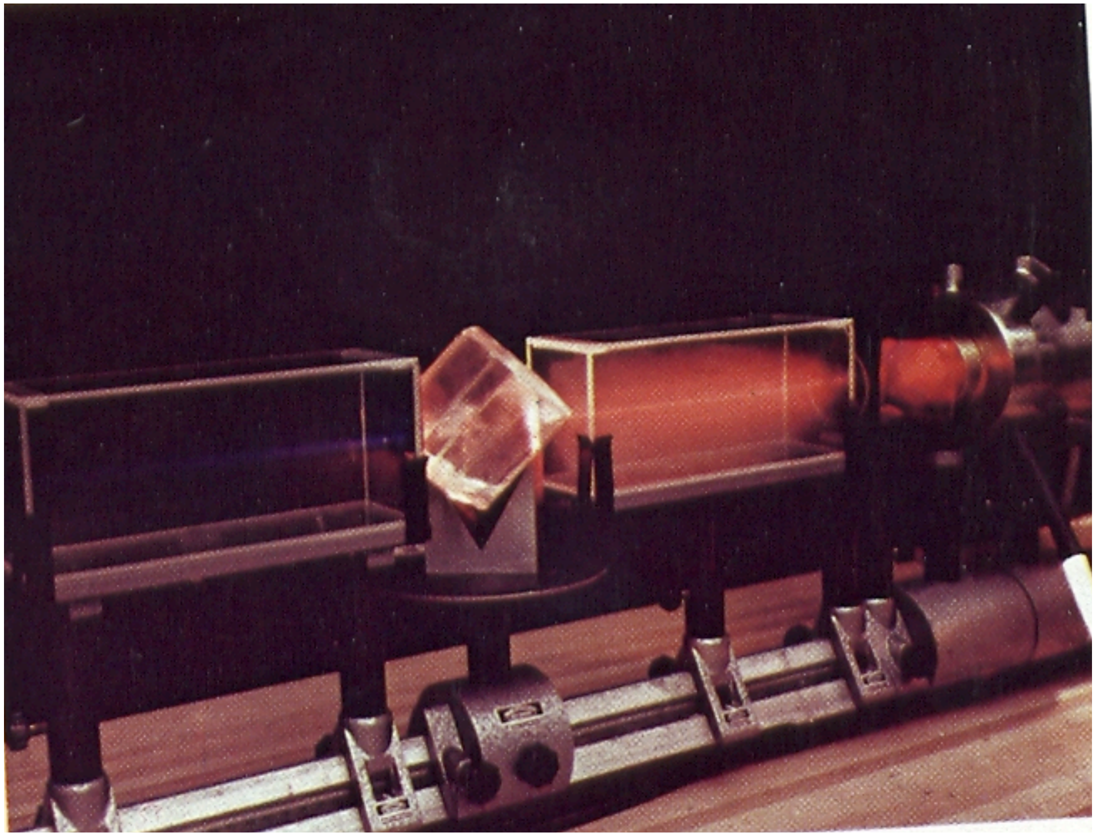
# Triad (or 3-wave) resonances

## Lecture 12



Second harmonic generation, a special case of a triad resonance, converts red light to blue (R.W. Terhune)

# Triad (or 3-wave) resonances

A. Derive the 3-wave equations
   - for a single resonant triad
   - for multiple triads
B. Mathematical structure of a single triad
   - ODEs
   - PDEs
C. What happens in multiple triads?
D. Application to capillary-gravity waves

# A. Derive the 3-wave equations

For dispersive waves of small amplitude, resonant triad interactions are the “first” nonlinear interactions to appear

(if they are possible).

Start with a physical system

(without dissipation)

$$N(u) = 0$$

with $N(0) = 0$

$$N(u) = 0 \quad \text{(nonlinear problem)}$$

Step 1:

Linearize about $u = 0$

$$u(\vec{x},t;\varepsilon) = \varepsilon[\sum_{k} A(\vec{k})e^{i\vec{k}\cdot\vec{x}-i\omega(\vec{k})t} + (cc)] + O(\varepsilon^2)$$

Find linearized dispersion relation:

(related to index of refraction in optics)

$$\omega(\vec{k})$$

# N($u$) = 0 (nonlinear problem)

## Step 2: Weakly nonlinear models

Q: Does $\omega(k)$ admit 3 pairs $\{\vec{k}, \omega(\vec{k})\}$

so

$$\vec{k}_1 \pm \vec{k}_2 \pm \vec{k}_3 = 0, \qquad \omega(\vec{k}_1) \pm \omega(\vec{k}_2) \pm \omega(\vec{k}_3) = 0\,?$$

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(Use graphical procedure due to
Ziman (1960), Ball (1964), and others.)

N($u$) = 0 (nonlinear problem)

## Step 2: Weakly nonlinear models

Q: Does $\omega(k)$ admit 3 pairs $\{\vec{k}, \omega(\vec{k})\}$

so

$$\vec{k}_1 \pm \vec{k}_2 \pm \vec{k}_3 = 0, \qquad \omega(\vec{k}_1) \pm \omega(\vec{k}_2) \pm \omega(\vec{k}_3) = 0\,?$$

If yes ➔ 3-wave equations (resonant triads)

If no ➔ 4-wave equations (resonant quartets)

# 3-wave equations, single triad

Suppose $\vec{k}_1 \pm \vec{k}_2 \pm \vec{k}_3 = 0, \qquad \omega(\vec{k}_1) \pm \omega(\vec{k}_2) \pm \omega(\vec{k}_3) = 0.$

Consider exactly one triad. Try

$$u(\vec{x},t;\varepsilon) = \varepsilon[\sum_{m=1}^{3} A_m \exp\{i\vec{k}_m \cdot \vec{x} - i\omega_m t\} + (cc)]$$

$$+\varepsilon^2[\sum_{m=1}^{3}\sum_{n=1}^{m} B_{mn}(t)\exp(i(\vec{k}_m + \vec{k}_n)\cdot \vec{x} - i(\omega_m + \omega_n)t\} + (cc)] + O(\varepsilon^3)$$

# 3-wave equations, single triad

Suppose $\vec{k}_1 \pm \vec{k}_2 \pm \vec{k}_3 = 0, \qquad \omega(\vec{k}_1) \pm \omega(\vec{k}_2) \pm \omega(\vec{k}_3) = 0.$

Consider exactly one triad. Try

$$u(\vec{x},t;\varepsilon) = \varepsilon[\sum_{m=1}^{3} A_m \exp\{i\vec{k}_m \cdot \vec{x} - i\omega_m t\} + (cc)]$$

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Bad! Find $B_{\mathrm{mn}}(t)$ grows like *t*.

$\Rightarrow \quad u(\vec{x},t;\varepsilon) = \varepsilon\{(bdd) + (\varepsilon t)(bdd) + O(\varepsilon^2)\}$

# 3-wave equations, single triad

Suppose $\vec{k}_1 \pm \vec{k}_2 \pm \vec{k}_3 = 0, \qquad \omega(\vec{k}_1) \pm \omega(\vec{k}_2) \pm \omega(\vec{k}_3) = 0.$

Exactly one triad. Use method of multiple scales:

$$u(\vec{x},t;\varepsilon) = \varepsilon[\sum_{m=1}^{3} A_m(\varepsilon\vec{x},\varepsilon t)\exp\{i\vec{k}_m \cdot \vec{x} - i\omega_m t\} + (cc)] + O(\varepsilon^2)$$

# 3-wave equations, single triad

Suppose $\vec{k}_1 \pm \vec{k}_2 \pm \vec{k}_3 = 0, \qquad \omega(\vec{k}_1) \pm \omega(\vec{k}_2) \pm \omega(\vec{k}_3) = 0.$

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➔ $$\boxed{\partial_\tau(A_m) + \vec{c}_m \cdot \nabla A_m = i\delta_m A_n^* A_l^*,}$$

$$m,n,l = 1,2,3$$

group velocity of m^th mode — real-valued constant

# 3-wave equations, single triad

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$$m,n,l = 1,2,3$$

group velocity of m[th] mode real-valued constant

(applications: capillary-gravity waves, internal waves;

$\chi_2$ materials in optics)

# B. Consider a single triad

PDE version

$$\boxed{\partial_\tau (A_m) + \vec{c}_m \cdot \nabla A_m = i\delta_m A_n^* A_l^*,}$$

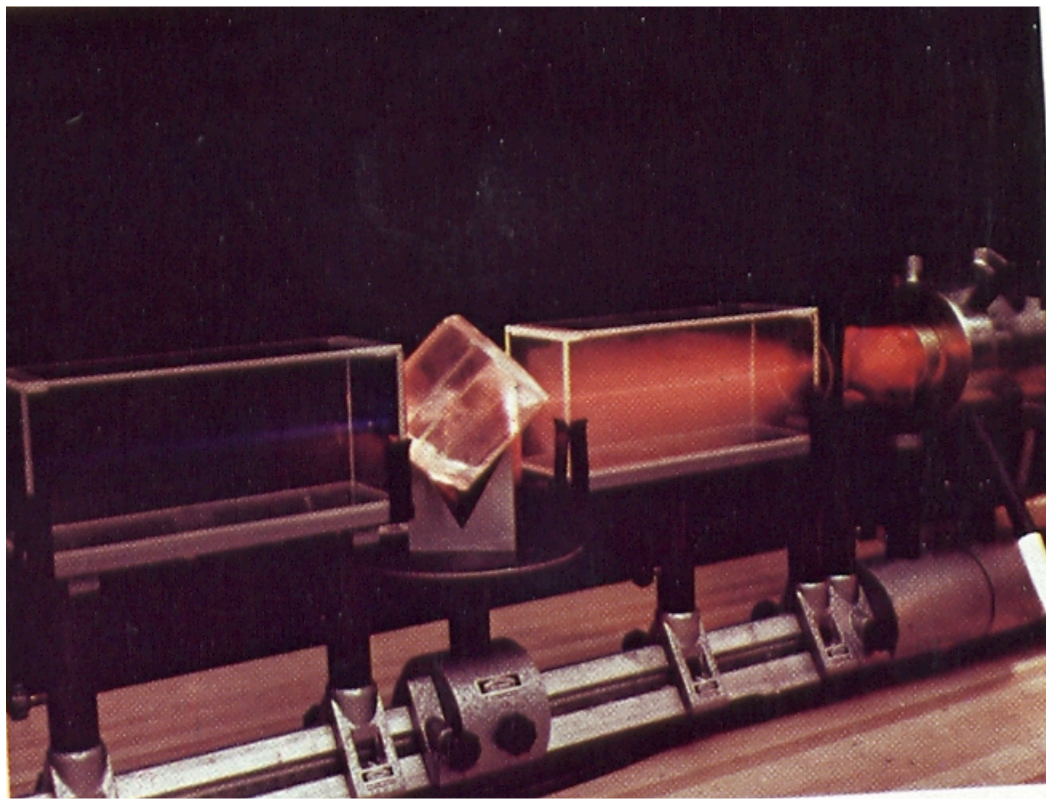
$$m, n, l = 1,2,3$$

ODE version

$$\boxed{A_1' = i\delta_1 A_2^* A_3^*, \quad A_2' = i\delta_2 A_3^* A_1^*, \quad A_3' = i\delta_3 A_1^* A_2^*.}$$

# Application of single triad, ODES

$$\boxed{A_1' = i\delta_1 A_2^* A_3^*, \quad A_2' = i\delta_2 A_3^* A_1^*, \quad A_3' = i\delta_3 A_1^* A_2^*.}$$



Second harmonic generation: $k+k = 2k$, $\omega+\omega = 2\omega$.

# Mathematical structure of single triad of ODEs

$$\boxed{A_1' = i\delta_1 A_2^* A_3^*, \quad A_2' = i\delta_2 A_3^* A_1^*, \quad A_3' = i\delta_3 A_1^* A_2^*.}$$

1) System is Hamiltonian

Conjugate variables: $\{q_j(\tau) = \dfrac{A_j(\tau)}{\sqrt{|\delta_j|}} sign(\delta_j), \quad p_j(\tau) = \dfrac{A^*{}_j(\tau)}{\sqrt{|\delta_j|}}\}$

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Hamiltonian:
$$H = i[A_1 A_2 A_3 + A_1^* A_2^* A_3^*]$$
$$= i\sqrt{|\delta_1\delta_2\delta_3|}[sign(\delta_1\delta_2\delta_3)q_1q_2q_3 + p_1p_2p_3]$$

Verify directly: $q_j' = \frac{\partial H}{\partial p_j}, \quad p_j' = -\frac{\partial H}{\partial q_j}, \quad j = 1,2,3$

$$\boxed{A_1' = i\delta_1 A_2^* A_3^*, \quad A_2' = i\delta_2 A_3^* A_1^*, \quad A_3' = i\delta_3 A_1^* A_2^*.}$$

2) Constants of the motion:

$$-iH = A_1 A_2 A_3 + A_1^* A_2^* A_3^*, \quad J_1 = \frac{|A_1|^2}{\delta_1} - \frac{|A_3|^2}{\delta_3}, \quad J_2 = \frac{|A_2|^2}{\delta_2} - \frac{|A_3|^2}{\delta_3}.$$

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3) Define Poisson bracket

$$\{F, G\} = \sum_{m=1}^{3} \left(\frac{\partial F}{\partial p_m}\frac{\partial G}{\partial q_m} - \frac{\partial F}{\partial q_m}\frac{\partial G}{\partial p_m}\right)$$

$$= \sum_{m=1}^{3} \delta_m \left(\frac{\partial F}{\partial A^*{}_m}\frac{\partial G}{\partial A_m} - \frac{\partial F}{\partial A_m}\frac{\partial G}{\partial A^*{}_m}\right)$$

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4) Show directly: $\{-iH, J_1\} = 0 = \{-iH, J_2\} = \{J_1, J_2\}$

$$\boxed{A_1' = i\delta_1 A_2^* A_3^*, \quad A_2' = i\delta_2 A_3^* A_1^*, \quad A_3' = i\delta_3 A_1^* A_2^*.}$$

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4) $$\{-iH, J_1\} = 0 = \{-iH, J_2\} = \{J_1, J_2\}$$

5) So what?

$$\boxed{A_1' = i\delta_1 A_2^* A_3^*, \quad A_2' = i\delta_2 A_3^* A_1^*, \quad A_3' = i\delta_3 A_1^* A_2^*.}$$

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4) $\{-iH, J_1\} = 0 = \{-iH, J_2\} = \{J_1, J_2\}$

5) So what?

If a Hamiltonian system of 6 real ODEs (or 3 complex ODEs) has (6/2 = 3) constants *in involution*, then the system is completely integrable.

The 3 constants (“action variables”) define a 3-dimensional surface in the 6-D phase space. Every solution of ODEs consists of straight-line motion on this surface.

$$\boxed{A_1' = i\delta_1 A_2^* A_3^*, \quad A_2' = i\delta_2 A_3^* A_1^*, \quad A_3' = i\delta_3 A_1^* A_2^*.}$$

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4) $\{-iH, J_1\} = 0 = \{-iH, J_2\} = \{J_1, J_2\}$

6) So what?

In the usual situation, $\delta_1$, $\delta_2$, $\delta_3$ do **not** all have the same sign.

Then the motion is necessarily bounded for all time, the 3-D surface is a *torus*, and the motion is either periodic or quasi-periodic in time. The entire solution can be written in terms of elliptic functions.

$$\boxed{A_1' = i\delta_1 A_2^* A_3^*, \quad A_2' = i\delta_2 A_3^* A_1^*, \quad A_3' = i\delta_3 A_1^* A_2^*.}$$

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4) $\{-iH, J_1\} = 0 = \{-iH, J_2\} = \{J_1, J_2\}$

7) So what?

In the *unusual* situation, $\delta_1$, $\delta_2$, $\delta_3$ all have the same sign.

Coppi, Rosenbluth & Sudan (1969) showed that $(A_1, A_2, A_3)$ can all blow up together, in finite time. This is the explosive instability. (See lecture 19.)

$$\boxed{A_1' = i\delta_1 A_2^* A_3^*, \quad A_2' = i\delta_2 A_3^* A_1^*, \quad A_3' = i\delta_3 A_1^* A_2^*.}$$

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8) ➔ For a single triad of ODEs, we know everything.

# Mathematical structure of a single triad of PDEs

$$\boxed{\partial_\tau(A_m) + \vec{c}_m \cdot \nabla A_m = i\delta_m A_n^* A_l^*,}$$

$$m,n,l = 1,2,3$$

group velocity of m^th mode

real-valued constant

# Mathematical structure of a single triad of PDEs

$$\boxed{\partial_\tau(A_m) + \vec{c}_m \cdot \nabla A_m = i\delta_m A_n^* A_l^*,}$$

$$m,n,l = 1,2,3$$

- Zakharov & Manakov (1976) showed that the system of PDEs is completely integrable!

# Mathematical structure of a single triad of PDEs

$$\boxed{\partial_\tau (A_m) + \vec{c}_m \cdot \nabla A_m = i\delta_m A_n^* A_l^*,}$$

$$m, n, l = 1, 2, 3$$

- Zakharov & Manakov (1976) showed that the system of PDEs is completely integrable!
- Kaup (1978) “solved” the initial-value problem in 1-D on $-\infty < x < \infty$, with restrictions
- Kaup, Reiman &Bers (1980) solved the initial-value problem in 3-D in all space, with restrictions
- Few physical applications of this theory are developed

# C.The opposite extreme: Zakharov's integral equation considers all possible interactions (resonant and non-resonant)

$$\partial_t A(\vec{k}) + i\omega(\vec{k})A(\vec{k})$$
$$= -i\iint [V(\vec{k},\vec{k}_1,\vec{k}_2)\delta(\vec{k}+\vec{k}_1+\vec{k}_2)A^*(\vec{k}_1)A^*(k_2) + perm.]d\vec{k}_1 d\vec{k}_2$$
$$-i\iiint [W(\vec{k},\vec{k}_1,\vec{k}_2,\vec{k}_3)\delta(\vec{k}+\vec{k}_1-\vec{k}_2-\vec{k}_3)A^*(\vec{k}_1)A(\vec{k}_2)A(\vec{k}_3)]d\vec{k}_1 d\vec{k}_2 d\vec{k}_3$$

Note: This equation acts on the **fast** time-scale
➔numerical integration is slow and expensive.

# D. What's missing from a single triad of ODEs?

In real life, the dynamics of a single triad of ODEs can require modification because of:

- Spatial variation of wave envelopes (requires PDEs instead)
- Multiple triad interactions

  (requires more interacting wave modes)
- Dissipation (makes the ODEs non-Hamiltonian)

# Multiple triads: an example

Consider a single triad of ODEs, energy conserved:

$$A_1' = i\delta_1 A_2^* A_3^*, \quad A_2' = i\delta_2 A_3^* A_1^*, \quad A_3' = i\delta_3 A_1^* A_2^*.$$

$$\delta_1 > 0, \quad \delta_2 > 0, \quad \delta_3 < 0.$$

Fact: If one mode has almost all the energy initially, only $A_3$ can share that energy with the other modes

Proof:

$$J_1 = \frac{|A_1|^2}{\delta_1} - \frac{|A_3|^2}{\delta_3}, \quad J_2 = \frac{|A_2|^2}{\delta_2} - \frac{|A_3|^2}{\delta_3},$$

# One example of multiple triads

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Fact (Hasselmann): The wave mode in a triad with the “different” interaction coefficient has the highest frequency in the triad.

# One example of multiple triads

$$A_1' = i\delta_1 A_2^* A_3^*, \quad A_2' = i\delta_2 A_3^* A_1^*, \quad A_3' = i\delta_3 A_1^* A_2^*.$$

$$\delta_1 > 0, \quad \delta_2 > 0, \quad \delta_3 < 0.$$

Conjecture (Simmons, 1967): For capillary-gravity waves, each wave mode can participate in a continuum of triad interactions.

The magnitude of the interaction coefficients does not vary much across this continuum, so expect that energy put into a single wave mode will generate broad-banded response

➔No selection mechanism

# multiple triads

Experimental
Tests:
Perlin & Hammack,1990
Perlin, Henderson,
Hammack, 1991

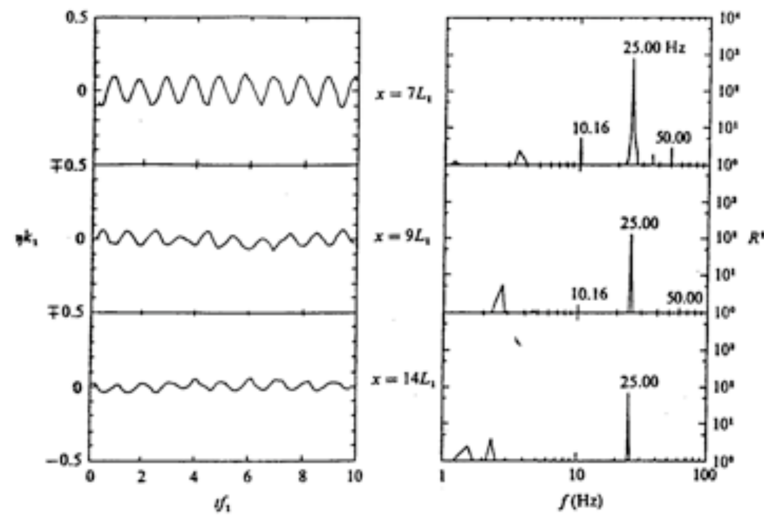



Figure 4(a). Temporal wave profiles and corresponding periodograms for the 25-Hz wavetrain of fig. 3(b).

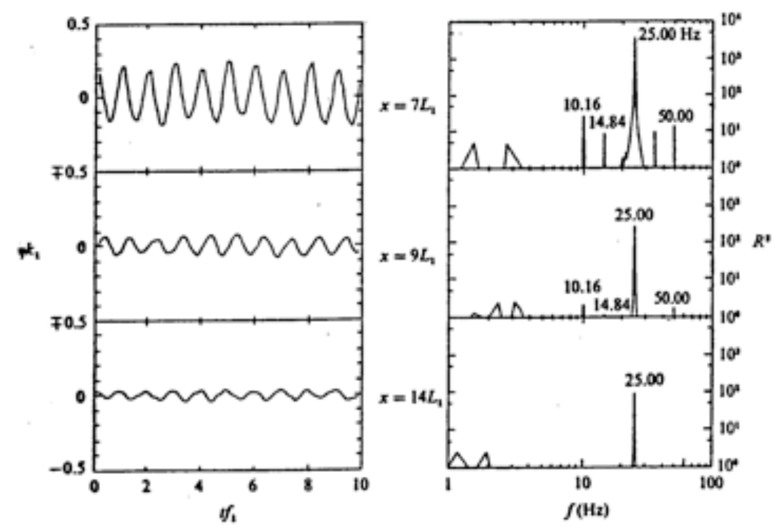
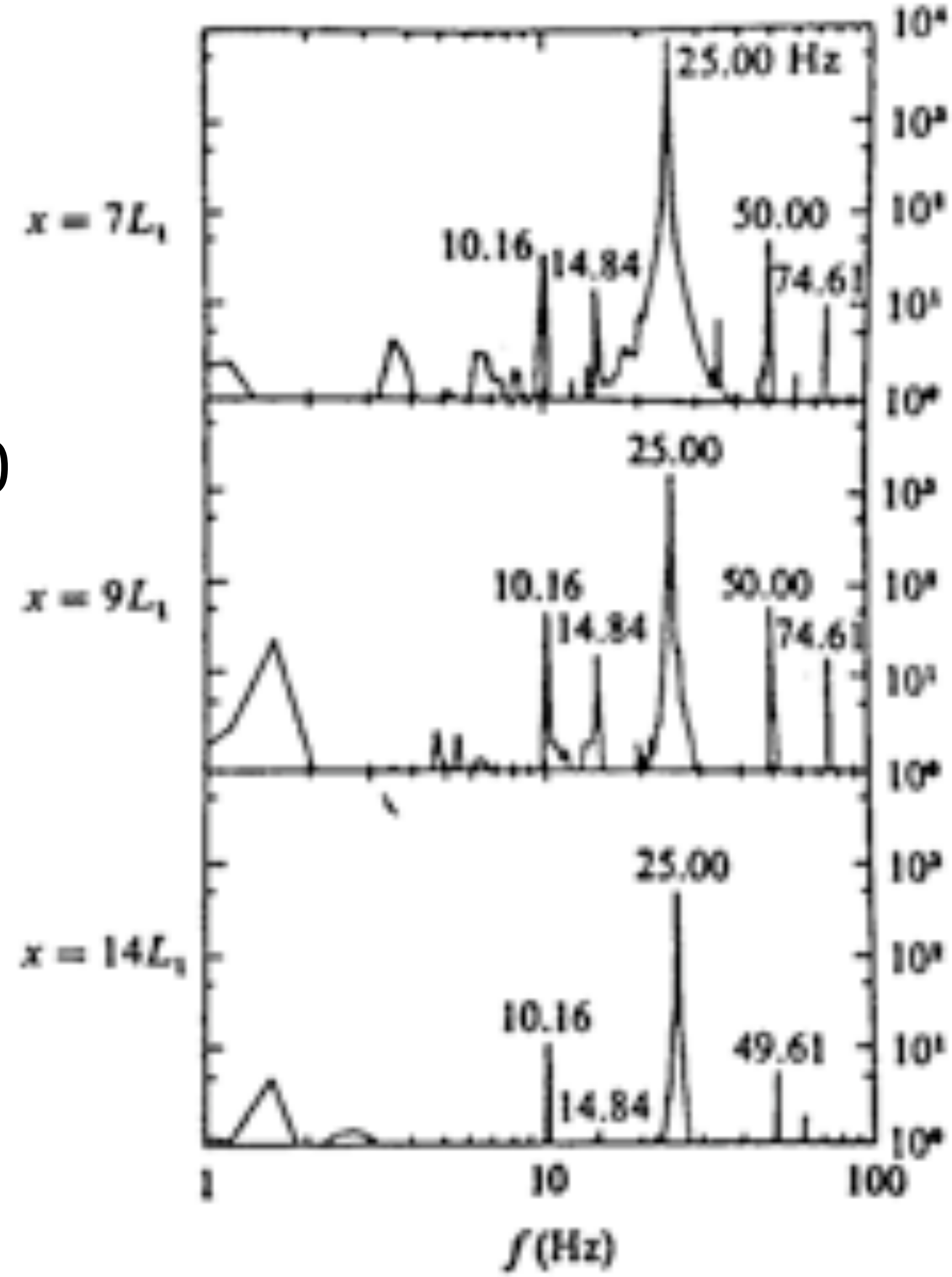



Figure 4(b). Temporal wave profiles and corresponding periodograms for the 25-Hz wavetrain of fig. 3(c).

# multiple triads

Experimental
Tests:
Perlin & Hammack,1990
Perlin, Henderson,
Hammack, 1991

Q: What is the
selection mechanism?
What causes it?

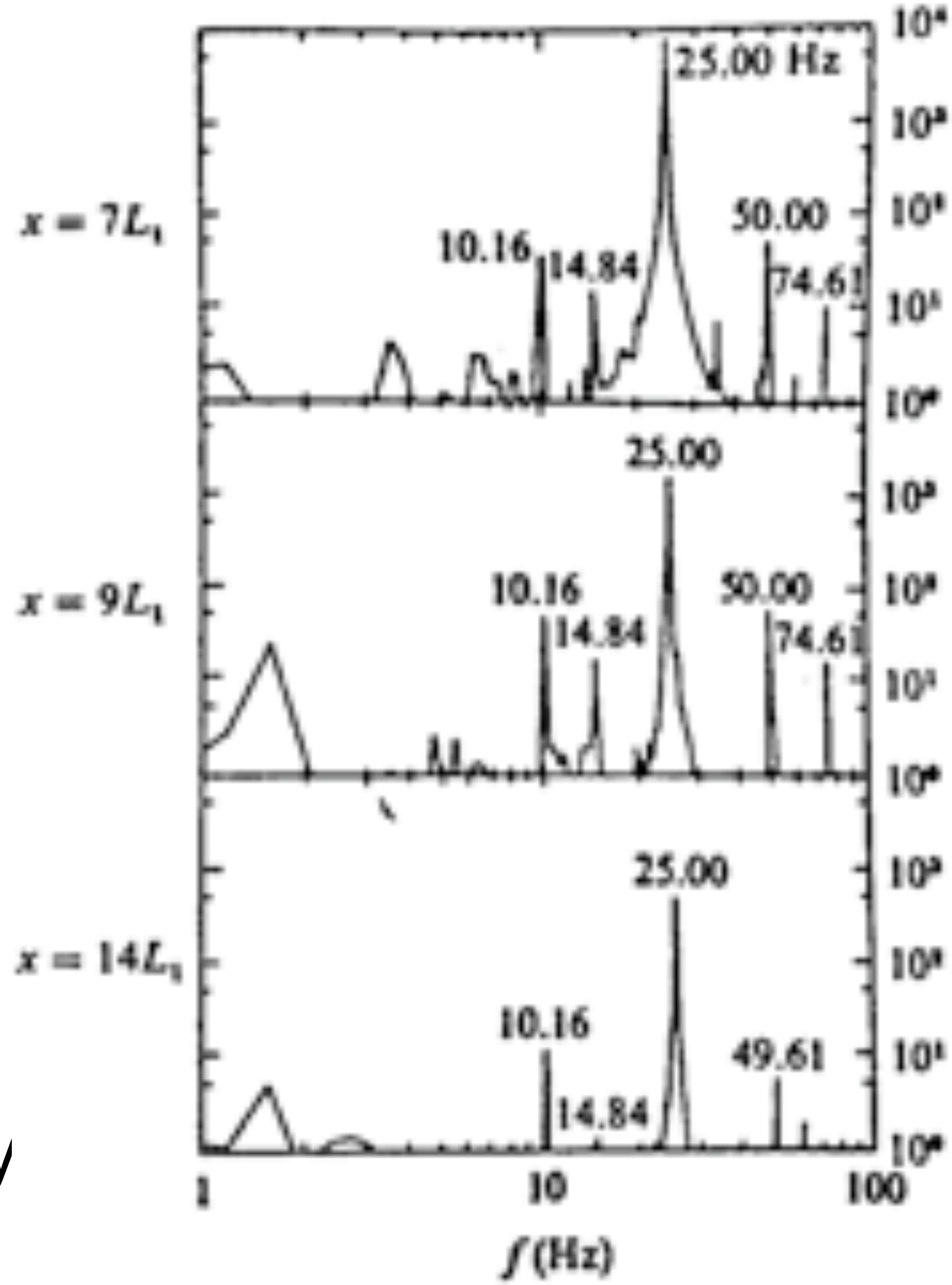


# multiple triads

Q: What is the selection mechanism? What causes it?

A: $60 = 25 + 35$
$35 = 25 + 10$
$25 = 10 + 15$

Only in last triad is 25 the highest frequency



# Einstein:

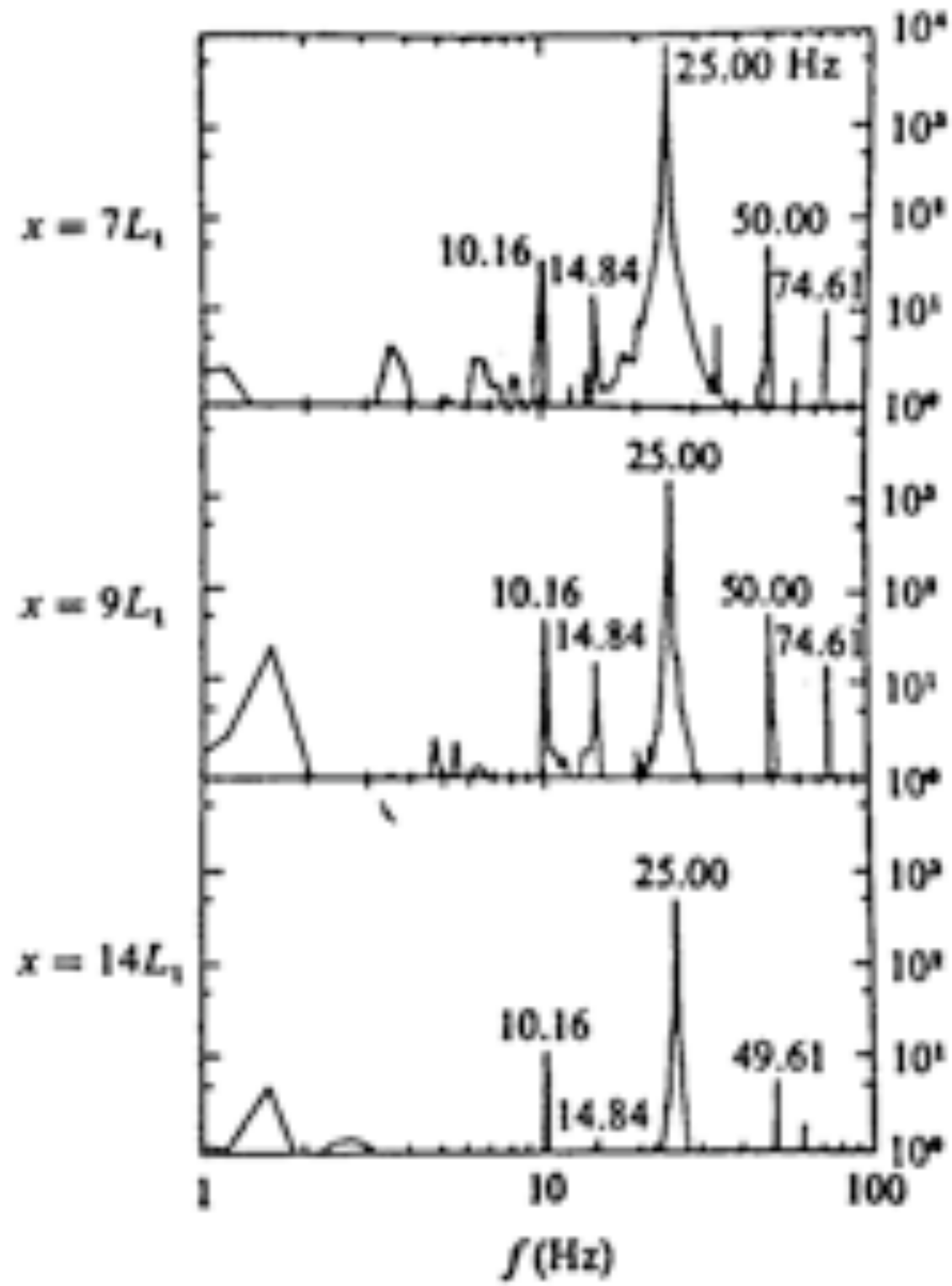
A good mathematical model of a physical problem should be as simple as possible, and no simpler.

60 = 25 + 35

35 = 25 + 10

25 = 10 + 15

+ dissipation



Thank you for your attention