The Explosive Instability due to 3-wave or 4-wave mixing

Lecture 19

joint work with

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Explosive instability due to 3-wave or 4-wave mixing

A. Review 3-wave mixing (resonant triads) and 4-wave (including NLS)
B. Explosive instability in ODEs
   3-wave mixing, 4-wave mixing
C. Open problem: a physical application of explosive instability due to 4-wave mixing
D. Effect of dissipation
E. Effect of spatial structure

A. Recall derivation of 3-wave equations

Start with physical problem that admits waves, but has no dissipation.

Linearize, and find dispersion relation, $\omega(k)$.

Recall derivation of 3-wave eq'ns

Start with physical problem that admits waves, but has no dissipation.

Linearize, and find dispersion relation, $\omega(k)$.

Q: Does $\omega(k)$ admit 3 pairs $\{\vec{k}, \omega(\vec{k})\}$

so

$$\vec{k}_1 \pm \vec{k}_2 \pm \vec{k}_3 = 0, \qquad \omega(\vec{k}_1) \pm \omega(\vec{k}_2) \pm \omega(\vec{k}_3) = 0\,?$$

If yes ➔ 3-wave equations (resonant triads)

If no ➔ 4-wave equations (resonant quartets)

3-wave equations

Suppose $\vec{k}_1 \pm \vec{k}_2 \pm \vec{k}_3 = 0, \qquad \omega(\vec{k}_1) \pm \omega(\vec{k}_2) \pm \omega(\vec{k}_3) = 0.$

Then $$u(\vec{x},t;\varepsilon) = \varepsilon \sum_{m=1}^{3} A_m(\varepsilon\vec{x},\varepsilon t) \cdot \exp\{i\vec{k}_m \cdot \vec{x} - i\omega(\vec{k}_m)t\} + O(\varepsilon^2)$$

and

$$\partial_\tau(A_m) + \vec{c}_m \cdot \nabla A_m = i\delta_m A_n^* A_l^*,$$

$$m,n,l = 1,2,3$$

3-wave mixing

(capillary-gravity waves; χ_2 materials)

3-wave equations

$$\partial_\tau(A_m) + c_m \cdot \nabla A_m = i\delta_m A_n^* A_l^*,$$

$$m,n,l = 1,2,3$$

- Coppi, Rosenbluth & Sudan (1969) showed that if $\{\delta_1, \delta_2, \delta_3\}$ all have the same sign, then $\{A_1, A_2, A_3\}$ can all blow up at the the same time, everywhere in space. This is the explosive instability.

3-wave equations

$$\partial_\tau(A_m) + c_m \cdot \nabla A_m = i\delta_m A_n^* A_l^*,$$

$$m,n,l = 1,2,3$$

- Coppi, Rosenbluth & Sudan (1969) showed that if $\{\delta_1, \delta_2, \delta_3\}$ all have the same sign, then $\{A_1, A_2, A_3\}$ can all blow up at the the same time, everywhere in space. This is the explosive instability.
- The blow-up occurs even with no spatial structure, so $A_m = A_m(\tau)$.

4-wave equations

- Start with $\omega(k)$
- Need

$$\vec{k}_1 \pm \vec{k}_2 \pm \vec{k}_3 \pm \vec{k}_4 = 0,$$

$$\omega(\vec{k}_1) \pm \omega(\vec{k}_2) \pm \omega(\vec{k}_3) \pm \omega(\vec{k}_4) = 0.$$

4-wave equations

- Start with $\omega(k)$
- Need

$$\vec{k}_1 \pm \vec{k}_2 \pm \vec{k}_3 \pm \vec{k}_4 = 0,$$

$$\omega(\vec{k}_1) \pm \omega(\vec{k}_2) \pm \omega(\vec{k}_3) \pm \omega(\vec{k}_4) = 0.$$

- Special case:

$$\{\vec{k} + \delta\vec{k}\} + \{\vec{k} - \delta\vec{k}\} - \vec{k} = \vec{k},$$

$$\{\omega(\vec{k}) + \delta\omega\} + \{\omega(\vec{k}) - \delta\omega\} - \omega(\vec{k}) = \omega(\vec{k}).$$

➔ nonlinear Schrödinger equation

Nonlinear Schrödinger equations

For one wave field (in 1-D):

$$i(\partial_\tau A + c\partial_\chi A) + \varepsilon[\alpha \cdot \partial_\chi{}^2 A + \gamma \,|\, A \,|^2\, A] = 0.$$

Nonlinear Schrödinger equations

For one wave field (in 1-D):

$$i(\partial_\tau A + c\partial_\chi A) + \varepsilon[\alpha \cdot \partial_\chi{}^2 A + \gamma \mid A \mid^2 A] = 0.$$

For two coupled wave fields (in 1-D):

$$i(\partial_\tau A + c_1\partial_\chi A) + \varepsilon[\alpha_1 \cdot \partial_\chi{}^2 A + A(\gamma_{11} \mid A \mid^2 + \gamma_{12} \mid B \mid^2)] = 0,$$
$$i(\partial_\tau B + c_2\partial_\chi B) + \varepsilon[\alpha_2 \cdot \partial_\chi{}^2 B + B(\gamma_{21} \mid A \mid^2 + \gamma_{22} \mid B \mid^2)] = 0.$$

(also called vector NLS)

NLS equations with 4-wave mixing

(in 1-D)

$$i(\partial_\tau A_m + c_m \partial_\chi A_m) + \varepsilon[\alpha_m \cdot \partial_\chi{}^2 A_m + A_m \sum_{n=1}^{4} \gamma_{mn} \mid A_n \mid^2$$

$$+\delta_m A_p^* A_q^* A_r^*] = 0,$$

$$m, p, q, r = 1,2,3,4$$

(Benney & Newell, 1967)

(gravity-driven surface water waves; χ_3 materials)

NLS equations with 4-wave mixing

(in 1-D)

$$i(\partial_\tau A_m + c_m \partial_\chi A_m) + \varepsilon[\alpha_m \cdot \partial_\chi{}^2 A_m + A_m \sum_{n=1}^{4} \gamma_{mn} \mid A_n \mid^2$$
$$+\delta_m A_p^* A_q^* A_r^*] = 0,$$
$$m, p, q, r = 1,2,3,4$$

Q: Does the "new" 4-wave mixing term permit new phenomena?

A: Yes - explosive instability

(even with no spatial structure, so NOT wave collapse)

B. Explosive instability with no spatial dependence

- 3-wave equations (1969)
- 4-wave equations (2007)

3-wave equations no spatial dependence

$$A_1'(\tau) = i\delta_1 A_2^* A_3^*, \quad A_2'(\tau) = i\delta_2 A_3^* A_1^*, A_3'(\tau) = i\delta_3 A_1^* A_2^*.$$

3 coupled, complex ODEs – $\{\delta_1, \delta_2, \delta_3\}$ known, real-valued

3-wave equations
no spatial dependence

$$A_1'(\tau) = i\delta_1 A_2^* A_3^*, \quad A_2'(\tau) = i\delta_2 A_3^* A_1^*, \; A_3'(\tau) = i\delta_3 A_1^* A_2^*.$$

3 coupled, complex ODEs – $\{\delta_1, \delta_2, \delta_3\}$ known, real-valued

Necessary and sufficient conditions for blow-up in finite time:

- at least two of $\{A_1(0), A_2(0), A_3(0)\}$ are non-zero;
- $\{\delta_1, \delta_2, \delta_3\}$ all have the same sign (and non-zero).
  (Coppi, Rosenbluth, Sudan, 1969)

$$A_1'(\tau) = i\delta_1 A_2^* A_3^*, \quad A_2'(\tau) = i\delta_2 A_3^* A_1^*, \quad A_3'(\tau) = i\delta_3 A_1^* A_2^*.$$

Necessary:

Manley-Rowe relations (constants of motion):

$$J_1 = \frac{|A_1|^2}{\delta_1} - \frac{|A_3|^2}{\delta_3}, \quad J_2 = \frac{|A_2|^2}{\delta_2} - \frac{|A_3|^2}{\delta_3}.$$

If sign (δ_m) $\neq$ sign (δ_n) for any $m \neq n$,

➔no blow-up.

(Results for 4-wave mixing are similar, but slightly more complicated.)

$$A_1'(\tau) = i\delta_1 A_2^* A_3^*, \quad A_2'(\tau) = i\delta_2 A_3^* A_1^*, \; A_3'(\tau) = i\delta_3 A_1^* A_2^*.$$

Sufficient: Suppose $\{\delta_1, \delta_2, \delta_3\}$ have same sign.

A 3-parameter family of singular solutions is

$$A_m(\tau) = \sqrt{\frac{\delta_m}{\delta_1\delta_2\delta_3}}\,\frac{e^{i\theta_m}}{(\tau_0 - \tau)}, \quad \theta_1 + \theta_2 + \theta_3 = \frac{\pi}{2} + 2\pi N.$$

The 3 free, real-valued parameters are $\{\tau_0, \theta_1, \theta_2\}$.

(This justifies name: “explosive instability”)

$$A_1'(\tau) = i\delta_1 A_2^* A_3^*, \quad A_2'(\tau) = i\delta_2 A_3^* A_1^*, \; A_3'(\tau) = i\delta_3 A_1^* A_2^*.$$

Q: General solution of the ODEs?

A: Assume $A_m(\tau) = \sqrt{\dfrac{\delta_m}{\delta_1\delta_2\delta_3}}\dfrac{e^{i\theta_m}}{(\tau_0 - \tau)}$ is the first term in a Laurent series, in the neighborhood of a pole:

$$A_m(\tau) = \sqrt{\frac{\delta_m}{\delta_1\delta_2\delta_3}}\frac{e^{i\theta_m}}{(\tau_0 - \tau)}[1 + \alpha_m(\tau_0 - \tau) + \beta_m(\tau_0 - \tau)^2 + \gamma_m(\tau_0 - \tau)^3 + O((\tau_0 - \tau)^4)]$$

Determine complex coefficients, order by order

$$A_1'(\tau) = i\delta_1 A_2^* A_3^*, \quad A_2'(\tau) = i\delta_2 A_3^* A_1^*, \; A_3'(\tau) = i\delta_3 A_1^* A_2^*.$$

General solution of the ODEs:

$$A_m(\tau) = \sqrt{\frac{\delta_m}{\delta_1\delta_2\delta_3}}\frac{e^{i\theta_m}}{(\tau_0 - \tau)}[1 + \alpha_m(\tau_0 - \tau) + \beta_m(\tau_0 - \tau)^2 + \gamma_m(\tau_0 - \tau)^3 + O((\tau_0 - \tau)^4)]$$

Find: $\alpha_m = 0$, $\mathrm{Im}(\beta_m) = 0$, $\mathrm{Re}(\beta_1 + \beta_2 + \beta_3) = 0$,

$$A_1'(\tau) = i\delta_1 A_2^* A_3^*, \quad A_2'(\tau) = i\delta_2 A_3^* A_1^*, \; A_3'(\tau) = i\delta_3 A_1^* A_2^*.$$

General solution of the ODEs:

$$A_m(\tau) = \sqrt{\frac{\delta_m}{\delta_1\delta_2\delta_3}}\frac{e^{i\theta_m}}{(\tau_0-\tau)}[1+\alpha_m(\tau_0-\tau)+\beta_m(\tau_0-\tau)^2$$
$$+\gamma_m(\tau_0-\tau)^3+O((\tau_0-\tau)^4)]$$

Find: $\alpha_m = 0$, $\mathrm{Im}(\beta_m) = 0$, $\mathrm{Re}(\beta_1 + \beta_2 + \beta_3) = 0$,
$\mathrm{Re}(\gamma_m) = 0$, $\mathrm{Im}(\gamma_1) = \mathrm{Im}(\gamma_2) = \mathrm{Im}(\gamma_3) = \gamma$.

$$A_1'(\tau) = i\delta_1 A_2^* A_3^*, \quad A_2'(\tau) = i\delta_2 A_3^* A_1^*, \; A_3'(\tau) = i\delta_3 A_1^* A_2^*.$$

General solution of the ODEs:

$$A_m(\tau) = \sqrt{\frac{\delta_m}{\delta_1\delta_2\delta_3}} \frac{e^{i\theta_m}}{(\tau_0 - \tau)} [1 + \alpha_m(\tau_0 - \tau) + \beta_m(\tau_0 - \tau)^2 + \gamma_m(\tau_0 - \tau)^3 + O((\tau_0 - \tau)^4)]$$

Find: $\alpha_m = 0$, $\mathrm{Im}(\beta_m) = 0$, $\mathrm{Re}(\beta_1 + \beta_2 + \beta_3) = 0$,
$\mathrm{Re}(\gamma_m) = 0$, $\mathrm{Im}(\gamma_1) = \mathrm{Im}(\gamma_2) = \mathrm{Im}(\gamma_3) = \gamma$.

6 real-valued, free constants: $\{\tau_0, \theta_1, \theta_2, \beta_1, \beta_2, \gamma\}$

$\Rightarrow$ every nontrivial solution of the ODEs near $\tau = \tau_0$ blows up at $\tau = \tau_0$.

$$A_1'(\tau) = i\delta_1 A_2^* A_3^*, \quad A_2'(\tau) = i\delta_2 A_3^* A_1^*, \; A_3'(\tau) = i\delta_3 A_1^* A_2^*.$$

General solution of the ODEs:

$$A_m(\tau) = \sqrt{\frac{\delta_m}{\delta_1\delta_2\delta_3}} \frac{e^{i\theta_m}}{(\tau_0 - \tau)} [1 + \beta_m (\tau_0 - \tau)^2 + i\gamma(\tau_0 - \tau)^3 + O((\tau_0 - \tau)^4]$$

Free constants: $\{\tau_0, \theta_1, \theta_2, \beta_1, \beta_2, \gamma\}$

Series converges absolutely if:

(i) $\beta_1 = \beta_2 = 0, \quad \left|\frac{\gamma(\tau_0 - \tau)^3}{3}\right| < 1$ or

(ii) $|\beta_n| \le B, \quad |\gamma| \le B^{\frac{3}{2}}, \quad |\tau_0 - \tau|^2 B < 1$

$$A_1'(\tau) = i\delta_1 A_2^* A_3^*, \quad A_2'(\tau) = i\delta_2 A_3^* A_1^*, \; A_3'(\tau) = i\delta_3 A_1^* A_2^*.$$

Q: What does it mean physically for all three wave trains to blow up in finite time?

$$A_1'(\tau) = i\delta_1 A_2^* A_3^*, \quad A_2'(\tau) = i\delta_2 A_3^* A_1^*, \; A_3'(\tau) = i\delta_3 A_1^* A_2^*.$$

Q: What does it mean physically for all three wave trains to blow up in finite time?

The end of the world?

A: Perhaps, but probably not.

The 3-wave equations evolve on a long time-scale (t = O(ε^{-1})). Blow-up in finite time usually means that assumptions in model have broken down at this time, or earlier.

Before model breaks down completely, there is ***significant*** energy transfer into the wave modes.

4-wave equations no spatial dependence

$$iA'_m + A_m \sum_{n=1}^{4} \Gamma_{mn} |A_n|^2 + \delta_m A_p^* A_q^* A_r^* = 0, \quad m,p,q,r = 1,2,3,4$$

4 coupled, complex ODEs – coefficients known, real-valued

Necessary and sufficient conditions for blow-up in finite time:

- at least three of $\{A_1(0), A_2(0), A_3(0), A_4(0)\}$ are non-zero;
- $\{\delta_1,\delta_2,\delta_3,\delta_4\}$ all have the same sign (and non-zero);
- $\left| \sum_{m,n=1}^{4} \Gamma_{mn} |\delta_n| \right| < 4\sqrt{\delta_1\delta_2\delta_3\delta_4}$

For NLS-type models with 4-wave mixing

- Self-focussing singularity (or wave collapse)
  - requires 2 or more spatial dimensions
  - a finite amount of energy collapses to a point
  - solution blows up at one point, in finite time
- Explosive instability
  - works in any number of spatial dimensions (including 0)
  - draws an infinite amount of energy from a background source
  - solution blows up everywhere, in finite time

C. An open problem

Find a physical example of the explosive instability due to 4-wave mixing, (with or) without spatial structure

$$iA'_m + A_m \sum_{n=1}^{4} \Gamma_{mn} \, | A_n |^2 + \delta_m A_p^* A_q^* A_r^* = 0,$$

$$m, p, q, r = 1,2,3,4$$

Final result for ODEs

when necessary conditions hold

- 3 wave mixing

$$A_m'(\tau) = i\delta_m A_p^* A_q^*, \qquad m,p,q = 1,2,3$$

A 6-parameter family of solutions all blow up

- 4-wave mixing

$$iA_m' + A_m \sum_{n=1}^{4} \Gamma_{mn} |A_n|^2 + \delta_m A_p^* A_q^* A_r^* = 0, \quad m,p,q,r = 1,2,3,4$$

An 8-parameter family of solutions all blow up

D. Can damping stop blow-up in ODES?

With no damping:

$$A'_m(\tau) = i\delta_m A_p^* A_q^*, \qquad m,p,q = 1,2,3$$

Add damping:

$$A'_m(\tau) = i\delta_m A_p^* A_q^* - \nu_m A_m, \qquad \nu_m \geq 0.$$

Simplest case: uniform damping $\nu_1 = \nu_2 = \nu_3 = \nu$

Can damping stop blow-up?

No damping:

(1)
$$A'_m(\tau) = i\delta_m A_p^* A_q^*,$$

Uniform damping:

(2)
$$A'_m(\tau) = i\delta_m A_p^* A_q^* - \nu A_m,$$

Can damping stop blow-up?

No damping:

(1)
$$A_m'(\tau) = i\delta_m A_p^* A_q^*,$$

Uniform damping:

(2)
$$A_m'(\tau) = i\delta_m A_p^* A_q^* - \nu A_m,$$

CHANGE VARIABLES:

$$T = \frac{1 - e^{-\nu\tau}}{\nu}, \qquad A_m(\tau) = e^{-\nu\tau}\alpha_m(T), \qquad m = 1,2,3$$

This maps (2) into (1), exactly (Miles, 1984)

Result: uniform damping provides a threshold for blow-up

3-wave mixing with uniform damping:

$$A'_m(\tau) = i\delta_m A_p^* A_q^* - \nu A_m ,$$

If

- $\{\delta_1, \delta_2, \delta_3\}$ all have the same sign
- $|A_m(0)| \geq \sqrt{\frac{\delta_m}{\delta_1\delta_2\delta_3}} \frac{\nu}{1-e^{-\nu\tau_0}}, \qquad m = 1,2,3$ (threshold)

then solutions blow-up in finite time.

(A similar result holds for 4-wave mixing)

Can non-uniform damping stop blow-up in ODES?

$$A_m'(\tau) = i\delta_m A_p^* A_q^* - \nu_m A_m , \qquad \nu_m \geq 0.$$

Seek a formal series solution:

$$A_m(\tau) = \sqrt{\frac{\delta_m}{\delta_1\delta_2\delta_3}} \frac{e^{i\theta_m}}{(\tau_0 - \tau)} [1 + \alpha_m(\tau_0 - \tau) + \beta_m(\tau_0 - \tau)^2 + ...]$$

This is too restrictive – singularity is not a pole

Effect of non-uniform damping

$$A'_m(\tau) = i\delta_m A_p^* A_q^* - \nu_m A_m ,$$

Try

$$A_m(\tau) = \sqrt{\frac{\delta_m}{\delta_1\delta_2\delta_3}}\frac{e^{i\theta_m}}{(\tau_0 - \tau)}[1 + \alpha_m(\tau_0 - \tau)$$
$$+b_m(\tau_0 - \tau)^2 \ln(\tau_0 - \tau) + \beta_m(\tau_0 - \tau)^2 + \ldots]$$

Find: $\{\alpha_m\}$ real, determined by $\{\nu_n\}$

$\{b_m\}$ real, determined by $\{\nu_n\}$

$\{\beta_m\}$ real, $\{\beta_1, \beta_2\}$ free

Continue…

Effect of non-uniform damping

$$A'_m(\tau) = i\delta_m A_p^* A_q^* - \nu_m A_m ,$$

Result (for ODE model):

- The ODEs are no longer completely integrable
- The singularity is no longer a pole

Effect of non-uniform damping

$$A_m'(\tau) = i\delta_m A_p^* A_q^* - \nu_m A_m ,$$

Result (for ODE model):

- The ODEs are no longer completely integrable
- The singularity is no longer a pole

But

- Blow-up persists for a 6-parameter family of (formal) solutions of the ODEs
- Damping does not quench blow-up, except perhaps by inserting a threshold for blow-up.

Thank you for your attention

E. Blow-up in PDES?

$$\partial_\tau(A_m) + \vec{c}_m \cdot \nabla A_m = i\delta_m A_p^* A_q^*, \qquad m,p,q = 1,2,3$$

Blow-up in PDES?

$$\partial_\tau (A_m) + \vec{c}_m \cdot \nabla A_m = i\delta_m A_p^* A_q^*, \qquad m,p,q = 1,2,3$$

Zakharov & Manakov, 1976

- 3-wave PDEs are completely integrable

Kaup, 1978

- Solved equations in 1-D on whole line (comp. sup.)
- Numerics + analysis to learn about blow up

Blow-up in PDES?

$$\partial_\tau(A_m) + \vec{c}_m \cdot \nabla A_m = i\delta_m A_p^* A_q^*, \quad m,p,q = 1,2,3$$

Zakharov & Manakov, 1976

3-wave PDEs are completely integrable

Kaup, 1978

Solved equations in 1-D on whole line (comp. sup.)

Numerics + analysis to learn about blow up

Unknown: periodic boundary conditions?

more than 1-D in space?

What about the PDEs?

$$\partial_\tau(A_m) + \vec{c}_m \cdot \nabla A_m = i\delta_m A_p^* A_q^*, \qquad m,p,q = 1,2,3$$

Alternative approach: variation of parameters

$$A_m(\tau) = \sqrt{\frac{\delta_m}{\delta_1\delta_2\delta_3}}\frac{e^{i\theta_m}}{(\tau_0 - \tau)}[1 + \beta_m(\tau_0 - \tau)^2 + i\gamma(\tau_0 - \tau)^3 + \ldots]$$

$\{\tau_0, \theta_1, \theta_2, \beta_1, \beta_2, \gamma\}$ are real-valued and free

What about the PDEs?

$$\partial_\tau(A_m) + \vec{c}_m \cdot \nabla A_m = i\delta_m A_p^* A_q^*, \qquad m,p,q = 1,2,3$$

Alternative approach: variation of parameters

$$A_m(\tau) = \sqrt{\frac{\delta_m}{\delta_1\delta_2\delta_3}}\frac{e^{i\theta_m}}{(\tau_0-\tau)}[1+\beta_m(\tau_0-\tau)^2 + i\gamma(\tau_0-\tau)^3 + \ldots]$$

$\{\tau_0,\ \theta_1,\ \theta_2,\ \beta_1,\ \beta_2,\ \gamma\}$ are real-valued and free

Suppose we allow $\{\beta_1 = \beta_1(x),\ \beta_2 = \beta_2(x),\ \gamma = \gamma(x)\}$?

What about the PDEs?

$$\partial_\tau (A_m) + \vec{c}_m \cdot \nabla A_m = i\delta_m A_p^* A_q^*, \qquad m,p,q = 1,2,3$$

Result:

$$A_m(\vec{x},\tau) = \sqrt{\frac{\delta_m}{\delta_1\delta_2\delta_3}} \frac{e^{i\theta_m}}{(\tau_0 - \tau)} [1 + \beta_m(\vec{x})(\tau_0 - \tau)^2$$
$$+\{g_m(\vec{x}) + i\gamma(\vec{x})\}(\tau_0 - \tau)^3 + ...]$$

$\{\beta_1(x), \beta_2(x), \gamma(x)\}$ are real-valued and arbitrary

$\{g_m(x)\}$ are real-valued, known (in terms of $\beta_m(x)$)

(Function space is arbitrary at this point.)

More of the same

$$\partial_\tau(A_m) + \vec{c}_m \cdot \nabla A_m = i\delta_m A_p^* A_q^*, \qquad m,p,q = 1,2,3$$

Allow $\{\theta_1(x),\ \theta_2(x),\ \beta_1(x),\ \beta_2(x),\ \gamma(x)\}$:

$$A_m(\vec{x},\tau) = \sqrt{\frac{\delta_m}{\delta_1\delta_2\delta_3}}\,\frac{e^{i\theta_m(\vec{x})}}{(\tau_0-\tau)}[1 + ia_m(\vec{x})(\tau_0-\tau)$$
$$+\{\beta_m(\vec{x}) + ib_m(\vec{x})\}(\tau_0-\tau)^2$$
$$+\{g_m(\vec{x}) + i\gamma(x)\}(\tau_0-\tau)^3 + \ldots]$$

This family of (formal) solutions of the PDEs admit 5 arbitrary functions of (x), in any function space.

The last step (V. Putkaradze)

$$\partial_\tau (A_m) + \vec{c}_m \cdot \nabla A_m = i\delta_m A_p^* A_q^*, \qquad m,p,q = 1,2,3$$

$$A_m(\vec{x},\tau) = \sqrt{\frac{\delta_m}{\delta_1\delta_2\delta_3}}\frac{\rho_m(\vec{x})e^{i\theta_m(\vec{x})}}{(\tau_0(\vec{x})-\tau)}[1 + ia_m(\vec{x})(\tau_0-\tau)$$
$$+\{\beta_m(\vec{x}) + ib_m(\vec{x})\}(\tau_0-\tau)^2$$
$$+\{g_m(\vec{x}) + i\gamma(x)\}(\tau_0-\tau)^3 + \ldots],$$

$$\rho_1^2(\vec{x}) = (1-\vec{c}_2\cdot\nabla\tau_0(\vec{x}))\cdot(1-\vec{c}_3\cdot\nabla\tau_0(\vec{x})),$$
$$\rho_2^2(\vec{x}) = (1-\vec{c}_3\cdot\nabla\tau_0(\vec{x}))\cdot(1-\vec{c}_1\cdot\nabla\tau_0(\vec{x})),$$
$$\rho_3^2(\vec{x}) = (1-\vec{c}_1\cdot\nabla\tau_0(\vec{x}))\cdot(1-\vec{c}_2\cdot\nabla\tau_0(\vec{x})).$$

What's left for 3-wave mixing?

$$\partial_\tau(A_m) + \vec{c}_m \cdot \nabla A_m = i\delta_m A_p^* A_q^*, \qquad m,p,q = 1,2,3$$

$$A_m(\vec{x},\tau) = \sqrt{\frac{\delta_m}{\delta_1\delta_2\delta_3}}\,\frac{\rho_m(\vec{x})e^{i\theta_m(\vec{x})}}{(\tau_0(\vec{x})-\tau)}[1 + ia_m(\vec{x})(\tau_0-\tau)$$
$$+\{\beta_m(\vec{x}) + ib_m(\vec{x})\}(\tau_0-\tau)^2$$
$$+\{g_m(\vec{x}) + i\gamma(x)\}(\tau_0-t)^3 + \ldots],$$

- Convergence of the series?

  (Cauchy-Kovalevskaya theory?)
- Constraints on the free functions for blow-up?
- Physical Application?

Can damping stop blow-up?

With no damping:

$$i\frac{d(A_m)}{dt} + A_m \sum_{n=1}^{4} \gamma_{mn} |A_n|^2 + \delta_m A_p^* A_q^* A_r^* = 0,$$

Add damping:

$$i\frac{d(A_m)}{dt} + A_m \sum_{n=1}^{4} \gamma_{mn} |A_n|^2 + \delta_m A_p^* A_q^* A_r^* + i\nu_m A_m = 0,$$

$$\nu_m \geq 0.$$

Simplest case: uniform damping

$$\nu_1 = \nu_2 = \nu_3 = \nu_4 = \nu$$

Can damping stop blow-up?

No damping:

(1)
$$i\frac{d(A_m)}{dt} + A_m \sum_{n=1}^{4} \gamma_{mn} |A_n|^2 + \delta_m A_p^* A_q^* A_r^* = 0,$$

Uniform damping:

(2)
$$i\frac{d(A_m)}{dt} + A_m \sum_{n=1}^{4} \gamma_{mn} |A_n|^2 + \delta_m A_p^* A_q^* A_r^* + i\nu \cdot A_m = 0,$$

Change variables:

$$T = \frac{1 - e^{-2\nu t}}{2\nu}, \quad A_m(t) = e^{-\nu t}\alpha_m(T), \quad m = 1,2,3,4$$

This maps (2) into (1), exactly.

Damping provides a threshold for blow-up

4-wave mixing with uniform damping:

$$i\frac{d(A_m)}{dt} + A_m \sum_{n=1}^{4} \gamma_{mn} |A_n|^2 + \delta_m A_p^* A_q^* A_r^* + i\nu \cdot A_m = 0$$

If

- $\{\delta_1, \delta_2, \delta_3, \delta_4\}$ all have the same sign
- $$\left| \sum_{m,n=1}^{4} \gamma_{mn} |\delta_n| \right| < 4\sqrt{\delta_1 \delta_2 \delta_3 \delta_4}$$
- $$|A_m(0)|^2 \geq \frac{\nu |\delta_m|}{\sqrt{\delta_1 \delta_2 \delta_3 \delta_4 - (\frac{1}{4} \sum_{m,n=1}^{4} \gamma_{mn} |\delta_n|)^2}}, \quad m = 1,2,3,4$$

Then solutions blow-up in finite time.