Waves on deep water, II

Lecture 14

Main question: Are there stable wave patterns that propagate with permanent form (or nearly so) on deep water?

Main approximate model:

$$i\partial_\tau A + \alpha\partial_\xi^2 A + \beta\partial_\zeta^2 A + \gamma |A|^2 A = 0$$

Nonlinear Schrödinger equation (NLS)

Waves on deep water

1. The story so far

- A uniform train of periodic waves is unstable on deep water, according to NLS and to experiments.

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1. The story so far

- A uniform train of periodic waves is unstable on deep water, according to NLS and to experiments
- The 1-D NLS equation is completely integrable!
- For focussing NLS in 1-D on $(-\infty,\infty)$, arbitrary initial data evolve into a finite number of envelope solitons, plus a modulated wavetrain that disperses (so its amplitude decays) as $\tau \to \infty$
- Envelope solitons are stable in 1-D NLS.

[For defocussing NLS, “dark solitons” are stable.]

Waves on deep water

Chapter 2:

Near recurrence of initial states

a) Lake, Yuen, Rungaldier & Ferguson, 1977 proposed (correctly) that with periodic boundary conditions, focussing NLS should exhibit near recurrence of initial states, just as KdV does.

Waves on deep water

b) What is “near recurrence of initial states” ?

Example from linearized equations on deep water, with periodic boundary conditions:

$$\eta(x,t) = \sum_{m=1}^{N} a_m \cos\{mx - \omega_m t + \phi_m\}, \quad \omega_m^2 = gm.$$

Waves on deep water

b) What is “near recurrence of initial states”?

Example from linearized equations on deep water, with periodic boundary conditions:

$$\eta(x,t) = \sum_{m=1}^{N} a_m \cos\{mx - \omega_m t + \phi_m\}, \quad \omega_m^2 = gm.$$

Frequencies are not rationally related: $\omega_m = \omega_1 \sqrt{m}$

➔ $\eta(x,t)$ is not periodic in time, but for finite N the solution returns close to its initial state, over and over again

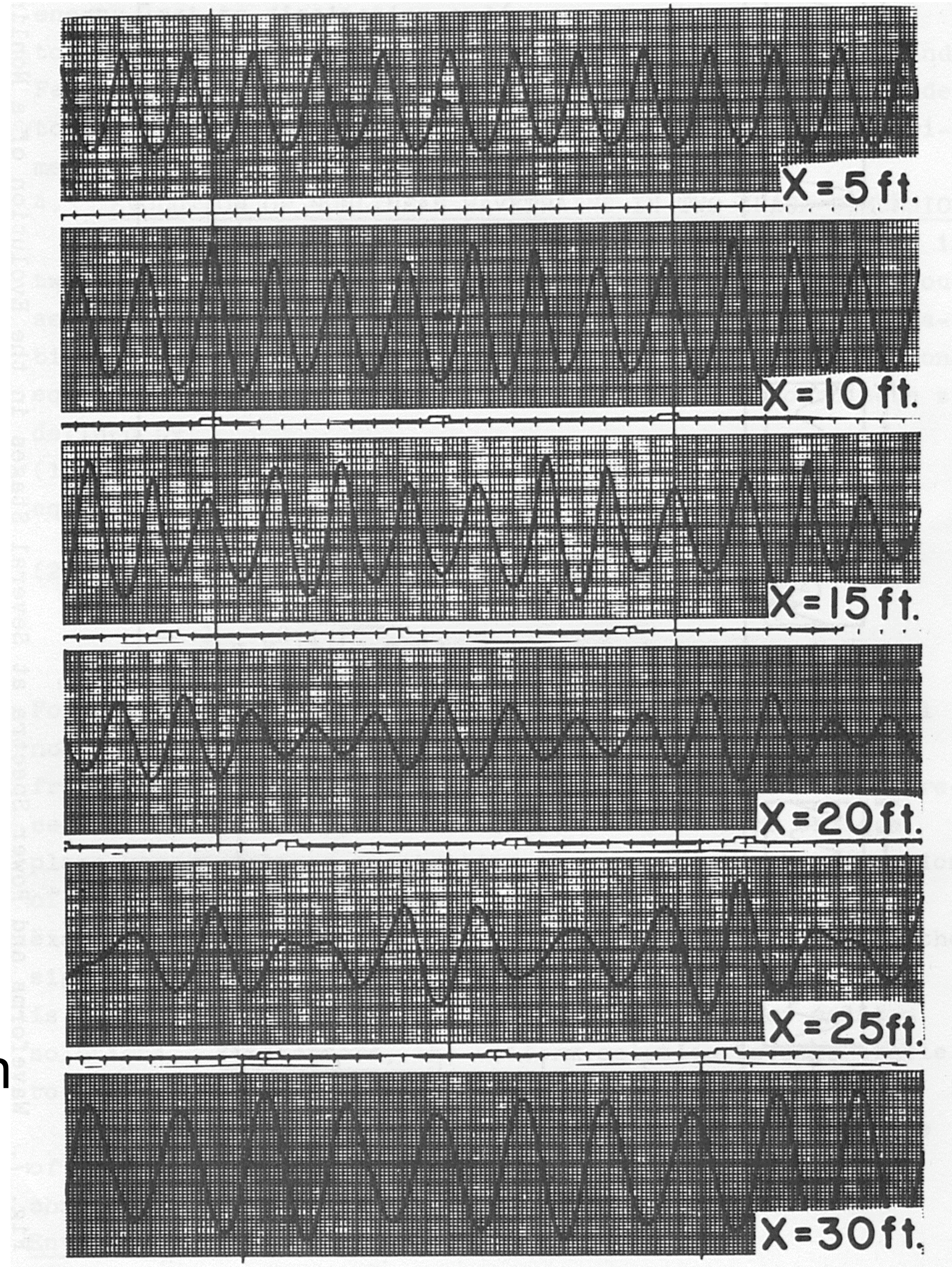
Experimental evidence of recurrence in deep water – Lake *et al*, 1977

Initial frequency:

ω = 3.6 Hz

λ = 12 cm

[First physical observation of FPU recurrence?]



Q: Stable wave patterns on deep water ?

A#1. NLS in 1-D with periodic b.c.:

- A uniform train of oscillatory plane waves is unstable
- But a continuous wave train exhibits near recurrence of initial states.

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A#1. NLS in 1-D with periodic b.c.:

- A uniform train of oscillatory plane waves is unstable
- But a continuous wave train exhibits near recurrence of initial states.

A#2. NLS in 1-D with localized initial data:

- Envelope solitons are stable

  (Envelope solitons have played an important role in communication through optical fibers)

Q: What about a 2-D free surface?
(so a 3-D fluid flow)

1. 2-D NLS: $$i\partial_\tau A + \partial_\xi^2 A + \beta\partial_\zeta^2 A + 2\sigma |A|^2 A = 0$$

- σ = +1 for envelope solitons
- σ = -1 for dark solitons

Q: What about a 2-D free surface?

(so a 3-D fluid flow)

1. 2-D NLS: $$i\partial_\tau A + \partial_\xi^2 A + \beta\partial_\zeta^2 A + 2\sigma |A|^2 A = 0$$

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2. Zakharov & Rubenchik, 1974

- σ = +1: for either sign of β, envelope solitons are unstable to 2-D perturbations
- σ = -1: for either sign of β, dark solitons are unstable to 2-D perturbations
- The unstable perturbations have long transverse wavelengths

  (Problem in water waves, but not necessarily in optical fibers)

Recall experiment by Hammack on envelope soliton

(a) 6 m from wavemaker
(b) 30 m from wavemaker

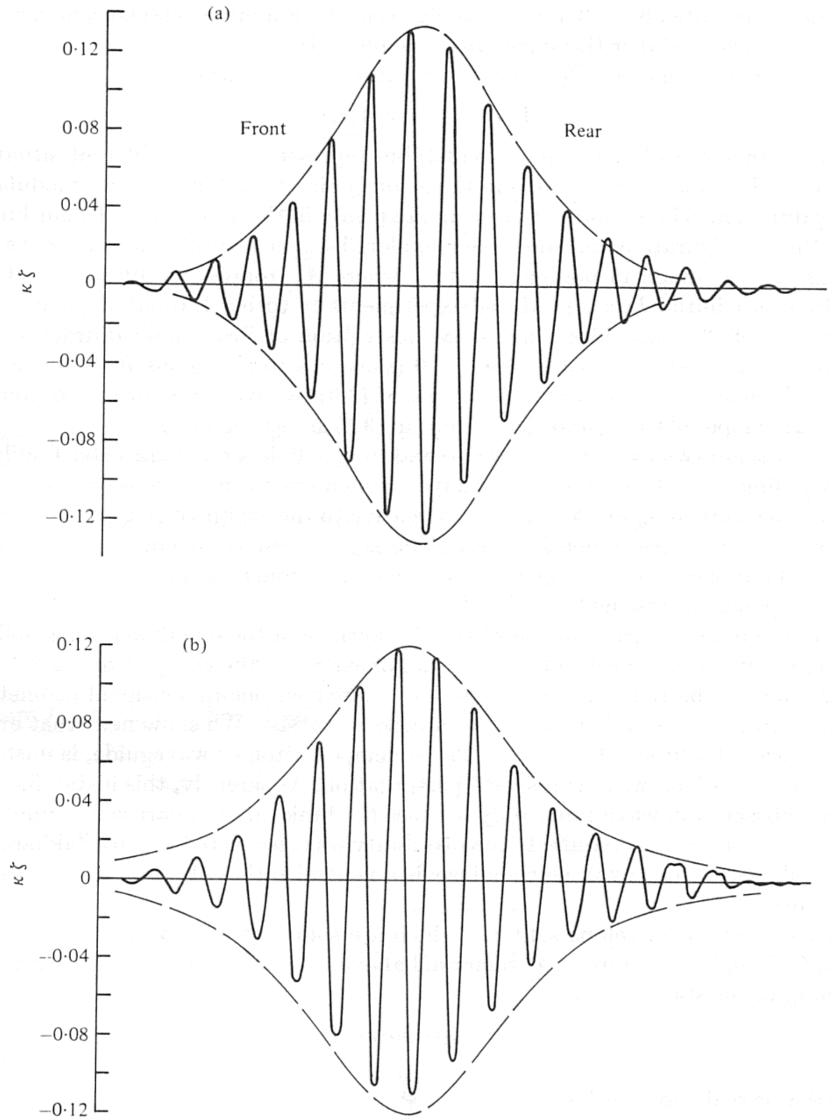

FIGURE 3. Measured surface displacement, showing evolution of envelope soliton at two downstream locations; $h = 1$ m, $kh = 4{\cdot}0$, $\omega = 1$ Hz, $\tilde{T} = 1{\cdot}0 \times 10^{-4}$; ——, measured history of surface displacement; – – –, theoretical envelope shape;

$$\kappa\zeta = \kappa a \operatorname{sech}(z),$$

$$z = [ag/\omega]\,(\nu/8\lambda)^{\frac{1}{2}}\,(C_g t - x);$$

(a) 6 m downstream of wave maker, $\kappa a = 0{\cdot}132$. (b) 30 m downstream of wave maker, $\kappa a = 0{\cdot}116$.

Hammack repeated the experiment, using the same wavemaker, in a wider tank

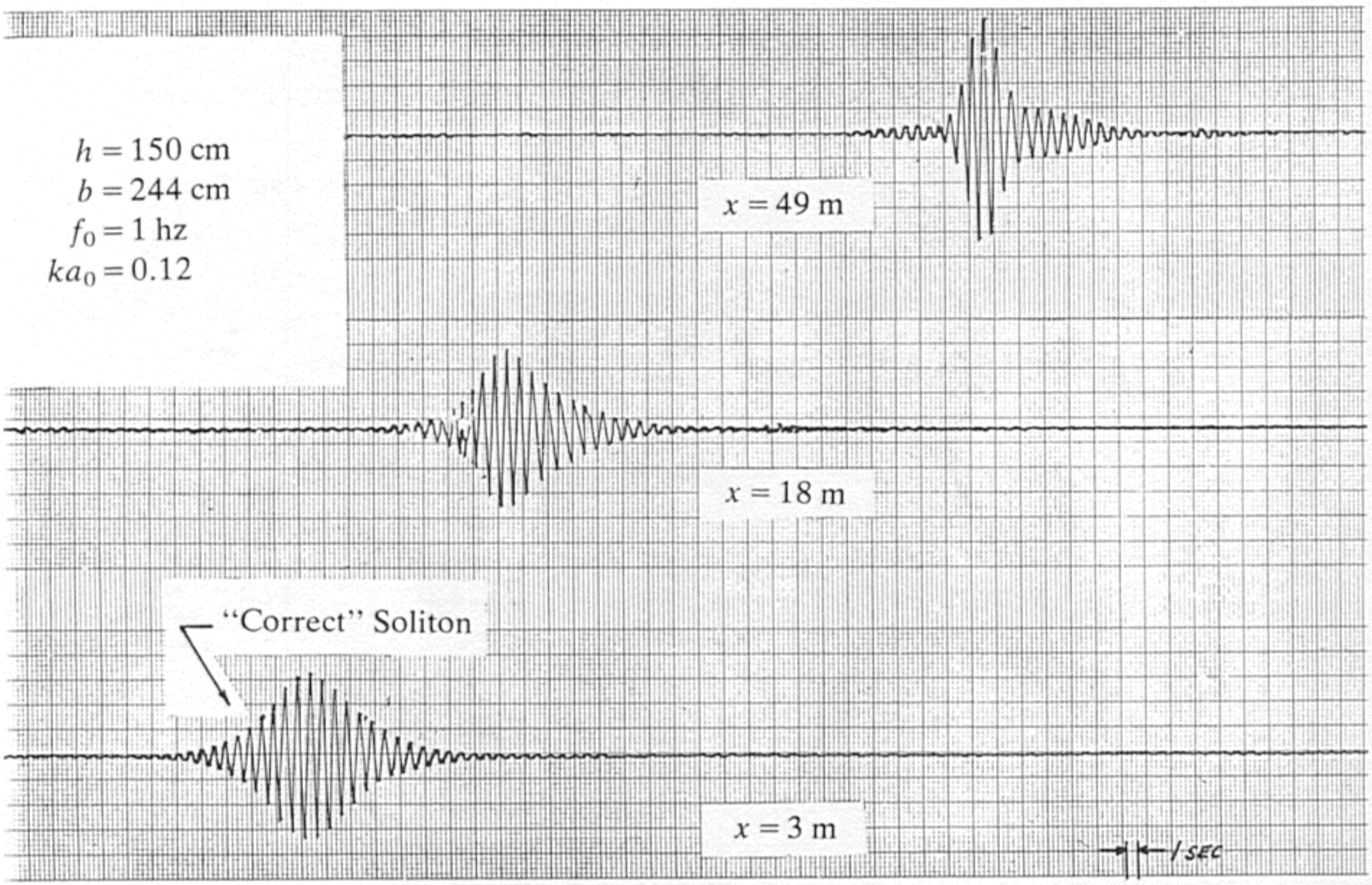

FIG. 4.18. *Evolution of water wave packet in a wide tank, showing the transverse instability that was absent in Fig. 4.16.* (*Courtesy of J. L. Hammack*).

Q: Stable patterns that propagate with (nearly) permanent form on 2-D surface in deep water?

A. The story continues - stay tuned

Intermission: wave collapse in 2-d

Zakharov & Synakh, 1976:

- Consider elliptic, focusing NLS in 2-D

$$i\partial_\tau A + \partial_\xi^2 A + \partial_\zeta^2 A + 2|A|^2 A = 0$$

(same signs for all coefficients ➔ not gravity waves)

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$$i\partial_\tau A + \partial_\xi^2 A + \partial_\zeta^2 A + 2|A|^2 A = 0$$

(same signs for all coefficients ➔ not gravity waves)

- Conserved quantities (finite list):

$$I_1 = \iint [|A|^2] d\xi d\zeta,$$

$$I_2 = \iint [A\partial_\xi A^* - A^*\partial_\xi A] d\xi d\zeta, \quad I_3 = \iint [A\partial_\zeta A^* - A^*\partial_\zeta A] d\xi d\zeta,$$

$$I_4 = H = \iint [|\nabla A|^2 - |A|^4] d\xi d\zeta.$$

Intermission: wave collapse in 2-d

$$i\partial_\tau A + \partial_\xi^2 A + \partial_\zeta^2 A + 2\,|A|^2\, A = 0$$

- Consider $J(\tau) = \iint [(\xi^2 + \zeta^2)\,|A|^2]\,d\xi d\zeta$

  If we interpret:

  $|A|^2(\xi, \zeta, \tau)$ as "mass density", then

  $I_1 = \iint [|A|^2]\,d\xi d\zeta$ is "total mass", and

  $J(\tau)$ is "moment of inertia".

  $J(\tau) \geq 0$.

Intermission: wave collapse in 2-d

$$i\partial_\tau A + \partial_\xi^2 A + \partial_\zeta^2 A + 2|A|^2 A = 0$$

- Consider $J(\tau) = \iint [(\xi^2 + \zeta^2)|A|^2] d\xi d\zeta$
- Compute $\frac{dJ}{d\tau}$ and $\frac{d^2 J}{d\tau^2}$
- Find: $\frac{d^2 J}{d\tau^2} = 8H = 8\iint [|\nabla A|^2 - |A|^4] d\xi d\zeta.$

➔ If $H < 0$, then $J(\tau) < 0$ in finite time. (Bad!)

This happens while I_1, I_2, I_3, H are conserved.

[Wave collapse has been important in nonlinear optics.]

Back to the main story

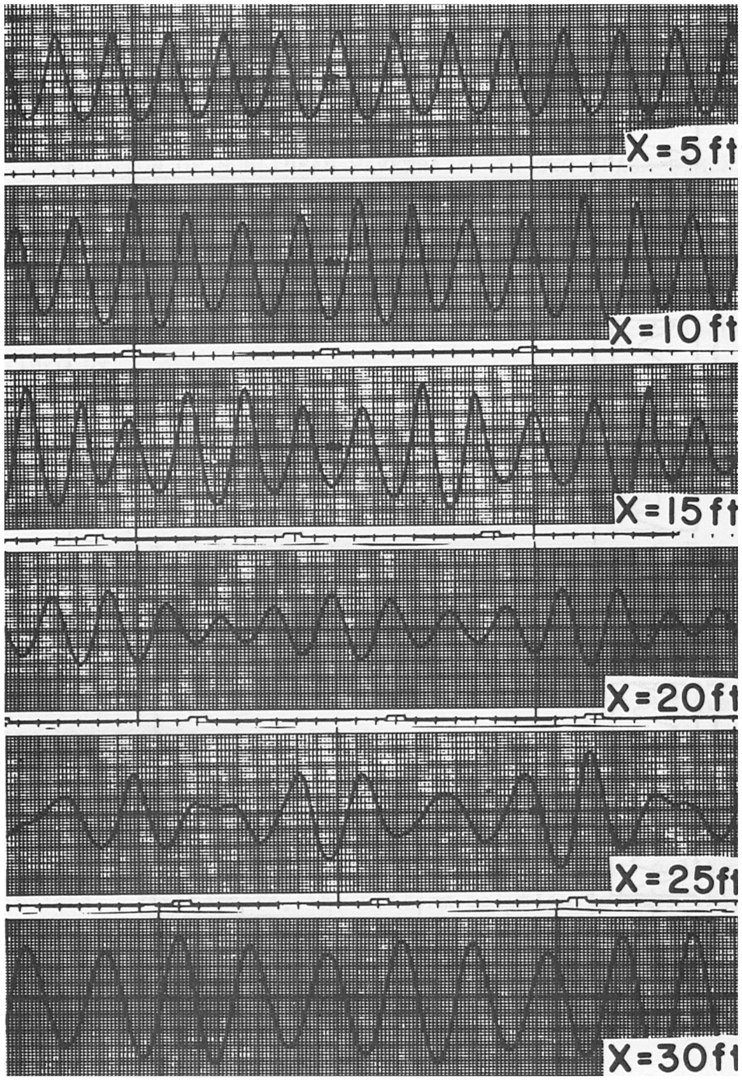
Q: Are there stable wave patterns that propagate with permanent form (or nearly so) on a 2-D free surface in deep water?

More complication:

Lake, Yuen, Rungaldier & Ferguson (1977)

Recall "near recurrence of initial states"

Lake, Yuen, Rungaldier & Ferguson



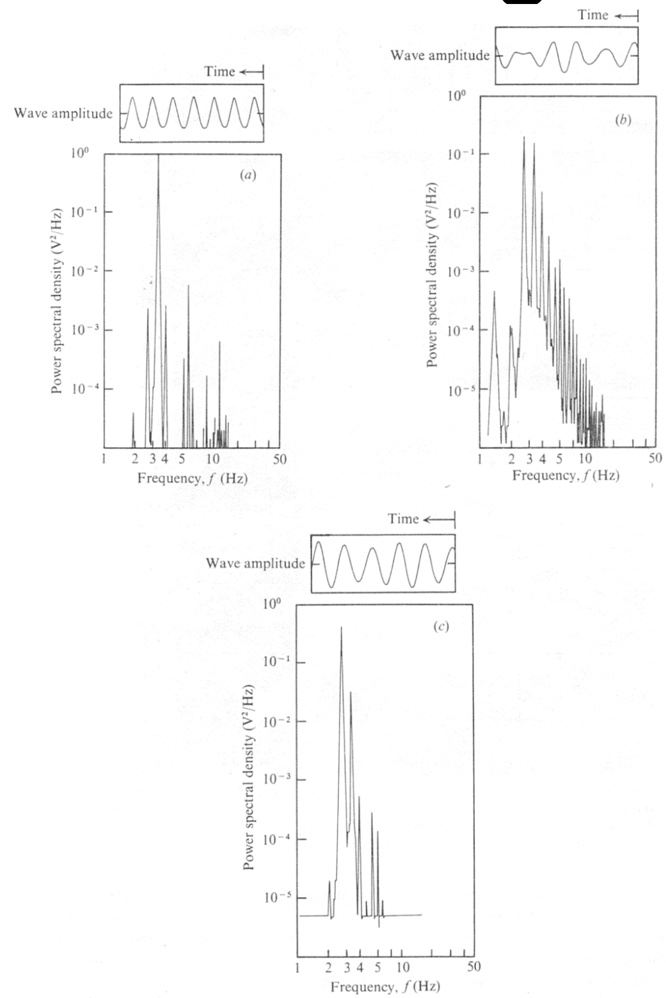


FIGURE 6. Evolution of a nonlinear finite amplitude wave train: wave forms and power spectral densities *vs.* propagation distance. (*a*) Initial stage of side-band growth, $x = 5$ ft, carrier wave with small amplitude modulation. (*b*) $x = 10$ ft, strong amplitude modulation, energy spread over many frequency components. (*c*) $x = 25$ ft, reduced amplitude modulation, return of energy to frequency components of original carrier wave, its side bands and harmonics. $f_0 = 3{\cdot}25$ Hz, $(ka)_0 = \delta = 0{\cdot}23$, $(ka)_{5\,ft} = 0{\cdot}29$.

Frequency downshifting – also seen in optics

Frequency downshifting – different from recurrence

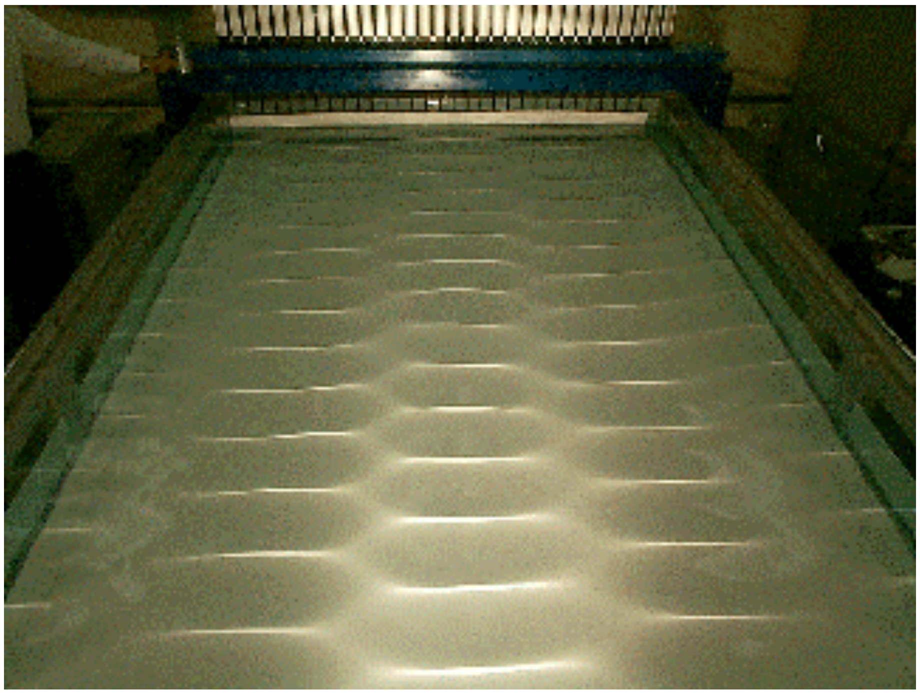
- Frequency downshifting does not occur in simulations based on NLS, in 1-D or 2-D
- It does not occur in simulations based on Dysthe's (1979) generalization of NLS
- It has been observed & studied in optics (Mollenauer, 1986; Gordon, 1986)
- My opinion: No satisfactory model of the process has been found

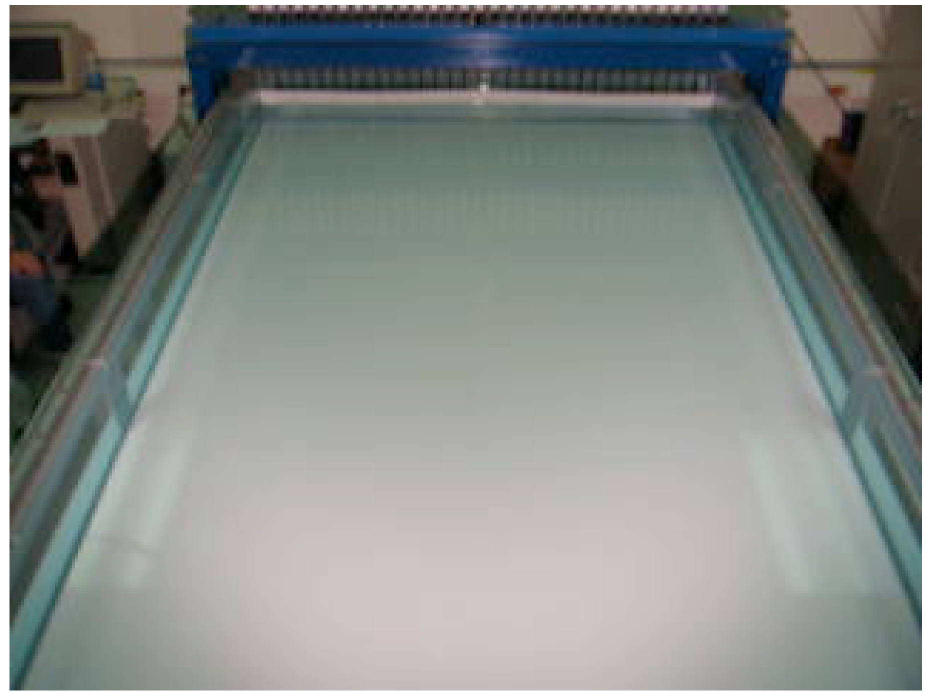
Q: Stable patterns that propagate with (nearly) permanent form on 2-D surface in deep water?

1990s – Joe Hammack built a new tank to study 2-D wave patterns (so 3-D fluid flows) on deep water

Experimental evidence of apparently stable wave patterns in deep water

(www.math.psu.edu/dmh/FRG)





3 Hz	frequency	4 Hz
17.3 cm	wavelength	9.8 cm

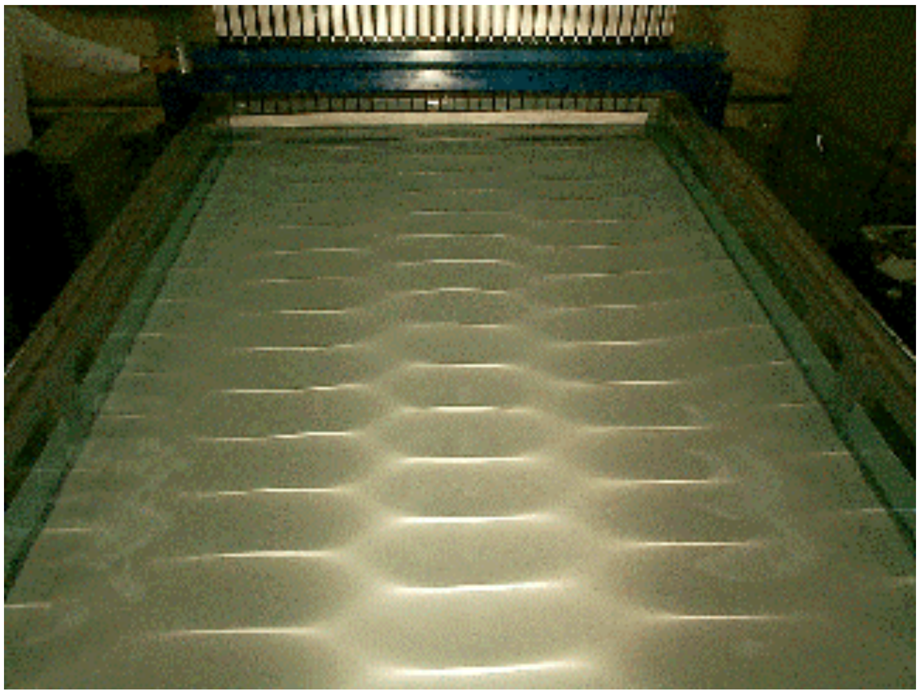
How to reconcile the experimental observations with Benjamin-Feir instability?

Options

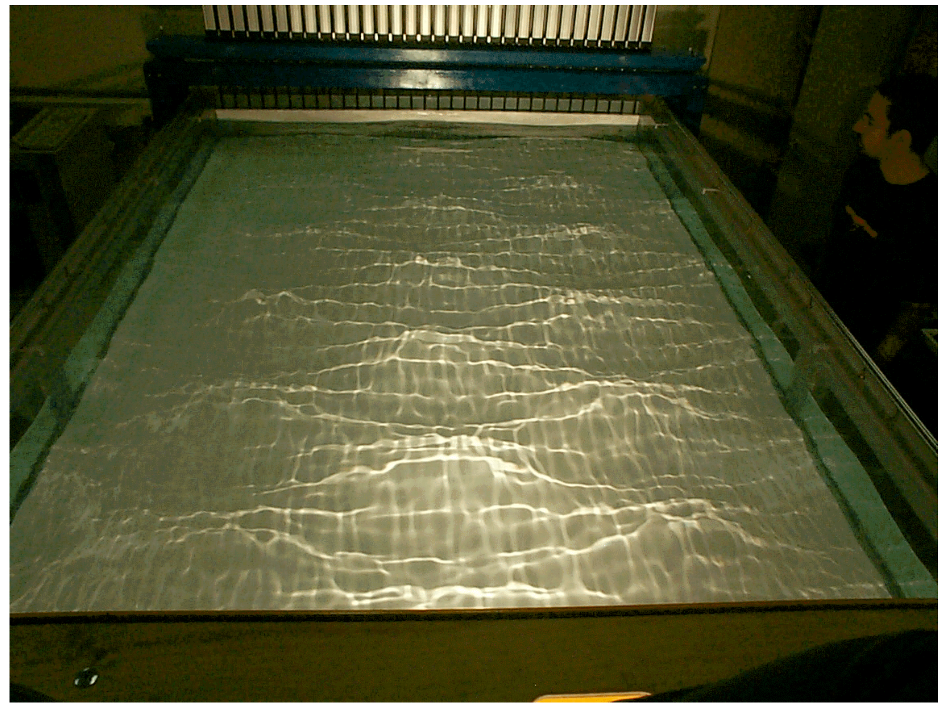
- Modulational instability afflicts 1-D plane waves, but not 2-D periodic patterns
- The Penn State tank is too short to observe the (relatively slow) growth of the instability
- Other (please specify)

More experimental results

(www.math.psu.edu/dmh/FRG)



3 Hz
old water



2 Hz
new water

Main results

- The modulational (or Benjamin-Feir) instability is valid for waves in deep water without dissipation

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- The modulational (or Benjamin-Feir) instability is valid for waves in deep water without dissipation

- But **any** amount of damping (of the right kind) stabilizes the instability (according to NLS & exp's)
- This dichotomy (with *vs*. without damping) applies to both 1-D plane waves and to 2-D periodic surface patterns
- Segur, Henderson, Carter, Hammack, Li, Pheiff, Socha, 2005
- Controversial

Stability *vs*. existence
in full water-wave equations

Recall:

- Craig & Nicholls (2000) prove that the full equations of (inviscid) water waves, with gravity and surface tension, admit solutions with 2-D, periodic surface patterns of permanent form on deep water.
- Iooss & Plotnikov (2008) prove the existence of such patterns for (some) pure gravity waves on deep water.

Neither paper considers stability.

Reconsider stability of plane waves in 1-D

$$i(\partial_t A + c_g \partial_x A) + \varepsilon[\alpha \partial_x^2 A + \gamma |A|^2 A \qquad] = 0$$

$$[\xi = t - \frac{x}{c_g}, X = \varepsilon \frac{x}{c_g}]$$

$$i\partial_X A + \alpha \partial_\xi^2 A + \gamma |A|^2 A \qquad = 0$$

Reconsider stability of plane waves in 1-D, with damping

$$i(\partial_t A + c_g \partial_x A) + \varepsilon[\alpha \partial_x^2 A + \gamma |A|^2 A + i\delta A] = 0$$

$$[\xi = t - \frac{x}{c_g}, X = \varepsilon \frac{x}{c_g}]$$

$$\delta \geq 0$$

$$i\partial_X A + \alpha \partial_\xi^2 A + \gamma |A|^2 A + i\delta A = 0$$

$$[A(\xi, X) = e^{-\delta X} \mathcal{A}(\xi, X)]$$

$$i\partial_X \mathcal{A} + \alpha \partial_\xi^2 \mathcal{A} + \gamma \cdot e^{-2\delta X} |\mathcal{A}|^2 \mathcal{A} = 0$$

NLS in 1-D, cont'd

$$i\partial_X \mathcal{A} + \alpha\partial_\xi^2 \mathcal{A} + \gamma \cdot e^{-2\delta X} \, |\mathcal{A}|^2 \, \mathcal{A} = 0$$

Hamiltonian equation, but $\frac{dH}{dX} \neq 0$

$$H = i\int [\alpha \, |\partial_\xi \mathcal{A}|^2 - \frac{\gamma}{2} e^{-2\delta X} \, |\mathcal{A}|^4] d\xi$$

Conjugate variables: A, A*

$$i\partial_X \mathcal{A} + \alpha\partial_\xi^2 \mathcal{A} + \gamma \cdot e^{-2\delta X} \, |\mathcal{A}|^2 \, \mathcal{A} = 0$$, cont'd

- Uniform (in ξ) wave train:

$$\mathcal{A} = \mathcal{A}_0 \exp\{i\gamma \, |\mathcal{A}_0|^2 \, (\frac{1 - e^{-2\delta X}}{2\delta})\}$$

- Perturb:

$$\mathcal{A}(\tau, X) = \exp\{i\gamma \, |\mathcal{A}_0|^2 \, (\frac{1 - e^{-2\delta X}}{2\delta})\}[|\mathcal{A}_0| + \mu(u + iv)]\} + O(\mu^2)$$

- …algebra..

$$\frac{d^2\hat{u}}{dX^2} + [\alpha m^2(\alpha m^2 - 2\gamma \cdot e^{-2\delta X} \, |\mathcal{A}_0|^2)] \cdot \hat{u} = 0$$

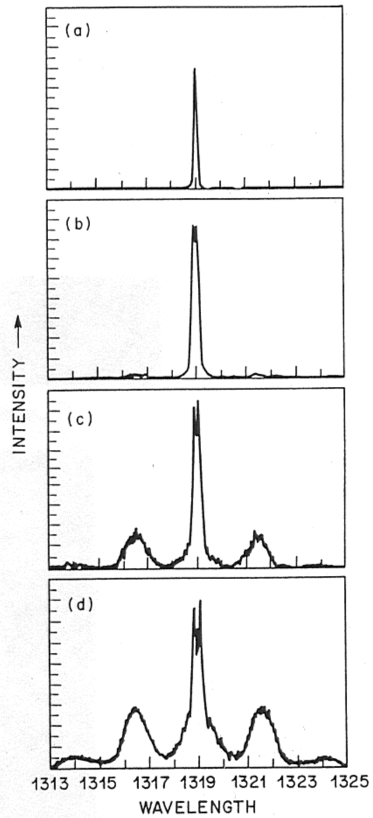

Fig.15.1 Experimental observation of modulational instability (Tai *et al.* 1986a). Input power level low (a); 5.5 W (b); 6.1 W (c); 7.1 W (d). For details see text.

If we eliminate σ_1 from (15.1.11) and (15.1.12) and construct the differential equation for the normalized side band amplitude $\overline{\rho}_1 = \rho_1/\rho_0$ (ρ_0 is given by (15.1.9)), we get

$$\frac{d^2\overline{\rho}}{dZ^2} - \Omega^2\left(\overline{\rho}_0 e^{-2\Gamma Z} - \frac{\Omega^2}{4}\right)\overline{\rho} = 0 \ . \qquad (15.2.1)$$

If we introduce a quantity R which designates the ratio of Ω^2 to ρ_0, $R = \Omega^2/\overline{\rho}_0$, R may be expressed in terms of engineering parameters as

$$R = \frac{\Omega^2}{\rho_0} = 1.1 \times 10^4 \frac{f^2 S}{P}(-\lambda^3 D) \ , \qquad (15.2.2)$$

$$\frac{d^2\hat{u}}{dX^2} + [\alpha m^2(\alpha m^2 - 2\gamma \cdot e^{-2\delta X} \, |\mathcal{A}_0|^2)] \cdot \hat{u} = 0$$

Hasegawa &Kodama

(1995)

$$\frac{d^2\hat{u}}{dX^2} + [\alpha m^2(\alpha m^2 - 2\gamma \cdot e^{-2\delta X} \, |\mathcal{A}_0|^2)] \cdot \hat{u} = 0, \text{ cont'd}$$

$$\frac{d^2\hat{u}}{dX^2} + [\alpha m^2(\alpha m^2 - 2\gamma \cdot e^{-2\delta X} \, |\mathcal{A}_0|^2)] \cdot \hat{u} = 0, \text{ cont'd}$$

- There is a growing mode if

$$[\alpha m^2(\alpha m^2 - 2\gamma \cdot e^{-2\delta X} \, |A_0|^2)] < 0$$

$$\frac{d^2\hat{u}}{dX^2} + [\alpha m^2(\alpha m^2 - 2\gamma \cdot e^{-2\delta X} \, |\mathcal{A}_0|^2)] \cdot \hat{u} = 0,$$ cont'd

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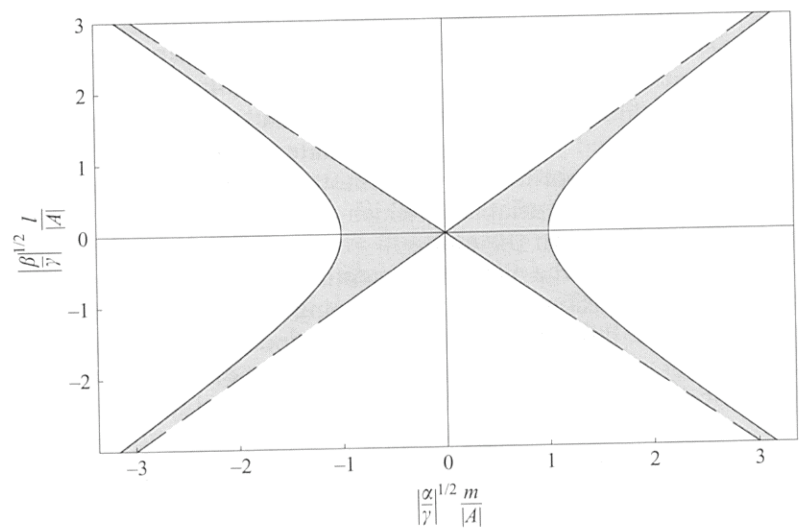


$$\frac{d^2\hat{u}}{dX^2} + [\alpha m^2(\alpha m^2 - 2\gamma \cdot e^{-2\delta X} \,|\mathcal{A}_0|^2)] \cdot \hat{u} = 0,$$ cont'd

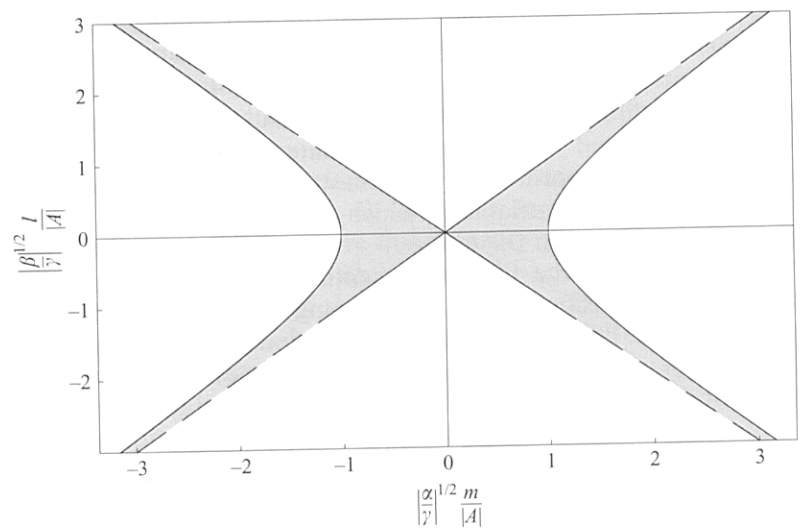
- There is a growing mode if

$$[\alpha m^2(\alpha m^2 - 2\gamma \cdot e^{-2\delta X} \,|A_0|^2)] < 0$$

- For any $\delta > 0$, growth stops eventually

➔No mode grows forever

➔Total growth is bounded



What is "linearized stability"?

(Lyapunov)

A uniform wave train solution is linearly stable if for every $\varepsilon > 0$ there is a $\Delta(\varepsilon) > 0$ such that if a perturbation (u,v) satisfies

$$\int [u^2(\xi,0) + v^2(\xi,0)]d\xi < \Delta(\varepsilon) \qquad \text{at } X = 0,$$

then necessarily

$$\int [u^2(\xi,X) + v^2(\xi,X)]d\xi < \varepsilon \qquad \text{for all } X > 0.$$

1-D NLS with damping, conclusion

$$\frac{d^2\hat{u}}{dX^2} + [\alpha m^2(\alpha m^2 - 2\gamma \cdot e^{-2\delta X} \,|\mathcal{A}_0|^2)] \cdot \hat{u} = 0$$

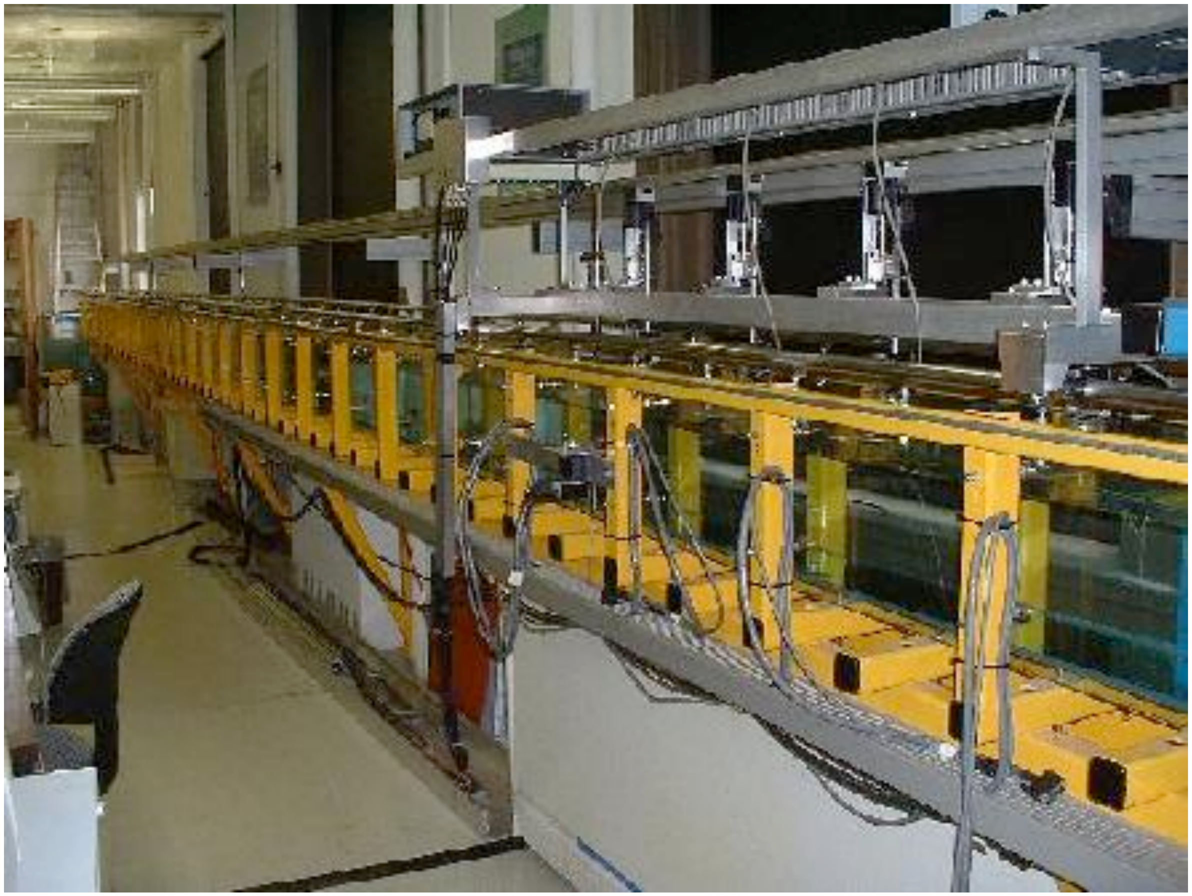
➔There is a universal bound, B: the total growth of any Fourier mode cannot exceed B

➔To demonstrate stability, choose $\Delta(\varepsilon)$ so that

$$\Delta(\varepsilon) < \frac{1}{B^2} \cdot \varepsilon$$

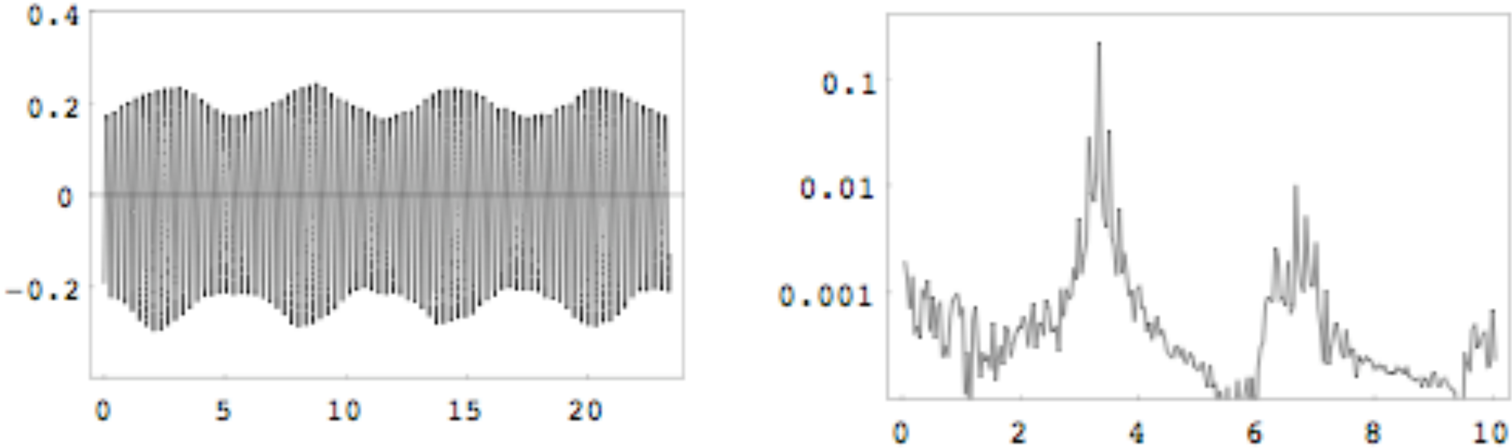
Nonlinear stability is similar, but more complicated

Experimental verification of theory

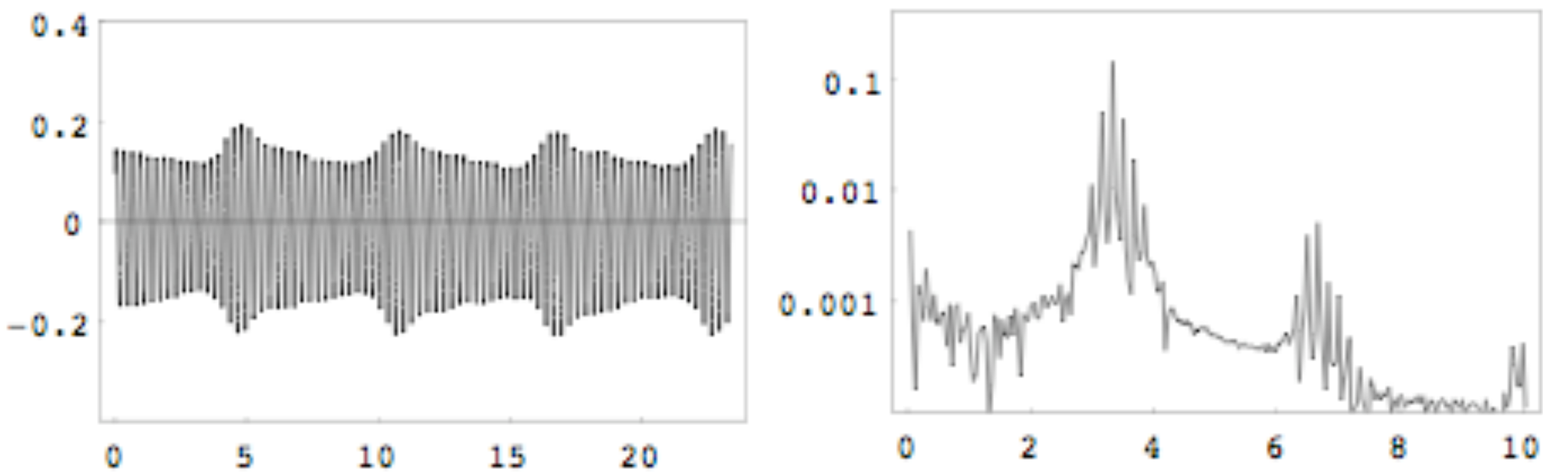


(old) 1-D tank at Penn State

Experimental wave records

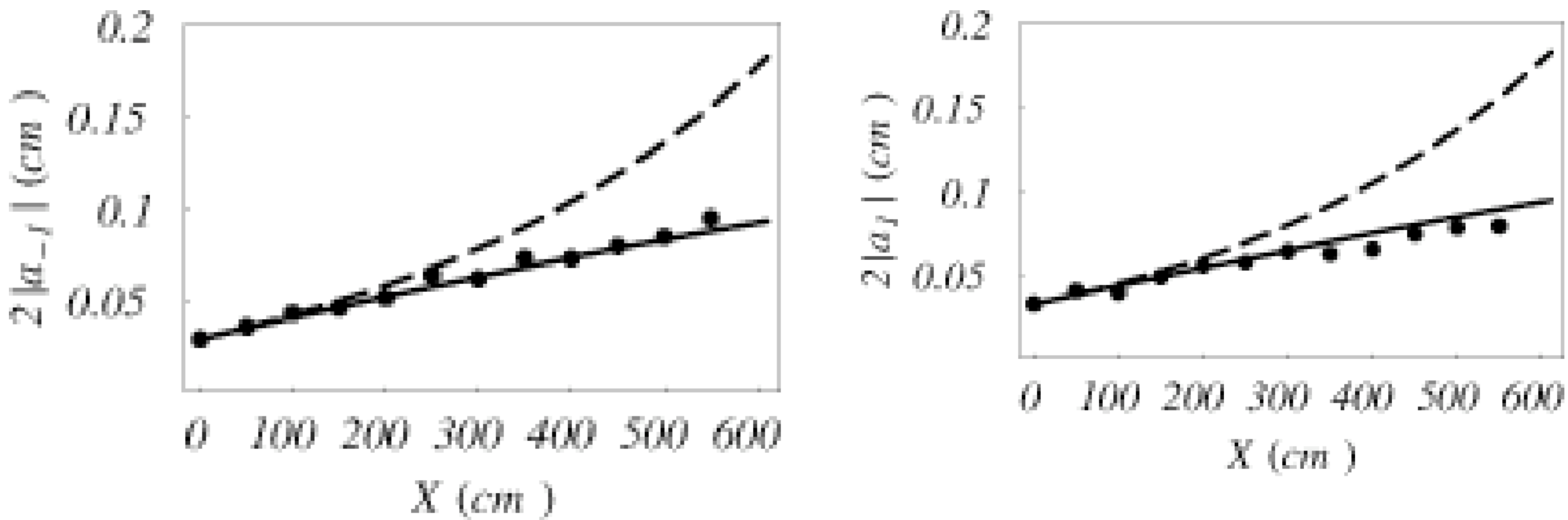


X_1



X_8

Amplitudes of seeded sidebands
(damping factored out of data)

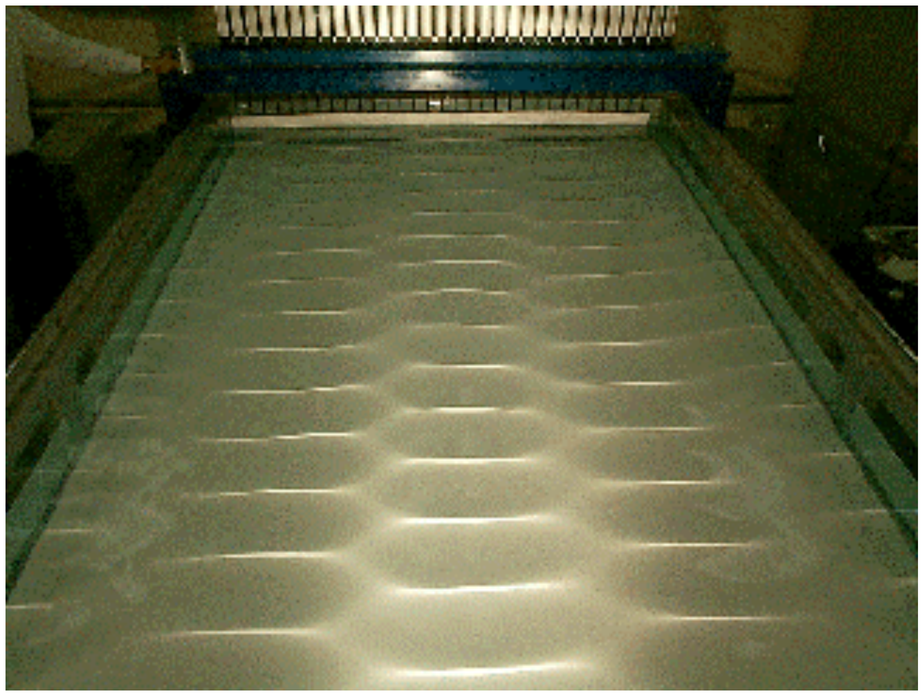


______ damped NLS theory

- - - Benjamin-Feir growth rate

• • • experimental data

Q: Are there stable wave patterns that propagate with permanent form (or nearly so) on deep water?



A: YES, in the presence of (weak) damping

Apparently NO, with no damping

Q: Stable wave patterns that propagate with nearly permanent form on deep water?

A: YES, in the presence of (weak) damping

Apparently NO, with no damping

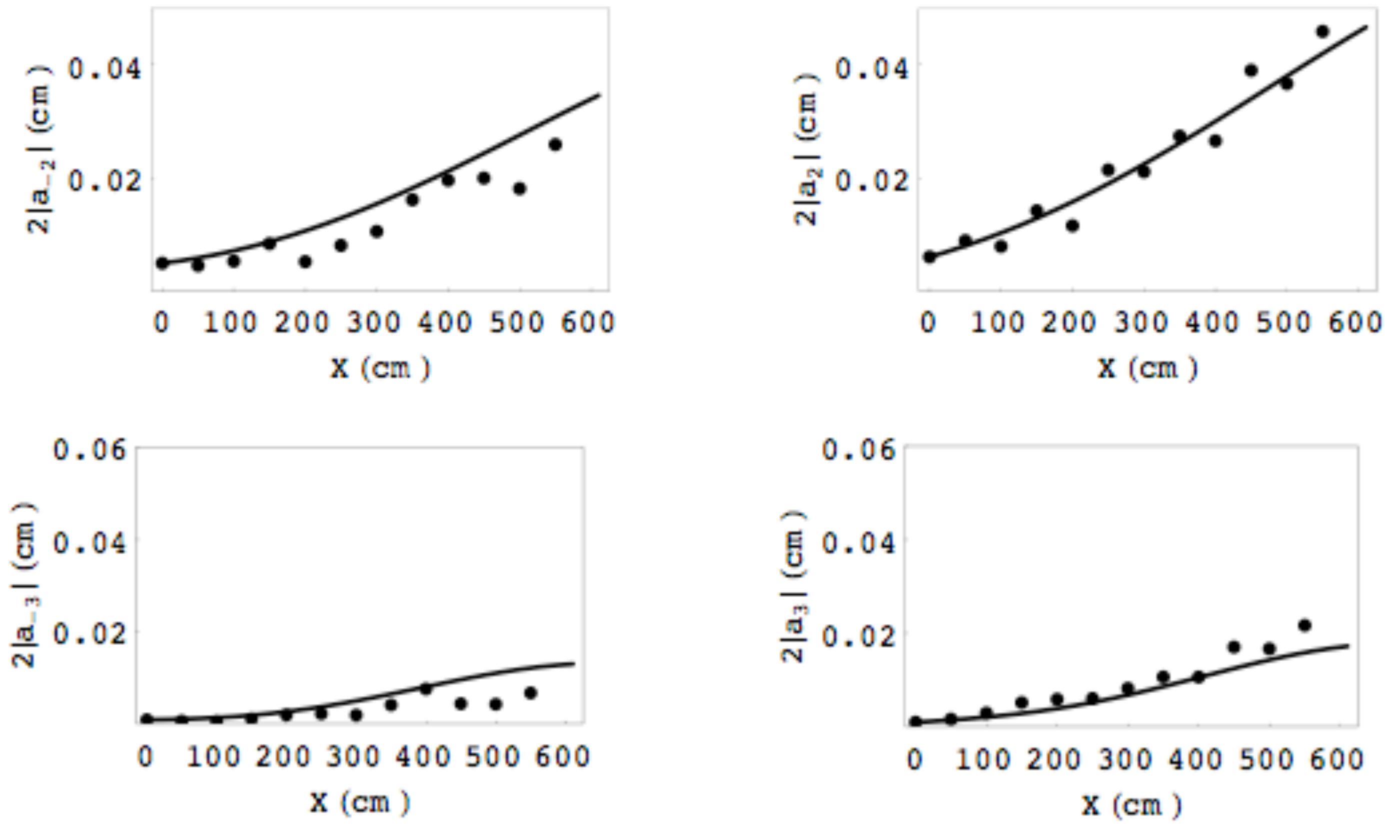
Q: Is this the final chapter of this story?

A: Almost certainly not.

- Downshifting is still unexplained. Its physical importance is largely unexplored.
- More surprises?

Amplitudes of unseeded sidebands

(damping factored out of data)



____ damped NLS theory

• • • experimental data

Numerical simulations of full water wave equations, plus damping

Wu, Liu & Yue

2006

A note on stabilizing the Benjamin-Feir instability 5

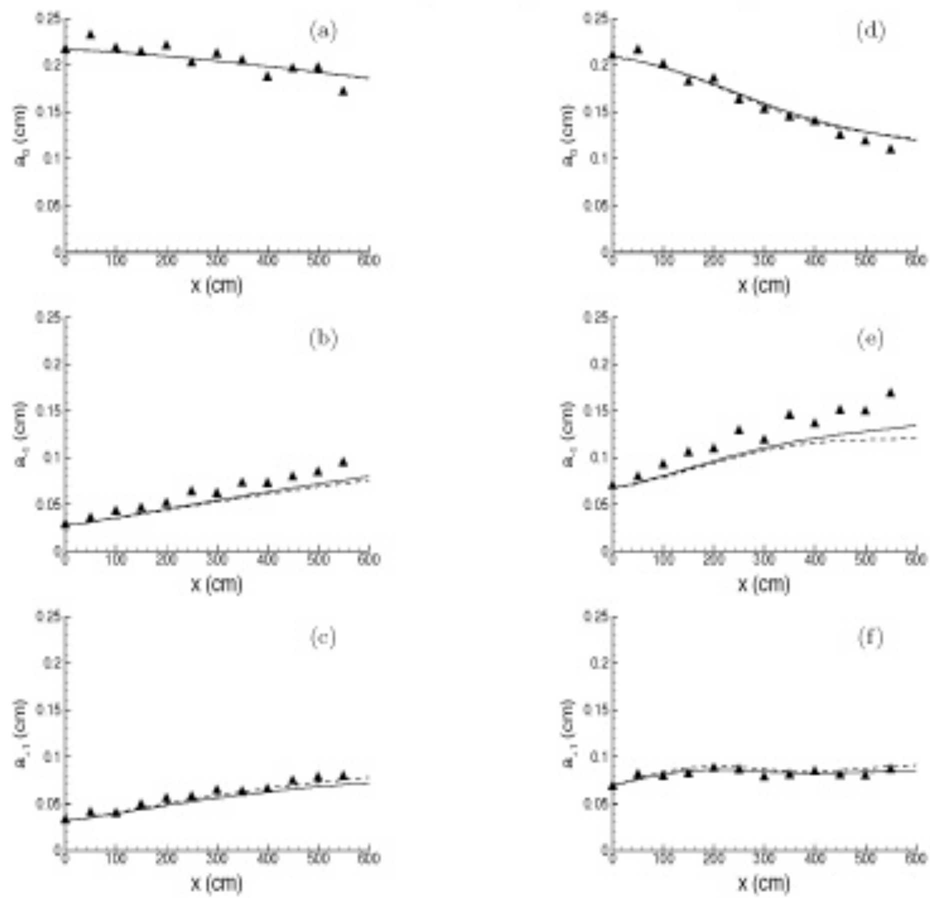


FIGURE 1. Comparisons of the HOS simulations (Model I: - - -; Model II: ——) with the experiments of **BF** (▲) for wave amplitudes in the decaying frame. (a), (d): carrier wave a_0; (b), (e): lower sideband a_{-1}; and (c), (f): upper sideband a_{+1}; as functions of distance from the wavemaker for the evolution of small-amplitude ((a), (b), (c)) and large-amplitude ((d), (e), (f)) wave trains. (x=0 is 128 cm from the wavemaker.)