Waves on deep water, I

Lecture 13

Main question: Are there stable wave patterns that propagate with permanent form (or nearly so) on deep water?

Main approximate model:

$$i\partial_\tau A + \alpha\partial_\xi^2 A + \beta\partial_\zeta^2 A + \gamma |A|^2 A = 0$$

Nonlinear Schrödinger equation (NLS)

Waves on deep water (I,II)

This lecture (13):

A. Sketch derivation of NLS for waves on deep water

B. Earlier work:

Waves with 1-D surface patterns on deep water
- existence and stability

Next lecture (14):

C. More recent work:

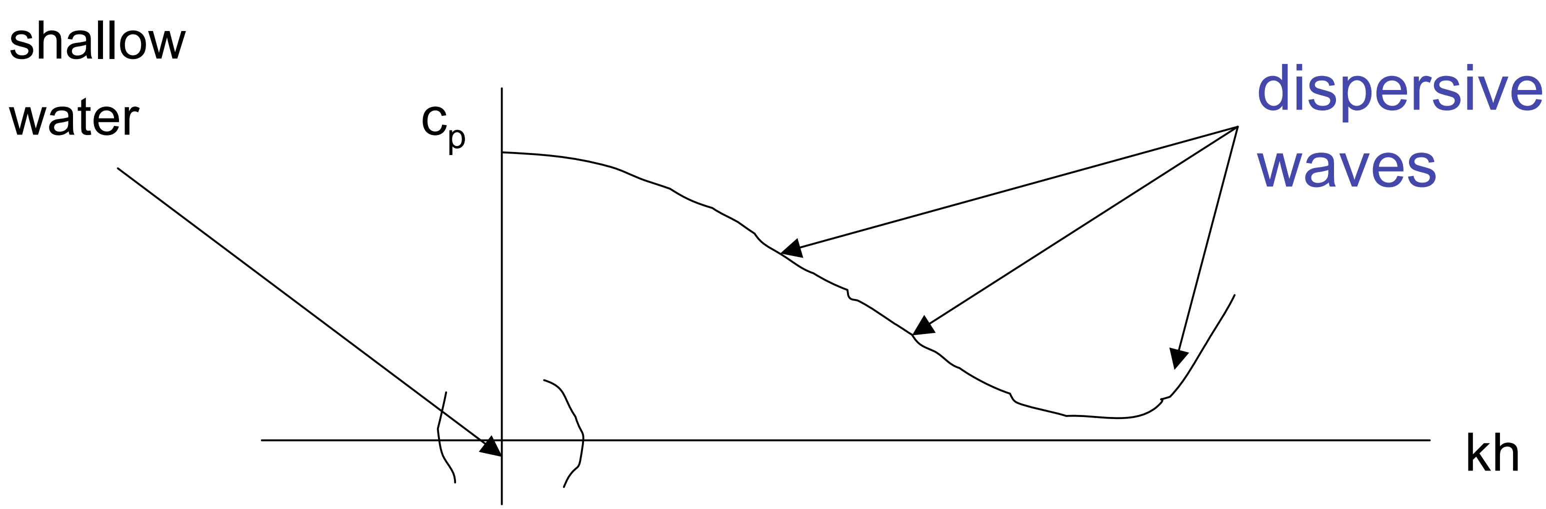
- Stability of waves with 2-D surface patterns
- Effect of small damping

A. Sketch derivation of NLS

NLS describes the slow evolution of a train (or packet) of dispersive waves:

- of small or moderate amplitude
- travelling in nearly the same direction
- with nearly the same frequency.

Linearized dispersion curve



Sketch derivation of NLS

1. Simplest case:

- Gravity waves only (no surface tension)
- Deep water ($kh \to \infty$)

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- Nearly monochromatic: $\vec{k} = (k_0, 0) + O(\varepsilon), \quad \omega = \sqrt{g k_0} + O(\varepsilon),$
- Small amplitude: $k_0 \|\eta\| = O(\varepsilon)$

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3. Look for: $\theta = k_0 x - \omega(k_0) t,$

$$\eta(x,y,t;\varepsilon) = \varepsilon[A(\varepsilon x, \varepsilon y, \varepsilon t, \varepsilon^2 t) e^{i\theta} + A^* e^{-i\theta}]$$
$$+ \varepsilon^2 [stuff_2] + \varepsilon^3 [stuff_3] + O(\varepsilon^4)$$

Sketch derivation of NLS

4. Insert formal expansions for

   $$\eta(x,y,t;\varepsilon), \quad \phi(x,y,z,t;\varepsilon)$$

   into full equations. Solve, order by order.
5. Algebra is fearsome. (Use maple or…)

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6. Find:

- At O(ε): $\omega^2 = gk$ (linearized dispersion relation)
- At O(ε^2): expansion becomes disordered unless

$$\frac{\partial A}{\partial(\varepsilon t)} + c_g \frac{\partial A}{\partial(\varepsilon x)} = 0, \quad (c_g = \text{group velocity})$$

(Wave envelope travels at group velocity)

Sketch derivation of NLS

7. Recall $\eta = \varepsilon[A(\varepsilon x,\varepsilon y,\varepsilon t,\varepsilon^2 t)e^{i\theta} + A^* e^{-i\theta}] + O(\varepsilon^2)$

Define: $\xi = (\varepsilon x) - c_g(\varepsilon t), \quad \zeta = \varepsilon y, \quad \tau = \varepsilon^2 t.$

8. At O(ε^3), expansion becomes disordered (again) unless $A(\xi,\zeta,\tau)$ satisfies

$$i\partial_\tau A + \alpha\partial_\xi^2 A + \beta\partial_\zeta^2 A + \gamma\,|A|^2 A = 0$$

Nonlinear Schrödinger equation in 2-D

(α, β, γ) are real numbers, defined by problem

B. History of NLS

$$i\partial_\tau A + \alpha\partial_\xi^2 A + \beta\partial_\zeta^2 A + \gamma\,|A|^2\,A = 0$$

This equation (or something equivalent) was derived by:

- Zakharov, 1968 water waves
- Ostrovsky, 1967 optics (& generalizations)
- Benjamin & Feir, 1967 water waves (nearby)
- Benney & Newell, 1967 general
- Whitham, 1967 his formulation was used by:
- Lighthill, 1965 had the basic idea
- Stokes, 1847 water waves
  (no spatial dependence)
- Zakharov & Ostrovsky, 2008 – historical review

Q: Waves of permanent form on deep water?

1. Stokes (1847): Consider a spatially uniform train of plane waves

$$\eta(x,t;\varepsilon) = \varepsilon[A(\varepsilon^2 t)e^{i\theta} + A^* e^{-i\theta}] + O(\varepsilon^2)$$

➔ $$i\partial_\tau A + \gamma |A|^2 A = 0, \quad \gamma = -4k_0^2$$

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$$A(\tau) = (A_0 e^{i\phi})e^{i\gamma|A_0|^2 \tau}$$

$$\eta(x,t;\varepsilon) = 2\varepsilon |A_0| \cos\{k_0 x - \omega(k_0)t - (2\varepsilon k_0 |A_0|)^2 t\} + O(\varepsilon^2)$$

↑ Stokes' nonlinear correction to frequency

Q: Do waves of permanent form exist on deep water?

* Stokes (1847) found a nonlinear correction for water waves of permanent form, with finite amplitude.

He did **not** prove that waves of permanent form exist.

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- Nekrassov (1921) and Levi-Civita (1925) proved that such waves exist on deep water.
- Struik (1926) extended their result to water of any (constant) depth.
- Amick & Toland (1981a,b) obtained optimal results about existence of waves of permanent form in 2-D.

Q: Why don't we see uniform wave trains on deep water?

Loch Ness

Lake Superior

Are plane waves stable on deep water?

Photos from

Benjamin, 1967

(L = 2.3 m, h = 7.6 m,

60 m between photos)

Benjamin & Feir:

"On the disintegration of wavetrains in deep water, Part 1", 1967

Stability of plane waves on deep water

Zakharov (1968):

- NLS $$i\partial_\tau A + \alpha\partial_\xi^2 A + \beta\partial_\zeta^2 A + \gamma \mid A \mid^2 A = 0$$
- For gravity waves on deep water, $\alpha < 0$, $\beta > 0$, $\gamma < 0$
- "Stokes' wave", $A(\xi,\zeta,\tau) = \mid A_0 \mid e^{i\gamma |A_0|^2 \tau}$

  represents a uniform train of plane waves with finite amplitude in deep water
- Linearize NLS about a Stokes' wave, and determine its linear stability in NLS

Linear stability of a Stokes' wave

NLS: $$i\partial_\tau A + \alpha\partial_\xi^2 A + \beta\partial_\zeta^2 A + \gamma |A|^2 A = 0$$

Linearize about a Stokes' wave:

$$A(\xi,\zeta,\tau) = e^{i\gamma|A_0|^2\tau}[|A_0| + \mu\cdot u(\xi,\zeta,\tau) + i\mu\cdot v(\xi,\zeta,\tau)] + O(\mu^2)$$

Linear stability of a Stokes' wave

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➔ $$\partial_\tau v = \alpha\partial_\xi^2 u + \beta\partial_\zeta^2 u + 2\gamma\, |A_0|^2\, u,$$
$$-\partial_\tau u = \alpha\partial_\xi^2 v + \beta\partial_\zeta^2 v.$$

Linear stability of a Stokes' wave

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$$-\partial_\tau u = \alpha\partial_\xi^2 v + \beta\partial_\zeta^2 v.$$

Linear PDEs, constant coefficients

Seek $u = U\cdot e^{im\xi + il\zeta + \Omega\tau} + (c.c.), \quad v = V\cdot e^{im\xi + il\zeta + \Omega\tau} + (c.c.).$

Linear stability of a Stokes' wave

Algebraic equation determines linear stability:

$$\Omega^2 + (\alpha m^2 + \beta l^2)(\alpha m^2 + \beta l^2 - 2\gamma \mid A_0 \mid^2) = 0$$

Re(Ω) > 0 ➔

linear instability

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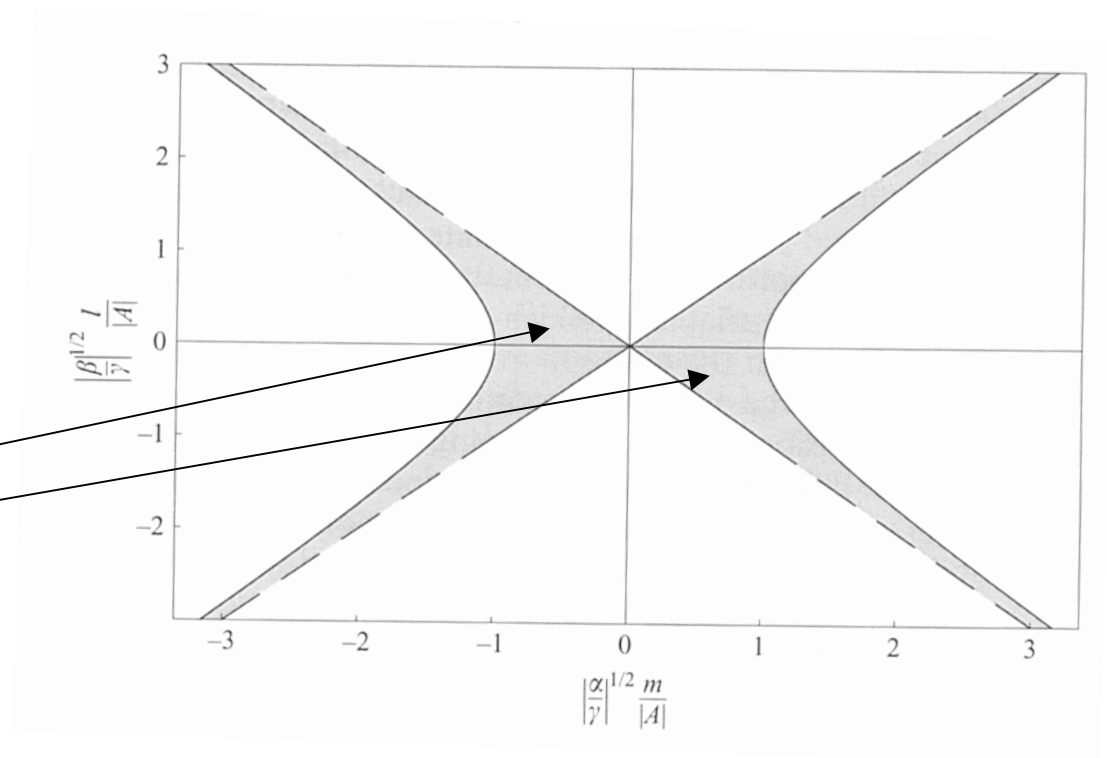
linear instability

Unstable
regions



Linear stability of a Stokes' wave

Result: In deep water, a uniform train of plane waves of finite amplitude is unstable!

The most unstable mode has a growth rate:

$$\Omega_{\mathrm{max}} = |\gamma|\,|A_0|^2.$$

(*Nonlinear instability* in the sense that the growth rate depends on $|A_0|$. As $|A_0| \to 0, \quad \Omega_{\mathrm{max}} \to 0$.)

Stability of a uniform wave train, according to NLS

$$i\partial_\tau A + \alpha\partial_\xi^2 A + \beta\partial_\zeta^2 A + \gamma \mid A \mid^2 A = 0$$

For other applications:

- $\alpha\beta < 0$, $\alpha\gamma > 0$ ➔ unstable
- $\alpha\beta < 0$, $\alpha\gamma < 0$ ➔ unstable
- $\alpha\beta > 0$, $\alpha\gamma > 0$ ➔ unstable
- $\alpha\beta > 0$, $\alpha\gamma < 0$ ➔ stable

Instability of a uniform train of water waves

-Benjamin & Feir

L = 2.3 m,

T = 1.2 s

Instability of a uniform train of EM waves in optical fiber -Tai, Hasegawa & Tomita (1986)

L = 1.3*10^{-6} m,

T = 4*10^{-15} s

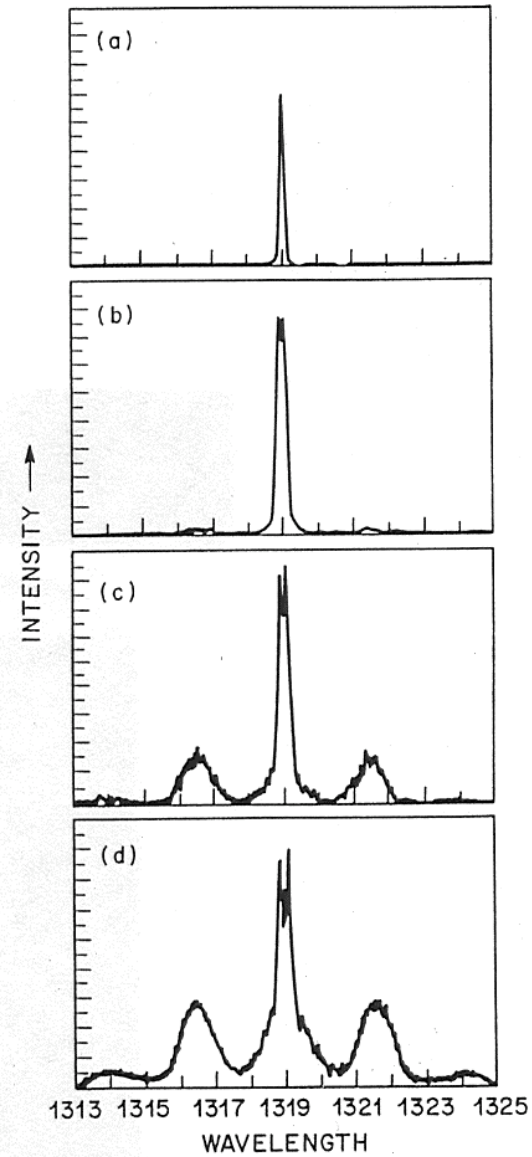


Fig.15.1 Experimental observation of modulational instability (Tai *et al.* 1986a). Input power level low (a); 5.5 *W* (b); 6.1 *W* (c); 7.1 *W* (d). For details see text.

Q: Are there stable wave patterns of permanent form on deep water?

- A#1 (Zakharov & others, about 1967)
  A uniform wave train is **not** stable

Q: Are there stable wave patterns of permanent form on deep water?

- A#1 (Zakharov & others, about 1967)
  A uniform wave train is **not** stable
- A#2 (Zakharov & Shabat, 1972)
  Consider NLS in 1-D

$$i\partial_\tau A = \partial_\xi^2 A + 2\sigma |A|^2 A$$

$\sigma = +1$ "focussing"
(gravity waves on deep water)
$\sigma = -1$ "defocussing"

Focussing NLS, in 1-D

$$i\partial_\tau A = \partial_\xi^2 A + 2\,|A|^2\,A$$

Look for travelling waves of "permanent form".

Special case:

$$A(\xi,\tau) = 2a \cdot e^{-i(2a)^2\tau}\,\mathrm{sec}\,h\{2a(\xi+\xi_0)\}$$

How does the free surface look?

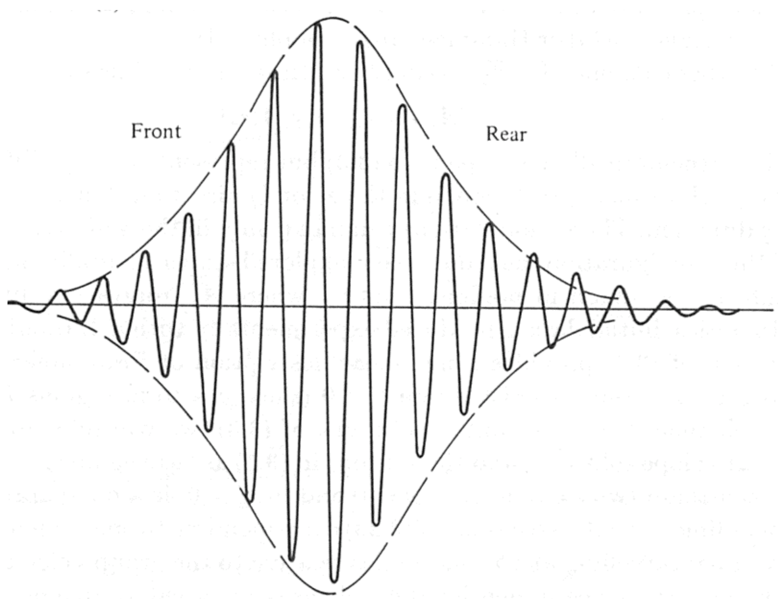
$$\eta(x,t;\varepsilon) = \varepsilon[Ae^{i\theta} + A^* e^{-i\theta}] + O(\varepsilon^2)$$

$$\eta = (2\varepsilon a)\,\mathrm{sec}\,h\{(2\varepsilon a)(x - c_g t)\}\cos\{kx - [\omega(k) + (2\varepsilon a)^2]t\}$$

wave packet with special shape

$$i\partial_\tau A = \partial_\xi^2 A + 2\,|A|^2\,A$$

$$\eta(x,t;\varepsilon) = 2(2\varepsilon a)\sec h\{(2\varepsilon a)(x - c_g t)\}\cos\{kx - [\omega(k) + (2\varepsilon a)^2]t\}$$



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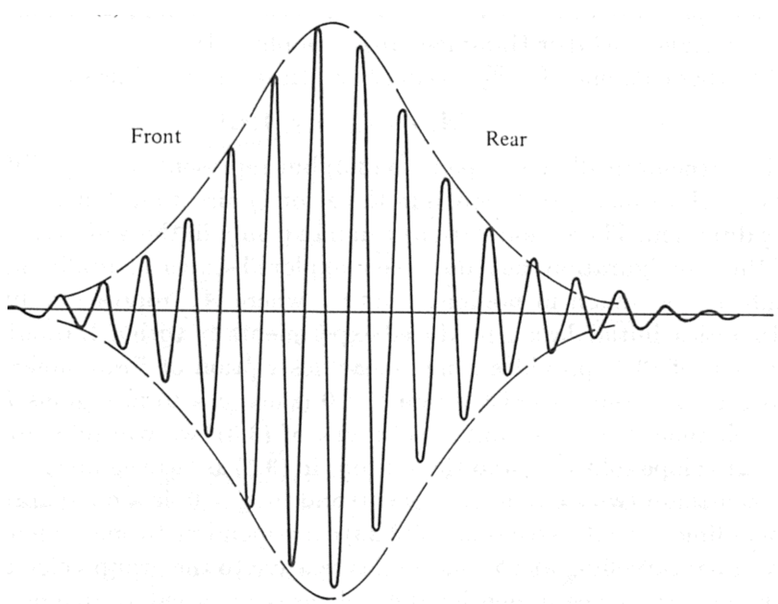


Nice, but so what?

$$i\partial_\tau A = \partial_\xi^2 A + 2\sigma |A|^2 A$$

Zakharov & Shabat (1972):

- NLS in 1-D is completely integrable, for either sign of σ! Just like KdV.
- Conservation laws

$$i\partial_\tau(|A|^2) = \partial_\xi(A^*\partial_\xi A - A\partial_\xi A^*),$$

$$i\partial_\tau(A^*\partial_\xi A - A\partial_\xi A^*) = \partial_\xi(\ldots),$$

$$i\partial_\tau(|\partial_\xi A|^2 + \sigma |A|^4) = \partial_\xi(\ldots),$$

$$\vdots$$

$$i\partial_\tau A = \partial_\xi^2 A + 2\sigma |A|^2 A$$

Zakharov & Shabat (1972):

- Scattering problem:

$$\partial_\xi v_1 = -i\lambda v_1 + A v_2,$$
$$\partial_\xi v_2 = -\sigma A^* v_1 + i\lambda v_2,$$

- Time-dependence:

$$\partial_\tau v_1 = [2i\lambda^2 + i\sigma |A|^2] v_1 + [2A\lambda + i\partial_\xi A] v_2,$$
$$\partial_\tau v_2 = [-2\sigma A^* \lambda + i\sigma\partial_\xi A^*] v_1 - [2i\lambda^2 + i\sigma |A|^2] v_2,$$

- Compatibility:

$$\partial_\xi \partial_\tau \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \partial_\tau \partial_\xi \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \Leftrightarrow \quad \boxed{i\partial_\tau A = \partial_\xi^2 A + 2\sigma |A|^2 A}$$

$$i\partial_\tau A = \partial_\xi^2 A + 2\sigma |A|^2 A$$

Zakharov & Shabat (1972):

For σ = +1 (focussing NLS):

- Any smooth initial data, $A(\xi,0)$, with $\int |A| d\xi < \infty$ evolves under focussing NLS into a N "envelope solitons", which persist forever, plus an oscillatory wavetrain that decays in amplitude as $\tau \to \infty$

- Envelope solitons are stable, within NLS in 1-D.

Stability of envelope solitons - experimental evidence

a) 6 m from wavemaker

b) 30 m from Wavemaker

$k = 4\ \mathrm{m}^{-1}$

- Hammack

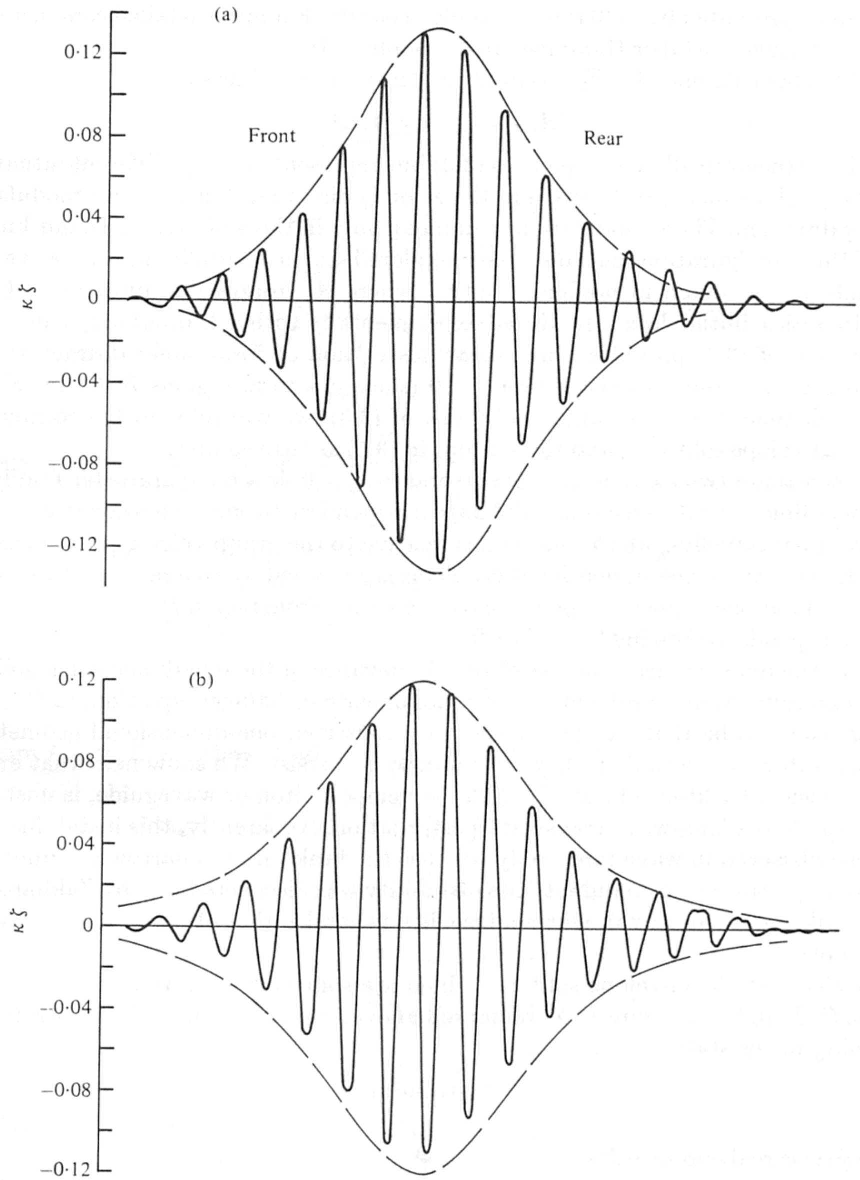


FIGURE 3. Measured surface displacement, showing evolution of envelope soliton at two downstream locations; $h = 1$ m, $kh = 4{\cdot}0$, $\omega = 1$ Hz, $\tilde{T} = 1{\cdot}0 \times 10^{-4}$; ——, measured history of surface displacement; – – –, theoretical envelope shape;

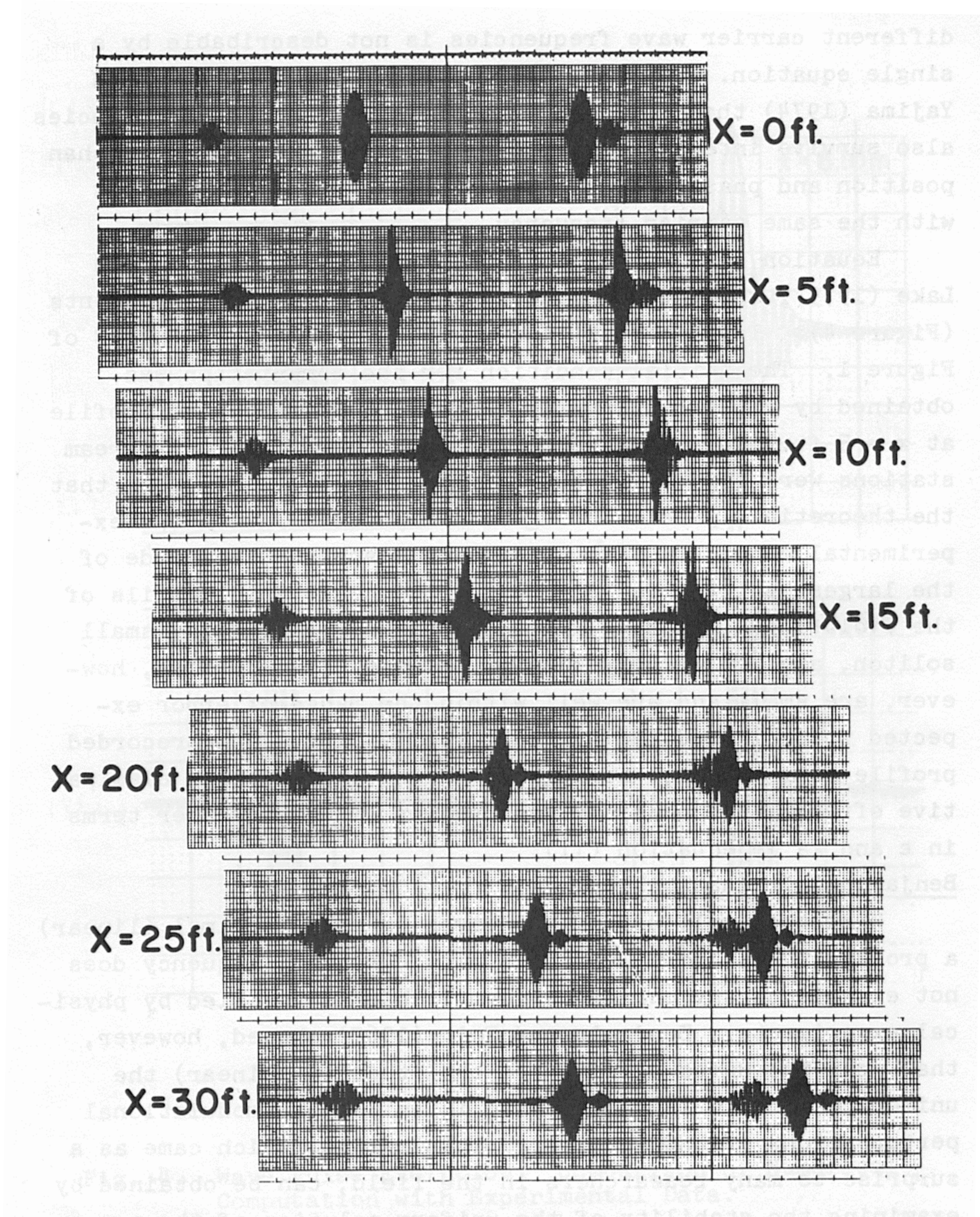
$$\kappa\zeta = \kappa a \operatorname{sech}(z),$$

$$z = [ag/\omega]\,(\nu/8\lambda)^{\frac{1}{2}}\,(C_g t - x);$$

(*a*) 6 m downstream of wave maker, $\kappa a = 0{\cdot}132$. (*b*) 30 m downstream of wave maker, $\kappa a = 0{\cdot}116$.

Stability of envelope solitons -

3 experiments by Yuen & Lake (1975)



Tentative conclusions

- According to NLS in 1-D or 2-D, a uniform train of plane waves is unstable in deep water.
- According to focussing NLS in 1-D with initial data in L_1, envelope solitons are stable in deep water.
- Experimental evidence seems to support these conclusions

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- According to NLS in 1-D or 2-D, a uniform train of plane waves is unstable in deep water.
- According to focussing NLS in 1-D with initial data in L_1, envelope solitons are stable in deep water.
- Experimental evidence seems to support these conclusions
- But wait for Lecture 14