Boundary Layers

Lecture 1 by Rebecca Dell

The idea of a 'boundary layer' dates back at least to the time of Ludwig Prandtl. In 1904, he said:

> I have set myself the task of investigating systematically the motion of a fluid of which the internal resistance can be assumed very small. In fact, the resistance is supposed to be so small that it can be neglected wherever great velocity differences or cumulative effects of the resistance do not exist. This plan has proved to be very fruitful, for one arrives thereby at mathematical formulations which not only permit problems to be solved but also give promise of providing very satisfactory agreement with observations.
>
> ...the investigation of a particular flow phenomenon is thus divided into two interdependent parts: there is on the one hand the *free fluid*, which can be treated as inviscid according to the vorticity principles of Helmholtz, and on the other hand the transition layers at the fixed boundaries, the movement of which is controlled by the free fluid, yet which in turn give the free movement its characteristic stamp by the emission of vortex sheets. (H. Rouse and S. Ince. *History of Hydrodynamics*. New York: Dover, 1957, p269.)

Prandtl's central observation was that the the motion of a fluid of small viscosity could be separated into two *interdependent* parts: the free fluid and the boundary layer. The mathematical description of each would be quite different, but each would affect the other by the necessity of matching the two flows together. Physically, the free fluid forces the outer edge of the boundary layer, while the boundary layer diffuses vorticity into the free fluid. For more information about Prandtl, see the article in *Physics Today*, 2005, v.58, no.12, 42-48.

To get an idea of how this interplay between the boundary layer and the free fluid manifests itself mathematically, let's consider a couple of governing equations for geophysical fluid flow. Consider the Navier–Stokes equation

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \rho F_i + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j}.$$

The final term expresses the effects of friction. If μ is small, as in most geophysical flows, the standard approximation is to neglect the final term. However, it is also the highest order term in the Navier–Stokes equation; neglecting it reduces the order of the equation and so reduces the number of boundary conditions that can be satisfied simultaneously. This is called a *singular perturbation*. We need to simplify our equation to make it mathematically tractable, but we must still satisfy our boundary conditions. To do this, it is necessary to find a way to retain the higher order derivatives *only where necessary*. The method for doing this is called boundary layer theory, which is a form of singular perturbation theory. In physical terms rather than mathematical terms, we apply boundary layer theory to understand the interplay between the localized viscous regions and the inviscid bulk of the fluid outside these regions.

To build intuition, let us consider another example. As we will learn in later chapters, the streamfunction of the steady, wind–driven ocean circulation is given by

$$\underbrace{\epsilon J(\Psi, \nabla^2\Psi)}_{\text{Nonlinear Advection of Vorticity}} \quad + \quad \underbrace{\frac{\partial \Psi}{\partial x}}_{\text{Planetary Vorticity Gradient}} \quad = \quad \underbrace{-r\nabla^2\Psi}_{\text{Bottom Friction}} \quad + \quad \underbrace{\nu\nabla^4\Psi}_{\text{Lateral Friction}} \quad + \quad \underbrace{T(x,y)}_{\text{Wind Forcing}} \quad .$$

In this equation, $\epsilon = \frac{U}{\beta L^2}$ is the Rossby number, and r and ν are nondimensional forms of the coefficients of bottom and lateral friction. Far from continental margins, in the free fluid, the dominant balance is between the advection of planetary vorticity and the wind forcing. The retention of only these two terms lowers the order of the equation. Any of the higher order terms—bottom friction, lateral friction, or advection— could be important in a restricted region of the flow along the boundaries. Generally, we think of boundary layers as a frictional effect, but it is important to realize that friction is not the sole source of boundary layers; inviscid nonlinear advection can create a boundary layer as well. This equation will be discussed in future lectures in great detail. For now, it is sufficient to recognize that boundary layer effects, such as the Gulf Stream, are clearly possible.

Looking ahead, the remaining lectures will adhere roughly to the following outline:

Linear Boundary Layer Theory : Ekman layers; boundary layers in a density–stratified fluid; boundary layer control of the interior; experimental applications

Coastal Bottom Boundary Layers : Boundary layers on the coastal shelf in cases of upwelling and downwelling; Observations (by S. Lentz)

Boundary Layers in the General Oceanic Circulation : Sverdrup theory; Stommel, Munk, and intertial boundary layers; inertial runaway; the thermocline and its boundary layer structure

1 Ekman Layers

Starting right in on linear boundary layer theory, let us derive the Ekman layer of a homogeneous fluid. The Ekman layer, described by Walfrid Ekman in 1902 in his doctoral dissertation, is a horizontal boundary layer in a rotating fluid. Such layers exist at the top and bottom of the ocean and at the bottom of the atmosphere. First, let's consider the steady equations of motion for an unstratified geophysical flow of uniform depth in a rotating coordinate frame

$$\begin{aligned}
u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - 2\Omega v &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \\
u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + 2\Omega u &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \\
u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + g \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0.
\end{aligned}$$

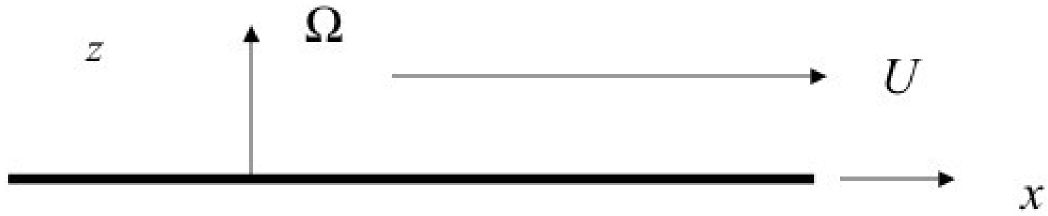

Figure 1: Ekman Layer Problem Setup

By omitting the energy equation, we implicitly assume that $\rho = \text{constant}$.

To derive the Ekman layer, we assume the fluid is flowing over an infinite flat plate, and far from the plate the fluid has velocity U, as shown in Figure 1. For simplicity, we assume that the far field velocity is constant in the down–stream direction, though it may vary in the cross–stream direction. We align our x–coordinate with the flow, so that $U = U(y)$, and we expect all quantities to be unvarying with x, ($\frac{\partial}{\partial x} = 0$). Additionally, we rescale all lengths by some typical length scale L and all velocities by some typical velocity U_0. We assume that a natural length scale can be found, for example from some boundary far away or from the lateral scale of variations in $U(y)$. The above system of equations then becomes

$$\begin{aligned} \epsilon\left(v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) - v &= \frac{E}{2}\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) && (1) \\ \epsilon\left(v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) + u &= -\frac{\partial p}{\partial y} + \frac{E}{2}\left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) && (2) \\ \epsilon\left(v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) &= -\frac{\partial p}{\partial z} + \frac{E}{2}\left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) && (3) \\ \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0. && (4) \end{aligned}$$

In these equations, our dimensionless parameters are the Rossby number, $\epsilon = \frac{U_0}{2\Omega L}$, and the Ekman number, $E = \frac{\nu}{\Omega L^2}$. Note that we have rescaled p by $\rho 2\Omega U_0$, and we have removed the hydrostatic component of the pressure, $-\rho g z$. We will also enforce two conditions at the lower boundary $z = 0$: no slip, that is $u = v = 0$; and no normal flow, that is $w = 0$.

We know that far from the boundary in the interior of the fluid, the velocity u_I must be given by

$$u_I = U(y).$$

If we assume that both ϵ and E are in some sense 'small', equation (2) tells us that in the interior

$$p_I = -\int^y U(y')\mathrm{y}'. \tag{5}$$

This is a statement of geostrophy—the pressure and the rotation are the dominant balance. Equations (1) and (4) then imply $v_I = 0$ and $w_I = 0$. This solution satisfies all of the governing equations, but it does not satisfy a no–slip condition applied at $z = 0$. In order to satisfy this condition, we turn to a boundary layer. The boundary layer is expected to be very thin, so it is more convenient to examine it in a 'stretched' coordinate—a coordinate in which the boundary layer thickness is $O(1)$. In the boundary layer, the dominant physical process will be a balance between rotation and viscosity. In equations (1) to (4) the rotational terms are already $O(1)$, and to make the viscosity terms the same order, each vertical derivative must be $O(\frac{1}{\sqrt{E}})$. Therefore, we rescale z as

$$z = \sqrt{E}\zeta.$$

This corresponds to rescaling heights by δ, where:

$$\delta = \sqrt{\frac{\nu}{\Omega}} \tag{6}$$

is the thickness of the boundary layer, and it arises, as we said, from the balance of viscosity ν and rotation Ω. Since we are rescaling our vertical coordinate, we must similarly rescale our vertical velocity; let us define $W(y,\zeta) \equiv \frac{w}{\sqrt{E}}$. The chain rule tells us

$$\frac{\partial}{\partial z} = \frac{1}{\sqrt{E}}\frac{\partial}{\partial \zeta} \qquad ; \qquad \frac{\partial^2}{\partial z^2} = \frac{1}{E}\frac{\partial^2}{\partial \zeta^2}.$$

Our governing equations (1) – (4) now become

$$\begin{aligned}
\epsilon\left(v\frac{\partial u}{\partial y} + W\frac{\partial u}{\partial \zeta}\right) - v &= \frac{1}{2}\left(E\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial \zeta^2}\right) && (7)\\
\epsilon\left(v\frac{\partial v}{\partial y} + W\frac{\partial v}{\partial \zeta}\right) + u &= -\frac{\partial p}{\partial y} + \frac{1}{2}\left(E\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial \zeta^2}\right) && (8)\\
\epsilon E\left(v\frac{\partial W}{\partial y} + W\frac{\partial W}{\partial \zeta}\right) &= -\frac{\partial p}{\partial \zeta} + \frac{E}{2}\left(E\frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial \zeta^2}\right) && (9)\\
\frac{\partial v}{\partial y} + \frac{\partial W}{\partial \zeta} &= 0. && (10)
\end{aligned}$$

Since we have included E in the scaling of ζ, in the limit where $E \to 0$ we remain in the boundary layer and friction remains important. If we again consider the Rossby number ϵ and the Ekman number E small with respect to one, these become:

$$\begin{aligned}
-v &= \frac{1}{2}\frac{\partial^2 u}{\partial \zeta^2} && (11)\\
+u &= -\frac{\partial p}{\partial y} + \frac{1}{2}\frac{\partial^2 v}{\partial \zeta^2} && (12)\\
0 &= -\frac{\partial p}{\partial \zeta} && (13)\\
\frac{\partial v}{\partial y} + \frac{\partial W}{\partial \zeta} &= 0. && (14)
\end{aligned}$$

Equation (13) tells us that we expect the pressure to be constant throughout the depth of the boundary layer, and so to be equal to its value in the inviscid interior, given by equation (5). The y–momentum equation (12) then becomes

$$u - U = \frac{1}{2}\frac{\partial^2 v}{\partial \zeta^2}. \tag{15}$$

The easiest way to solve the coupled equations (11) and (15) is to define a complex variable $\Lambda = (u - U) + iv$. The two equations can then be expressed as a single complex equation

$$\frac{\partial^2 \Lambda}{\partial \zeta^2} = 2i\Lambda.$$

This second–order equation has two solutions, but we discard the unbounded solution, leaving $\Lambda = \Lambda_0 e^{-(1+i)\zeta}$. Retaining only the real part of this, we find

$$\begin{aligned} u &= U + e^{-\zeta} \left(-A \cos \zeta + B \sin \zeta\right) \\ v &= e^{-\zeta} \left(A \sin \zeta + B \cos \zeta\right). \end{aligned}$$

Our no–slip boundary condition gives $A = U(y)$ and $B = 0$, so our final solution is

$$u = U(y) \left(1 - e^{-\zeta} \cos \zeta\right) \tag{16}$$

$$v = U(y) e^{-\zeta} \sin \zeta. \tag{17}$$

These solutions, normalized by $U(y)$, are plotted in Figure 2. They have the satisfying characteristic that as you move far from the boundary (as $\zeta \to \infty$), they approach the far–field solution of $u = U$ and $v = 0$. These same solutions are shown in Figure 3 as a hodograph, which traces out the direction of the total horizontal velocity vector. At the boundary $z = \zeta = 0$, the flow is at an angle of 45° to the far–field flow, and as ζ increases the velocity traces out a spiral, known as the Ekman spiral. It is interesting to note that Ekman began his investigations into boundary layers in rotating frames because Fridtjof Nansen observed from the deck of the *Fram* that icebergs in the Nordic Seas tend to move at an angle of 45° to the wind. Here we see that same observation arise in the mathematics.

At this point it is worth pausing briefly to consider physically what is happening. As fluid flows over the frictional plate, vorticity is generated at the boundary. This vorticity diffuses upward into the bulk of the fluid, tilting lines of planetary vorticity, 2Ω. At the same time, this diffusion is balanced and cancelled by the tilting of lines of constant vorticity caused by the rotation of the frame. Far from the plate, these two effects balance completely. The thickness of the boundary layer δ is the distance over which the vorticity shed from the bottom boundary moves before being balanced and cancelled by the tilting of planetary vorticity. It is a diffusive and inertial scale.

Now let us return to our calculated solution for velocity in the boundary layer, equations (16) and (17). As we are in a rotating frame, we generally expect flow to be along lines of constant pressure, called isobars. In the far field, pressure and velocity are both purely functions of y, and we see flow in the x–direction. However, in the boundary layer $v \neq 0$; we get flow *across* isobars. The combination of bottom friction and a rotating frame creates flow down the pressure gradient and perpendicular to the free fluid velocity. The total transport of this perpendicular velocity is given by

$$\int_0^\infty v \mathrm{d}z = \delta \int_0^\infty v \mathrm{d}\zeta = \frac{\delta}{2} U(y).$$

This perpendicular transport is known as the Ekman flux. If $U(y)$ is not constant, then our Ekman layer solution is horizontally divergent or convergent. By continuity (equation 14), this induces vertical motion, forcing fluid out of or into the boundary layer. This induced vertical motion is also referred to as the Ekman pumping. We find

$$W = -\frac{1}{2} \frac{\partial U}{\partial y} [1 - e^{-\zeta} \left(\cos \zeta + \sin \zeta\right)].$$

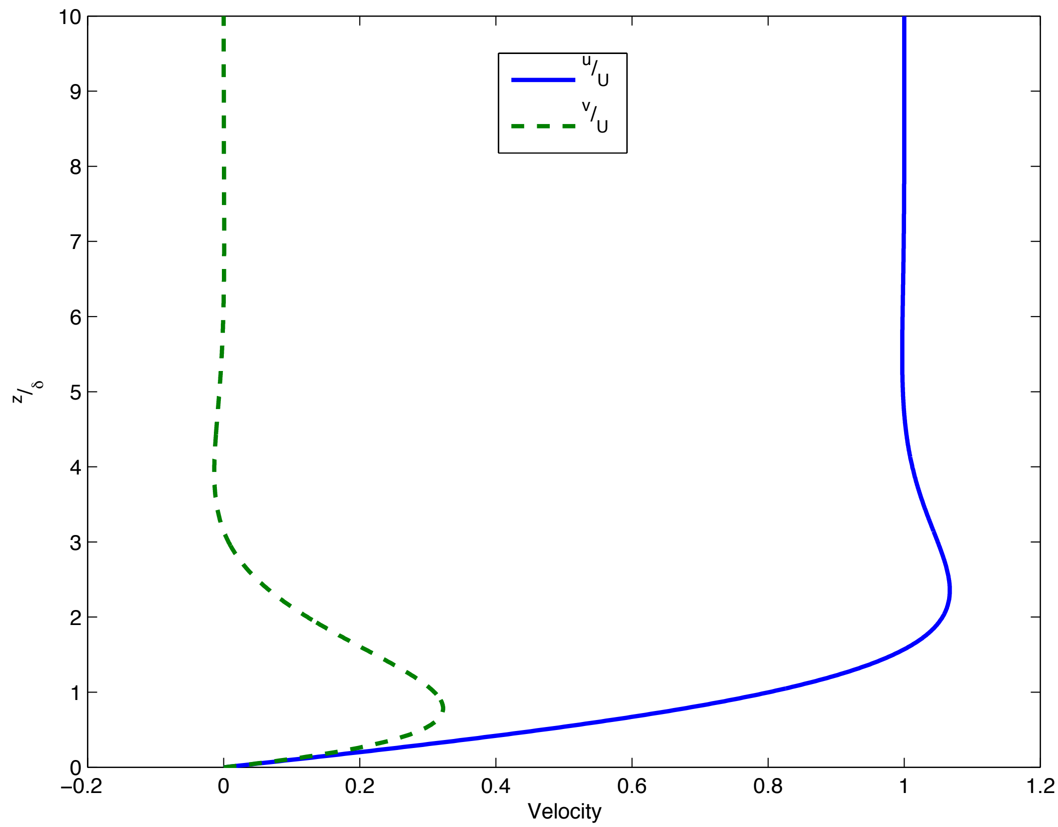

Figure 2: Velocity components in the boundary layer. Note the overshoot of the velocity profile at $u \approx 2$ and the oscillatory nature of the profile. This reflects the underlying inertial wave dynamics.

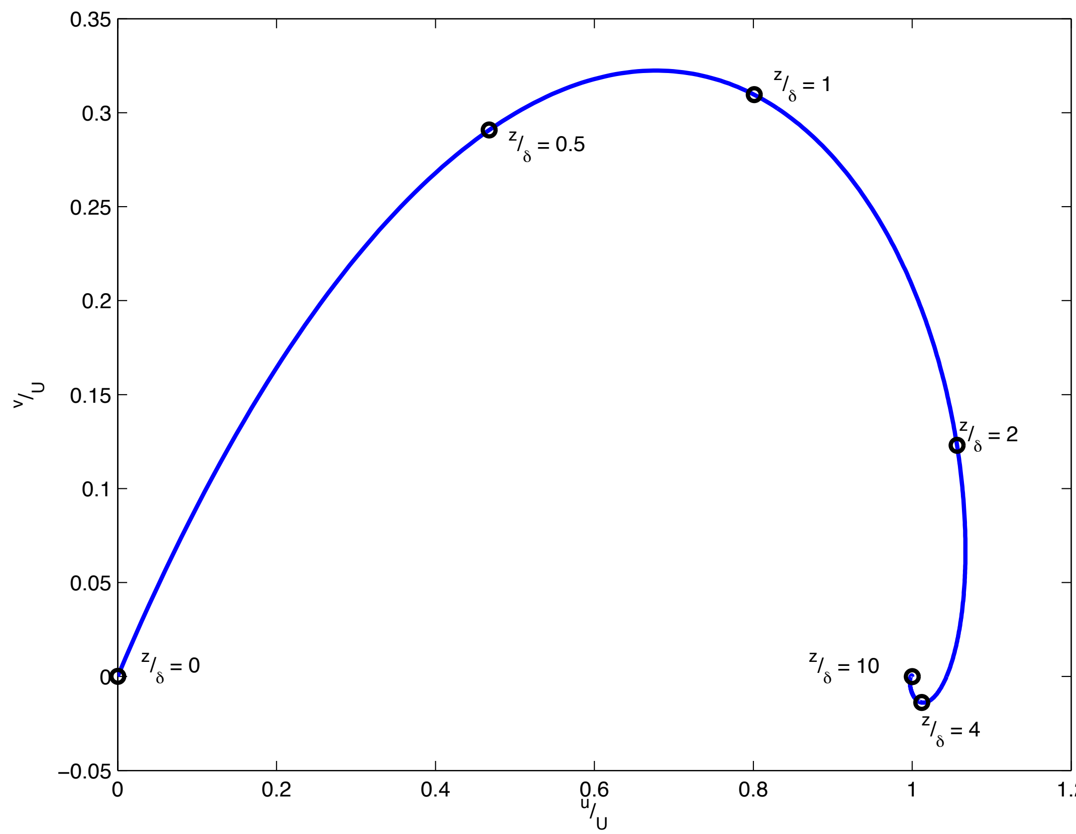

Figure 3: Ekman layer hodograph. This shows the line traced by the velocity vector as distance from the plate increases. Note that at the plate the fluid starts with a velocity at 45° to the forcing.

To derive this, we used the boundary condition that $W(\zeta = 0) = 0$. As we move far away from the bottom boundary, the lower boundary condition on the interior flow is

$$w_I(y, 0) = \sqrt{E}W(y, \zeta \to \infty) = -\frac{1}{2}\sqrt{E}U_y.$$

The combination of rotation and dissipation forces convergence in the boundary layer, which in turn creates a vertical velocity throughout the interior of the fluid. This vertical velocity is proportional to the vorticity of the interior flow. In the ocean, Ekman pumping is driven by layers at the surface and bottom. The surface Ekman pumping is on the order of 10^{-6}ms^{-1}, but it is enough to force the large–scale circulation of the entire ocean.

2 Spin Down

The vertical velocity we just calculated can have many substantial effects, including spinning down the fluid through vorticity conservation. If Ekman pumping forces fluid out of the boundary layer and into the overlying fluid, it will vertically compress the vortex tube of that fluid, an effect sometimes referred to as 'vortex squashing'. Conservation of circulation requires that the radial velocity in a vortex tube declines as the radius of the tube expands. This is described by the vorticity equation

$$\frac{d\omega}{dt} = \omega \cdot \nabla u + \text{Dissipation}.$$

For small ϵ in the interior, this reduces to

$$\epsilon\frac{d\omega}{dt} = \frac{\partial w}{\partial z}. \tag{18}$$

ω is the vertical component of the vorticity, given by

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\frac{\partial U}{\partial y}.$$

Equation (18) comes from taking the full vorticity equation, expanding in ϵ, and throwing out the higher order terms. We can integrate this equation over the depth of the fluid from the top of the boundary layer up. We know that ω is independent of z by the Taylor–Proudman effect for homogeneous fluids. Therefore, the left hand side is unchanged by integration. The right hand side simply gives us the difference between the vertical velocity at the bottom of our domain—which we know—and the vertical velocity at the top of our domain—which we assume is zero. We get

$$\begin{aligned}\epsilon\frac{d\omega}{dt} &= -\sqrt{E} \times w(z = 0) \\ &= \sqrt{E}\frac{1}{2}\frac{dU}{dy} = -\frac{\omega}{2}\sqrt{E}.\end{aligned}$$

The solution to this ODE is just an exponential decay, with non–dimensional decay time

$$T = \frac{2\epsilon}{\sqrt{E}}.$$

In dimensional units, this is

$$T = \frac{L}{U_0}\frac{2U_0}{2\Omega L}\sqrt{\frac{\Omega L^2}{\nu}} = \frac{L}{\sqrt{\Omega\nu}}.$$

This is the 'spin–down time' of the system, the characteristic time it takes for the vorticity from the bottom boundary to diffuse into the free fluid. It is so–called because it is the time scale over which the fluid would come to rest with respect to the bottom boundary. It is long compared with the rotation period of the system Ω, but it is *short* compared with the diffusion time of the system, $\frac{L^2}{\nu}$. This scale tells us that the larger the rotation rate, the less important viscosity becomes. However, the larger the rotation rate, the more rapidly the fluid is expected to spin down. This is because the coupling between the boundary layer and the interior is *inertial*, not viscous. The coupling of the inviscid vorticity dynamics of the interior and the viscous dynamics of the boundary layer is through vortex stretching.

3 Nonlinear Modifications of Ekman Layer

So far we have discussed only the linear theory of rotating boundary layers, however we can expect that introducing nonlinearity will both add terms and potentially change the structure of the solution. As we saw in equation (6), the thickness of the boundary layer is roughly given by the ratio of the effect of viscosity and rotation

$$\delta \sim \sqrt{\frac{\text{viscosity}}{\text{rotation}}}.$$

When we consider nonlinearity, we add the effects of advection, manifest as local vorticity. We therefore might guess that the thickness of the nonlinear boundary layer is given by something like

$$\delta = \sqrt{\frac{2\nu}{f+\omega}}.$$

In this $f = 2\Omega$ is the so–called Coriolis parameter and ω is the local or relative vorticity as before. However, the addition of relative vorticity has two competing effects. It causes the boundary layer to be thinner by the above equation, but at the same time it induces positive vertical velocity in the interior. This vertical velocity carries vorticity from the lower boundary upward, thickening the region affected by the presence of the boundary—that is thickening the boundary layer. A priori it is not obvious which of these effects is stronger and if the BL will get thicker or thinner. An exercise to develop intuition about problems of this type is to calculate the boundary layer flow over a non-rotating plate which has a uniform downward velocity through its surface.

To gain a quantitative insight into the net effect of this nonlinearity, we want to concentrate our attention on the scale at which it is most relevant, in the transition region between the Ekman layer and the free fluid interior. In this transitional region, we expect inertial effects—and therefore relative vorticity—to be as important as viscosity. In order for viscous and inertial effects to both be $O(1)$ in equations (7) – (10), we must again rescale our vertical coordinate, this time by the Rossby number ϵ

$$Z = \epsilon\zeta = \frac{\epsilon}{\sqrt{E}}z.$$

We now have three scales: the Ekman scale, where viscosity and rotation balance; the transitional scale, where viscosity and inertia balance; and the large scale of the free fluid, which we treat as invsicid. We assume that these scales are so well separated that we can treat Z and ζ as independent variables. By the chain rule, we then find

$$\frac{\partial}{\partial z} = \sqrt{\frac{1}{E}}\frac{\partial}{\partial \zeta} + \epsilon\sqrt{\frac{1}{E}}\frac{\partial}{\partial Z}.$$

Our equations of motion (1)–(4) become

$$\begin{aligned}
\epsilon\left(v\frac{\partial u}{\partial y} + W\frac{\partial u}{\partial \zeta}\right) - v &= \frac{1}{2}\left(E\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial \zeta^2} + 2\epsilon\frac{\partial^2 u}{\partial \zeta \partial Z}\right) \\
\epsilon\left(v\frac{\partial v}{\partial y} + W\frac{\partial v}{\partial \zeta}\right) + u &= -\frac{\partial p}{\partial y} + \frac{1}{2}\left(E\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial \zeta^2} + 2\epsilon\frac{\partial^2 v}{\partial \zeta \partial Z}\right) \\
\epsilon E\left(v\frac{\partial W}{\partial y} + W\frac{\partial W}{\partial \zeta}\right) &= -\frac{\partial p}{\partial \zeta} - \epsilon\frac{\partial p}{\partial Z} + \frac{E}{2}\left(E\frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial \zeta^2} + 2\epsilon\frac{\partial^2 W}{\partial \zeta \partial Z}\right) \\
.\frac{\partial v}{\partial y} + \frac{\partial W}{\partial \zeta} + \epsilon\frac{\partial W}{\partial Z} &= 0
\end{aligned}$$

We now expand every variable in powers of ϵ, for example

$$\begin{aligned}
u &= u_0 + \epsilon u_1 + \ldots \\
p &= p_0 + \epsilon p_1 + \ldots \\
&\vdots \quad .
\end{aligned}$$

To lowest order, the problem we find is the linear problem we have already solved. If we repeat the solution procedure from the first section of this lecture, we find

$$u_0 = U(y) - A(y,Z)e^{-\zeta}\cos\zeta + B(y,Z)e^{-\zeta}\sin\zeta \tag{19}$$

$$v_0 = A(y,Z)e^{-\zeta}\sin\zeta + B(y,Z)e^{-\zeta}\cos\zeta. \tag{20}$$

Note that we now allow our coefficients A and B to vary with Z. Mathematically, this is a consequence of treating Z and ζ as independent variables. Physically, this is possible because changes with Z are so slow on the ζ scale that the coefficients still act as though they were constant. From the no–slip condition, we have the conditions on the coefficients that $A(Z=0) = U$ and $B(Z=0) = 0$. Again as in the first section, we can use the convergence of u_0 and v_0 to calculate the induced vertical velocity

$$W_0 = C(Z) + \frac{1}{2}\frac{\partial A}{\partial y}e^{-\zeta}(\cos\zeta + \sin\zeta) + \frac{1}{2}\frac{\partial B}{\partial y}e^{-\zeta}(\cos\zeta - \sin\zeta).$$

$C(Z)$ is a constant of integration. To find $C(Z)$, we must consider the far–field flow that the vertical velocity matches to. By vorticity conservation, $\frac{\partial w}{\partial z} = 0$ to order ϵ in the interior from equation (18), so $w =$ constant. This means that $C(Z) = C(0) = -\frac{1}{2}\frac{\partial U}{\partial y}$ in order that the vertical velocity at the bottom of the interior flow matches that at the top of the boundary layer. This gives

$$W_0 = \frac{1}{2}\frac{\partial U}{\partial y} + \frac{1}{2}\frac{\partial A}{\partial y}e^{-\zeta}\left(\cos\zeta + \sin\zeta\right) + \frac{1}{2}\frac{\partial B}{\partial y}e^{-\zeta}\left(\cos\zeta - \sin\zeta\right).$$

However, to find A and B we must move on to the next higher order problem. As before, we can express it most compactly using complex notation

$$\frac{\partial^2 \Lambda_1}{\partial \zeta^2} - 2i\Lambda_1 = Ru + iRv. \tag{21}$$

where $\Lambda_1 = u_1 + iv_1$, and Ru and Rv are the nonlinear terms from the governing equations

$$\begin{aligned} Ru &\equiv 2\left(v_0 \frac{\partial u_0}{\partial y} + W_0 \frac{\partial u_0}{\partial \zeta}\right) - 2\frac{\partial^2 u_0}{\partial \zeta \partial Z} \\ Rv &\equiv 2\left(v_0 \frac{\partial v_0}{\partial y} + W_0 \frac{\partial v_0}{\partial \zeta}\right) - 2\frac{\partial^2 v_0}{\partial \zeta \partial Z}. \end{aligned}$$

We can see that some of the terms of Ru and Rv have the same form as the homogeneous solution of the left–hand side of equation (21), proportional to $e^{-(1+i)\zeta}$. This is a kind of resonance between the homogeneous solutions and the forcing functions. Forcing functions like these are called secular terms, and they give rise to terms of the form $\zeta e^{-(1+i)\zeta}$ in these solutions. They will grow linearly, and eventually lead to u_1 and v_1 to become the same size as u_0 and v_0, when $\epsilon\zeta = Z$ is $O(1)$. At this point, our expansion in powers of ϵ would be invalid. Therefore, we must eliminate these terms by setting their coefficients to zero.