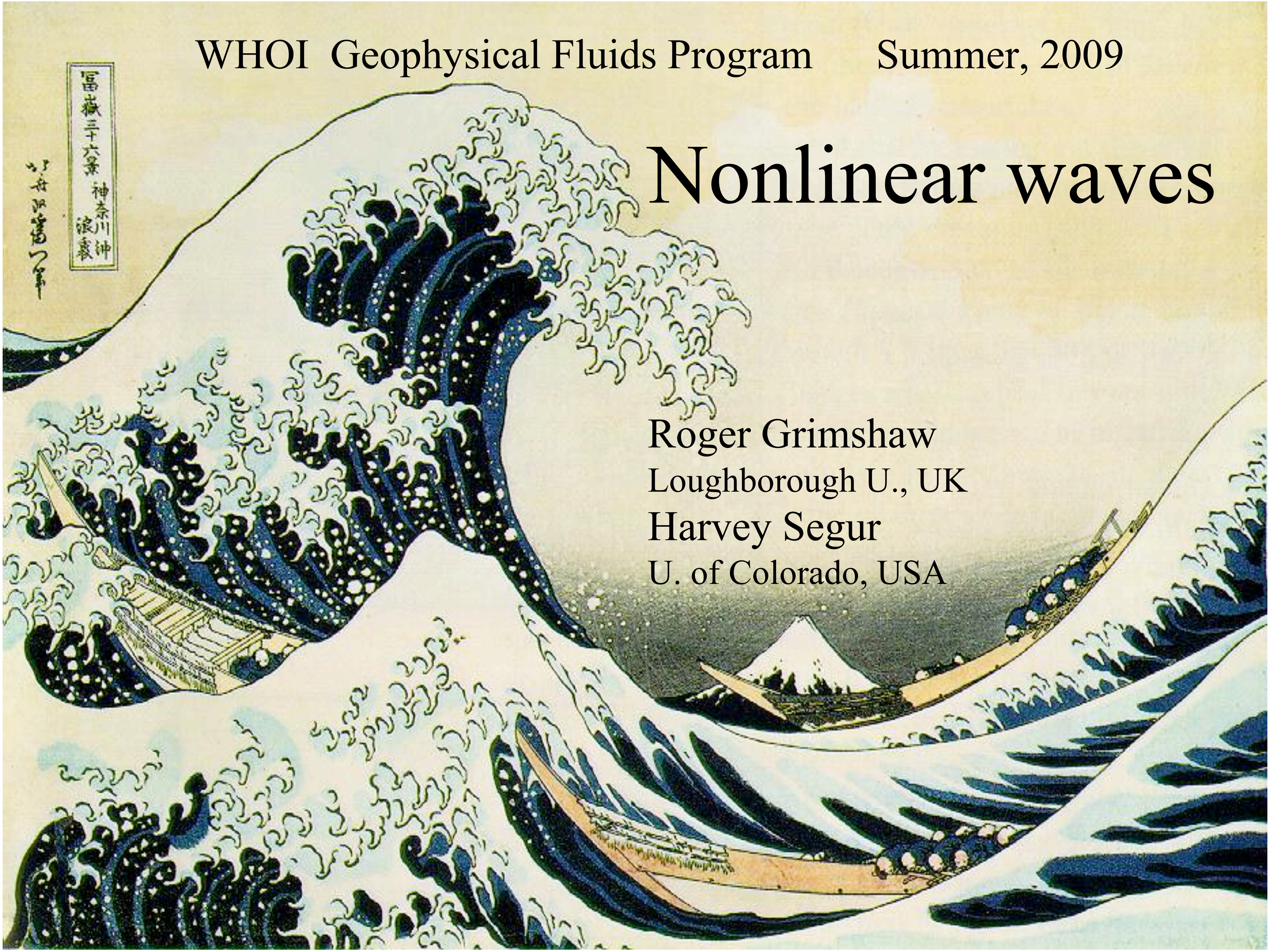


WHOI Geophysical Fluids Program Summer, 2009

# Nonlinear waves

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U. of Colorado, USA

# Introduction to water waves

Lecture 1

Q: What’s so special about water waves?

# Why study water waves?

a) Woods Hole Oceanographic Institute

b) For the Program on Nonlinear Waves:

Water waves provide a concrete physical example of a dynamical system rich enough to exhibit many of the mathematical concepts that have been developed in recent years:

linear stability, nonlinear stability

solitons, complete integrability

chaos, sensitive dependence on initial data

singularities, blow-up in finite time

deterministic vs. probabilistic models

c) Water waves evolve on a “human” time-scale, so we can observe many of these concepts in physical experiments

# Introduction to water waves

Q: What are "water waves"?

A (for my lectures): Waves in the water that you see or feel at the beach or in a boat

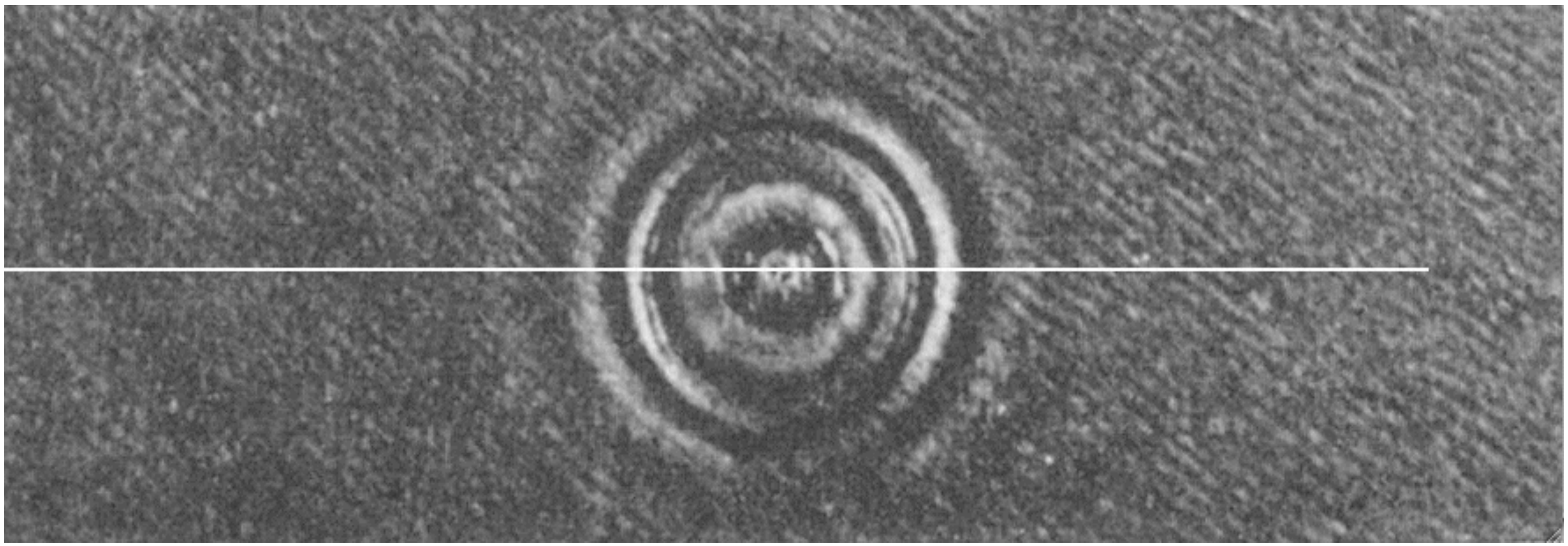
(sometimes called "surface water waves")

# Properties of water waves

- Surface water waves have their maximum displacement at the free surface
- Waves propagate, with little dissipation
  - ask a baby in a bath-tub
  - Snodgrass *et al* (1966)
- Approximately periodic: 0.1 s < $T$ < 25 s.
- Approximate maximum speed: $c = \sqrt{gh}$
- Water waves are “dispersive”:

  Long waves travel faster than short waves
  (for gravity-induced waves)

# Long waves travel faster than short waves (for gravity-induced waves)



from Stoker's *Water Waves* (1957)

# Ocean waves I am ignoring

- Sound waves (pressure waves) in water

  Speed of sound in water (at 10° C): 1450 m/sec

  Speed of 2004 tsunami: < 200 m/sec

  Pressure waves create initial conditions for surface waves

- Internal waves

  Due to variations in fluid density

  Period of surface waves: seconds

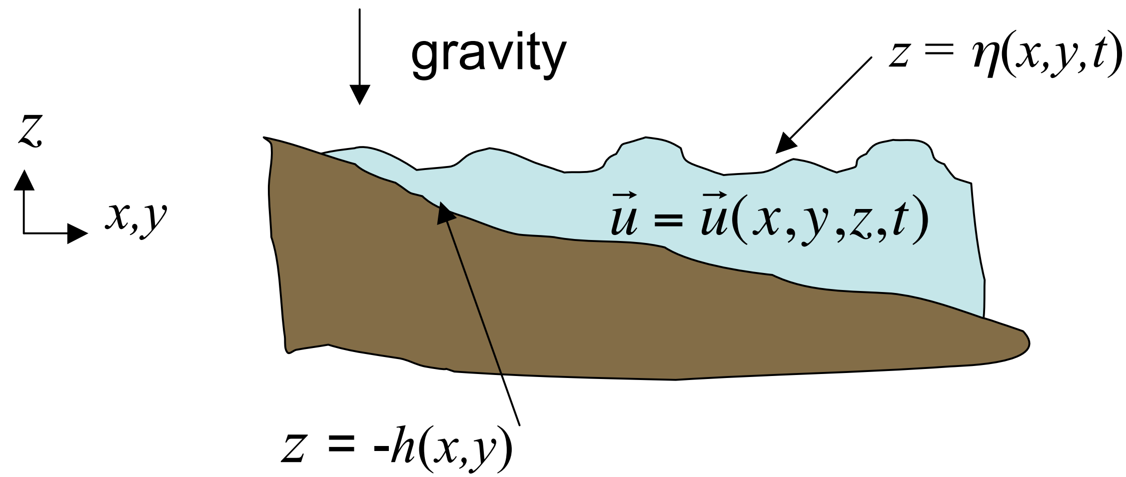
  Period of internal waves: hours

- Inertial waves (including Rossby waves)

  Due to rotation of earth

  Period of inertial waves ≥ 12 hours

# Derive the governing equations
## (following Stokes, 1847)




Find: $\eta(x,y,t)$ – position of free surface, and

$\vec{u}(x,y,z,t)$ – fluid velocity, for

$-h(x,y) < z < \eta(x,y,t)$, for $t > 0$

and for all $\{x,y\}$ with $\eta + h > 0$

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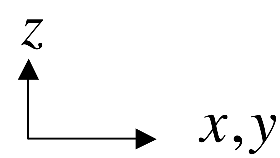
- [ one more coming… ]

# Derive the governing equations

## 2. Coordinate system(s)

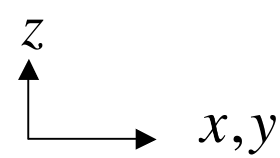



- $[x, y, z]$ denote fixed, laboratory coordinates
- Denote the current location of a fixed fluid particle by

$$\vec{x}(t) = \{x(t), y(t), z(t)\}$$

# Derive the governing equations

## 2. Coordinate system(s)

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– Velocity of this fluid particle is $\vec{u}(\vec{x}, t) = (u, v, w) = \left(\frac{Dx}{Dt}, \frac{Dy}{Dt}, \frac{Dz}{Dt}\right)$

– so $\left(\frac{D}{Dt}\right)$ means "following the fluid particle"

# Derive the governing equations

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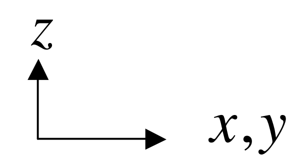


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– $\left(\frac{D}{Dt}\right)$ means “following the fluid particle”

– acceleration of the fluid particle (in $z$-direction) is

$$\frac{Dw}{Dt}(t, \vec{x}(t)) = \frac{\partial w}{\partial t} + \frac{\partial w}{\partial x} \cdot \frac{Dx}{Dt} + \frac{\partial w}{\partial y} \cdot \frac{Dy}{Dt} + \frac{\partial w}{\partial z} \cdot \frac{Dz}{Dt},$$

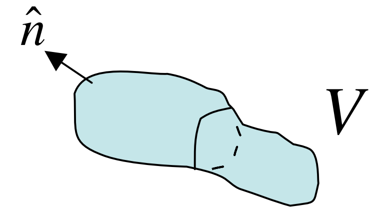
$$\frac{Dw}{Dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}.$$

# Derive the governing equations

3. Physics

(a) Mass conservation of fluid with constant density

Total mass flow rate from V

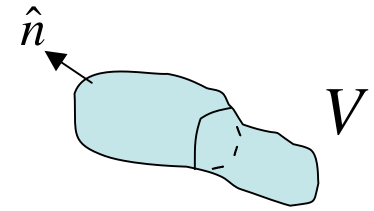



$$= \oiint_{\partial V} [(\rho)\vec{u} \cdot \hat{n}]ds = 0$$

# Derive the governing equations

## 3. Physics

### (a) Mass conservation of fluid with constant density

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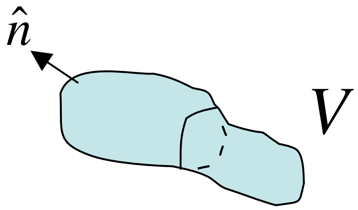


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Divergence theorem ➔

$$= \rho \iiint_V [\nabla \cdot \vec{u}] dv = 0$$

Valid for all choices of V ➔

$$\boxed{\nabla \cdot \vec{u} = 0}$$

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(b) Aside:

Define vorticity: $\vec{\omega}$ = curl( $\vec{u}$ )

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Suppose (for some reason) $\vec{\omega} \equiv 0$

➔ there is a velocity potential, $\phi(\vec{x},t)$, with $\vec{u} = \nabla\phi$

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Then $\nabla \cdot \vec{u} = 0 \quad \Leftrightarrow \quad \nabla^2 \phi = 0$

The velocity of the fluid is found by solving LaPlace' eq'n!

What could be simpler?

# Derive the governing equations

## 3. Physics

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The velocity of the fluid is found by solving LaPlace' eq'n!

What could be simpler?

Q: Is there any reason to believe $\vec{\omega} \equiv 0$ ?

# Derive the governing equations

## 3. Physics

### (c) Back to reality

Navier-Stokes equations:

$$\rho \frac{D\vec{u}}{Dt} + \nabla p + \rho g \hat{z} = \mu \nabla^2 \vec{u}$$

density acceleration pressure gravity viscous forces

# Derive the governing equations

## 3. Physics

### (c) Back to reality

Navier-Stokes equations:

$$\rho \frac{D\vec{u}}{Dt} + \nabla p + \rho g \hat{z} = \cancel{\mu \nabla^2 \vec{u}}$$

Neglect viscous forces ➔ Euler's equations

$$\frac{D\vec{u}}{Dt} + \frac{1}{\rho} \nabla p + g\hat{z} = 0$$

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Neglect viscous forces ➔ Euler's equations

$$\frac{D\vec{u}}{Dt} + \frac{1}{\rho}\nabla p + g\hat{z} = 0$$

Take curl ➔ $$\boxed{\frac{D\vec{\omega}}{Dt} = \partial_t \vec{\omega} + \vec{u} \cdot \nabla \vec{\omega} = \vec{\omega} \cdot \nabla \vec{u}}$$

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Take curl ➔ $$\boxed{\frac{D\vec{\omega}}{Dt} = \partial_t \vec{\omega} + \vec{u} \cdot \nabla \vec{\omega} = \vec{\omega} \cdot \nabla \vec{u}}$$

If $\vec{\omega} \equiv 0$ at $t$ = 0, then $\vec{\omega} \equiv 0$ for all $t$.

# Derive the governing equations

## 3. Physics

(d) Last assumption: irrotational flow

$$\vec{\omega} \equiv 0 \qquad \text{at } t = 0.$$

➔ $\vec{u} = \nabla\phi, \quad \nabla^2 \phi = 0$ everywhere in fluid.

This gives the simplest mathematical model, and the one studied the most. Dropping the assumption gives a more general model. (For internal waves, this is not valid.)

# Derive the governing equations

## 3. Physics

### (e) Bernoulli's Law:

Show $\dfrac{D\vec{u}}{Dt} = \partial_t \vec{u} + \nabla(\tfrac{1}{2}\,|\,\vec{u}\,|^2) - \vec{u} \times \vec{\omega}$

# Derive the governing equations

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For an irrotational flow,

$$\frac{D\vec{u}}{Dt} = \partial_t \vec{u} + \nabla(\tfrac{1}{2}|\vec{u}|^2) = \nabla\{\partial_t \phi + \tfrac{1}{2}|\nabla\phi|^2\}$$

# Derive the governing equations

## 3. Physics

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Euler ➔ $$\nabla\{\partial_t \phi + \tfrac{1}{2}|\nabla\phi|^2 + \frac{p}{\rho} + gz\} = 0$$

➔ $$\boxed{\partial_t \phi + \tfrac{1}{2}|\nabla\phi|^2 + \frac{p}{\rho} + gz = F(t)}$$

# Derive the governing equations

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### (e) Bernoulli's Law:

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➔ $$\boxed{\partial_t \phi + \tfrac{1}{2}|\nabla\phi|^2 + \frac{p}{\rho} + gz = \cancel{F(t)}}$$ = 0 in fluid

# Derive the governing equations

3. Physics

(f) What happens on the bottom boundary of fluid?

– No flow through the (solid) bottom boundary:

➔ on $z = -h(x,y)$,

$$\vec{u} \cdot \nabla\{z + h(x,y)\} = 0 \quad \text{or}$$

$$\partial_z \phi + \nabla\phi \cdot \nabla h = 0 \quad \text{or}$$

$$\partial_n \phi = 0$$

# Derive the governing equations

3. Physics

(g) What happens at the free surface?

– With no surface tension and no wind

$$p = p_{air} \; (= 0) \qquad \text{on} \quad z = \eta(x,y,t)$$

# Derive the governing equations

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- With surface tension, usual model:

$$p = p_{air} + \underset{\text{const}}{\sigma}\underbrace{\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}_{\text{mean curvature}}$$

# Derive the governing equations

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➔ $$\boxed{p = 0 - \sigma\left[\nabla \cdot \left(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right)\right]} \quad \text{on} \quad z = \eta(x,y,t).$$

# Derive the governing equations

## 3. Physics

(g) What happens at the free surface?

– 2 boundary conditions on $z = \eta(x,y,t)$:

* Kinematic: a particle on the surface remains there

$$\frac{D}{Dt}\{\eta(x(t),y(t),t) - z\} = 0$$

➔ $$\boxed{\partial_t \eta + \nabla\phi \cdot \nabla\eta = \partial_z \phi}$$

# Derive the governing equations

## 3. Physics

### (g) What happens at the free surface?

– 2 boundary conditions on $z = \eta(x,y,t)$:

* Kinematic: a particle on the surface remains there

$$\frac{D}{Dt}\{z(t) - \eta(x(t), y(t), t)\} = 0$$

➔ $$\boxed{\partial_t \eta + \nabla\phi \cdot \nabla\eta = \partial_z \phi}$$

* Dynamic: Use surface pressure in Bernoulli's Law

$$\boxed{\partial_t \phi + \frac{1}{2}|\nabla\phi|^2 + g\eta = \frac{\sigma}{\rho}\nabla\cdot\{\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\}}$$

(+ wind?)

# Equations of motion

for an irrotational flow, with no forcing from wind:

$$\partial_t \eta + \nabla\phi \cdot \nabla\eta = \partial_z \phi, \qquad \text{on } z = \eta(x,y,t)$$

$$\partial_t \phi + \frac{1}{2}|\nabla\phi|^2 + g\eta = \frac{\sigma}{\rho}\nabla \cdot \{\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\}, \qquad \text{on } z = \eta(x,y,t),$$

$$\nabla^2\phi = 0 \qquad -h(x,y) < z < \eta(x,y,t),$$

$$\partial_z \phi + \nabla\phi \cdot \nabla h = 0, \qquad \text{on } z = -h(x,y).$$

# Extra lecture -optional

Monday afternoon (when?):

Start with the nonlinear equations of water waves, linearize them, and explore some fundamental concepts that emerge from the linearized equations.