Boundary Layers: Sloping bottoms in a stratified, rotating fluid

Lecture 5 by Iva Kavčič

In oceanic coastal regions, e.g. on the shelf regions between the coast and the deep ocean, the bottom generally slopes and the fluid is stratified.
We have already seen the way the thermal boundary layers on vertical walls can control the interior flow and how the Ekman layers on horizontal boundaries can do the same for rotating fluids. Sloping boundaries are a type of a hybrid of these two.

1 The model

We begin with the schematic of the bottom boundary layer (hereafter: BL), shown in Fig. 1. Fluid

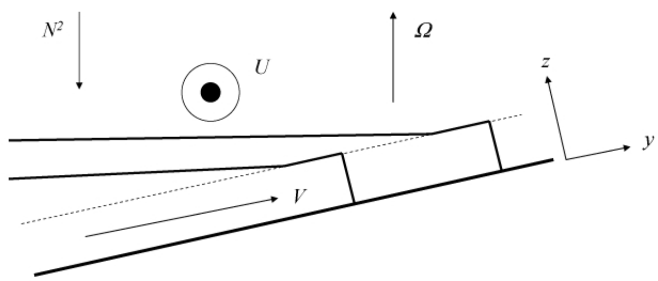

Figure 1: A schematic of the bottom BL, upwelling case

is stratified with the density gradient $\partial\rho/\partial z'$. Here z' denotes the direction of the true vertical (Fig. 2), aligned with the direction of gravity, g, and planetary rotation, Ω. The bottom is in direction y of the slant coordinate frame (y, z), rotated counterclockwise with the angle θ with respect to the reference coordinate frame (y', z'), (Fig. 2). The density gradient, $\partial\rho/\partial z'$, produces the buoyancy

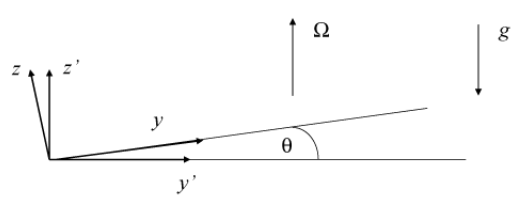

Figure 2: The reference, (y', z'), and the slant coordinate frame, (y, z)

force in the true vertical, with the frequency, (Fig. 1),

$$N^2 = -\frac{g}{\rho}\frac{\partial \rho}{\partial z'}. \tag{1}$$

As in the linear Ekman layer problem, the flux is to the left (from high to low pressure) of the along isobar flow component U (Fig. 1). Here this results in canceling of the density gradient component perpendicular to the bottom ($\partial\rho/\partial z$) due to the upslope transport of the heavier fluid by the cross isobar flow component V, and formation of the mixed BL (Fig. 1). The density gradient component along the slope ($\partial\rho/\partial y$) can be derived as

$$\frac{\triangle\rho}{\triangle z'} = \frac{\triangle\rho}{\triangle y \sin\theta}, \tag{2}$$

$$g\frac{\partial\rho}{\partial y} = -\rho N^2 \sin\theta. \tag{3}$$

From (3) we see that fluid moving up the slope a distance $\triangle y = \triangle z'/\sin\theta$, (Fig. 3), will produce a density anomaly

$$\triangle\rho = -\frac{1}{g}N^2\rho_0 \sin\theta \triangle y. \tag{4}$$

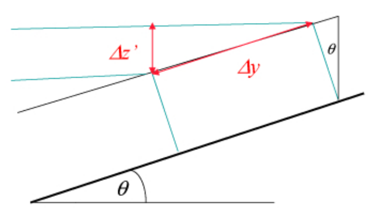

Figure 3: The distances in reference and slant coordinate frame

From (3) and the thermal wind relation

$$f\frac{\partial U}{\partial z} = \frac{g}{\rho}\frac{\partial\rho}{\partial y}, \tag{5}$$

we see that over a depth of the bottom BL of the order

$$H = -\frac{fU}{N^2\sin\theta} \tag{6}$$

it would be possible to adjust the speed of the current to zero without Ekman layers and their dissipation, i.e. currents could flow long distances without decay. Here f is the Coriolis parameter, $f = 2\Omega\cos\theta \approx 2\Omega$, for θ small.

Following MacCready and Rhines, MacCready and Rhines (1993), we write the equations in the slant coordinate frame (Fig. 2):

$$v = v'\cos\theta + w'\sin\theta \tag{7}$$

$$w = -v'\sin\theta + w'\cos\theta \tag{8}$$

Far from the lower boundary the temperature is:

$$T_\infty = \triangle T_v z'/D, \tag{9}$$

with $\triangle T_v$ being the mean temperature difference in the true vertical, z' (Fig. 2). The buoyancy frequency now can be defined as:

$$N = \left(\frac{g\alpha\triangle T_v}{D}\right)^{1/2}. \tag{10}$$

The temperature equation is then:

$$\begin{aligned} T &= \triangle T_v z'/D + \vartheta(y,z) && (11)\\ &= \triangle T_v (z\cos\theta + y\sin\theta)/D + \vartheta(y,z), && (12) \end{aligned}$$

where the $\vartheta(y,z)$ is the temperature perturbation. We assume that $\vartheta \to 0$ as $z \to \infty$.

2 The steady-state solution

As in the case of stratified fluid, we investigate the steady-state behavior of the flow in the BL and interior. Here we simplify the problem by searching for the solutions independent of y, i.e. only functions of z. As $z \to \infty$, $u \to U$, and a constant v is independent of y. The above, together with the assumption of incompressibility, gives $w \equiv 0$. Furthermore, nonlinear terms in equations vanish identically.
The governing equations of motions then are:

$$\begin{aligned} 2\Omega\cos\theta u &= -\frac{1}{\rho_0}\frac{\partial\tilde{p}}{\partial y} + Av_{zz} + b\sin\theta, && (13)\\ -2\Omega\cos\theta v &= Au_{zz}, && (14)\\ 2\Omega\sin\theta u &= -\frac{1}{\rho_0}\frac{\partial\tilde{p}}{\partial z} + b\cos\theta, && (15)\\ vN^2\sin\theta &= \kappa b_{zz}. && (16) \end{aligned}$$

Here A is the momentum mixing coefficient, κ is the thermal diffusivity and $\tilde{p}(y,z)$ is the pressure perturbation. Buoyancy perturbation, b, is given by

$$b = \alpha g\vartheta, \tag{17}$$

where α is the coefficient of thermal expansion, and Coriolis parameter is $f = 2\Omega\cos\theta$.
The boundary conditions (hereafter: BC) at the lower boundary ($z = 0$) are:

$$\begin{aligned} u(z=0) &= 0, && (18)\\ v(z=0) &= 0, && (19)\\ b_z(z=0) &= -N^2\cos\theta. && (20) \end{aligned}$$

Here (20) represents the insulating BC at $z = 0$.
Since u, v and b are independent of y, derivation of (15) with respect to y gives:

$$\partial^2\tilde{p}/\partial y\partial z = 0. \tag{21}$$

Then, from (13) and (14) we derive the boundary layer equation:

$$-\frac{f^2}{A}v = Av_{zzzz} + b_{zz}\sin\theta. \tag{22}$$

If the fluid were homogeneous ($\partial\rho/\partial z' = 0$) or if the bottom were flat ($\theta = 0$) we would recover the Ekman layer problem. Using the thermal equation (16) to eliminate b in favor of v yields:

$$v_{zzzz} + 4q^4 v = 0, \tag{23}$$

$$q^4 = \frac{1}{4}\left[\frac{f^2}{A^2} + \frac{N^2}{A\kappa}\sin^2\theta\right]. \tag{24}$$

The general form of the solution of (23) is $v \sim \exp(rz)$, which gives

$$r_{1,\ldots,4} = \pm(1 \pm i)\,q. \tag{25}$$

Since it is not physical for the solutions to grow exponentially in space, we keep the ones with $Re\,(r) < 0$. Their linear combination is also the solution of (23):

$$v = C\exp(-qz)\cos(qz) + B\exp(-qz)\sin(qz) \tag{26}$$

From (24) we can see that if the bottom is flat ($\theta = 0$) the BL scale is the Ekman layer thickness. If the bottom is vertical, i.e. if $\theta = \pi/2$, the scale is the buoyancy layer thickness. Applying the BC (19) gives $C = 0$. Then, from the thermal equation (16) follows

$$b_z = -B\frac{N^2\sin\theta}{2q\kappa}\exp(-qz)\left(\cos(qz) + \sin(qz)\right), \tag{27}$$

and from (14)

$$u = -\frac{f}{2Aq^2}B\exp(qz)\cos(qz) + U_\infty. \tag{28}$$

The insulating BC, (20), gives $B = 2q\kappa\cot\theta$, while the non slip condition on u, (18), yields

$$U_\infty = -\frac{f}{Aq}\kappa\cot\theta, \tag{29}$$

giving the solution for u

$$u = U_\infty\left[1 - \exp(-qz)\cos(qz)\right]. \tag{30}$$

Therefore, we see that the flow at infinity is not arbitrary - it is the part of the solution. Moreover, equation (29) gives the two limiting cases for u on horizontal (for $\theta = 0$, $U_\infty \to \infty$) and vertical ($U_\infty = 0$ for $\theta = \pi/2$) bottom.
Similarly to the linear Ekman problem, only the frictionally driven flow up the slope (v component) contributes to the total flux (stream function)

$$\Psi(z) = \int_0^z v\,dz = \kappa\cot\theta\left\{1 - \exp(qz)\left[\cos(qz) + \sin(qz)\right]\right\}. \tag{31}$$

The total, as $z \to \infty$, is:

$$\Psi(\infty) = \kappa\cot\theta = \frac{Aq}{f}U_\infty. \tag{32}$$

This also follows directly from the integral of the thermal equation (16), together with the insulating BC (20).
Now, we see that the boundary layer controls the interior through the dependence of the U_∞ on the thermal diffusion and the slope (29). This result, while at first glance non intuitive, is really just a manifestation of the control mechanisms we have already met in our discussion of the linear flow in the cylinder, although here in a more extreme form.
Unlike in the linear Ekman layer case, now we are not any more able to drive the system as we would like and establish some arbitrary equilibrium velocity along the isobaths. We continue our presentation by investigating the possibility of initially specifying a different far field along shore flow, and monitor its evolution in time. Further reference can be found in the work of MacCready, Rhines and Garrett, Garrett and Rhines (1993).

3 The slow diffusion equation

We now derive the "slow diffusion equation" (hereafter: SDE) from the time-dependant system of driving equations (13)-(16):

$$\begin{aligned}
\frac{\partial v}{\partial t} + 2\Omega\cos\theta u &= -\frac{1}{\rho_0}\frac{\partial \tilde{p}}{\partial y} + Av_{zz} + b\sin\theta, &(33)\\
\frac{\partial u}{\partial t} - 2\Omega\cos\theta v &= Au_{zz}, &(34)\\
2\Omega\sin\theta u &= -\frac{1}{\rho_0}\frac{\partial \tilde{p}}{\partial z} + b\cos\theta, &(35)\\
\frac{\partial b}{\partial t} + vN^2\sin\theta &= \kappa b_{zz}. &(36)
\end{aligned}$$

As in the case of Ekman layer in a stratified fluid, we apply the scaling: $y = Ly',\ z = Dz',\ t = \frac{D^2}{\kappa}t'$, where L, D and D^2/κ are the characteristic length, depth ant time scales, respectively, and y', z' and t' are the non-dimensional variables. The velocity components are scaled with $(u, v) = U(u', v')$, pressure is scaled with respect to the geostrophic balance, $\tilde{p} = \rho_0 fUL\tilde{p}'$, and buoyancy with $b = \frac{fUL}{D}b'$.
After defining the BL thickness as $\delta = D/L$ and dropping the primes, the dimensional system (33)-(36) can be written in the non-dimensional form:

$$\begin{aligned}
\frac{E}{2\sigma}\frac{\partial v}{\partial t} + u &= -\frac{\partial \tilde{p}}{\partial y} + \frac{E}{2}v_{zz} + b\frac{\sin\theta}{\delta}, &(37)\\
\frac{E}{2\sigma}\frac{\partial u}{\partial t} - v &= \frac{E}{2}u_{zz}, &(38)\\
-\delta\tan\theta u &= -\frac{\partial \tilde{p}}{\partial z} + b\cos\theta, &(39)\\
\frac{E}{2\sigma}\frac{\partial b}{\partial t} + \frac{\sin\theta}{\delta}Sv &= \frac{E}{2\sigma}b_{zz}, &(40)
\end{aligned}$$

where $f = 2\Omega\cos\theta$ is the Coriolis parameter, $E = \frac{2A}{fD^2}$ is the Ekman number, and $S = \frac{N^2\delta^2}{f^2}$ is the stability parameter.

In the interior (as in the case of stratified fluid before) it is reasonable to neglect friction and to assume stationarity. Also, $\tan\theta$ is small. Therefore, the system (37)-(40) reduces to:

$$u_I = -\frac{\partial p_I}{\partial y} + b_I \frac{\sin\theta}{\delta}, \tag{41}$$

$$b_I \cos\theta = \frac{\partial p_I}{\partial z}. \tag{42}$$

Eliminating v between (38) and (40) yields:

$$\frac{1}{\sigma}\frac{\partial}{\partial t}\left[u_I + \frac{\delta}{S\sin\theta} b_I\right] = \frac{1}{2}\frac{\partial^2}{\partial z^2}\left[u_I + \frac{\delta}{\sigma S\sin\theta} b_I\right]. \tag{43}$$

Eliminating the pressure in (41) and (42) and noting that b is independent of y gives:

$$\frac{\partial u_I}{\partial t} = \frac{\partial b_I}{\partial z}\left[\frac{\sin\theta}{\delta}\right], \tag{44}$$

and the SDE is then:

$$\frac{\partial}{\partial t}\left(\frac{\partial u_I}{\partial z}\right) = \sigma \frac{1 + \frac{\delta^2}{\sigma S \sin^2\theta}}{1 + \frac{\delta^2}{S\sin^2\theta}} \frac{\partial^2}{\partial z^2}\left(\frac{\partial u_I}{\partial z}\right). \tag{45}$$

We introduce the modified stability parameter, S_*, as:

$$S_* \equiv S\frac{\sin^2\theta}{\delta^2} = \frac{N^2}{f^2} sin^2\theta. \tag{46}$$

The effective diffusion coefficient then becomes:

$$\mu_{diff} = \sigma\left(\frac{\frac{1}{\sigma} + S_*}{1 + S_*}\right), \tag{47}$$

in the non-dimensional form, whereas its dimensional form is given with:

$$(\mu_{diff})_{dimensional} = A\left(\frac{\frac{1}{\sigma} + S_*}{1 + S_*}\right). \tag{48}$$

From both (47) and (48) we can see that if $\sigma > 1$, the diffusion coefficient would be smaller than in the absence of stratification.
Now, if u is independent of z and t as $z \to \infty$, we obtain the final form of the SDE:

$$\frac{\partial u_I}{\partial t} = mu_{diff}\frac{\partial^2 u_I}{\partial z^2}. \tag{49}$$

To obtain the BC for SDE we need to consider the BL at sloping bottom, i.e. find BC such as to match the boundary.
We introduce the BL coordinate:

$$\zeta = zE^{-1/2}, \ \lim_{z\to 0}\zeta = 0. \tag{50}$$

Then (labeling correction variables with "e") the BL equations for correction functions are:

$$
\begin{aligned}
u_e &= -\frac{\partial p_e}{\partial y} + \frac{1}{2} v_{z\zeta\zeta} + b_e \frac{\sin\theta}{\delta}, && (51)\\
-v_e &= \frac{1}{2} u_{e\zeta\zeta}, && (52)\\
-\delta \tan\theta u_e &= -\frac{1}{E^{1/2}} \frac{\partial p_e}{\partial \zeta} + b_e \cos\theta, && (53)\\
\frac{\sin\theta}{\delta} v_e &= \frac{1}{2\sigma S} b_{e\zeta\zeta}. && (54)
\end{aligned}
$$

These are the same steady equations, (13)-(16), and BC, (18)-(20) we dealt with before, yielding the equation for v_e:

$$
\begin{aligned}
\frac{\partial^4 v_e}{\partial \zeta^4} + 4 q_e^4 v_e &= 0, && (55)\\
q_e^4 &= \left[1 + \sigma S \frac{\sin^2\theta}{\delta^2}\right]. && (56)
\end{aligned}
$$

Like for the steady-state case, the BL solution is then:

$$
\begin{aligned}
v_e &= C \exp(-q_e\zeta) \cos(q_e\zeta) + B \exp(-q_e\zeta) \sin(q_e\zeta) && (57)\\
b_e &= \sigma S \frac{\sin\theta}{\delta} \left\{ \exp(-q_e\zeta) \left[(C - B)\sin(q_e\zeta) - (C + B)\cos(q_e\zeta)\right] \right\} && (58)\\
u_e &= \frac{1}{q_e^2} \left[C \exp(-q_e\zeta) \sin(q_e\zeta) - B \exp(-q_e\zeta) \cos(q_e\zeta)\right] && (59)\\
& && (60)
\end{aligned}
$$

Matching conditons between BL and the interior:

$$
\begin{aligned}
u_I + u_e &= 0 && (61)\\
v_I + v_e &= 0 && (62)
\end{aligned}
$$

Here: $v_I \sim O(E) \to B = 0$, and

$$
\frac{\partial b_I}{\partial z} + E^{-1/2} \frac{\partial b_e}{\partial \zeta} = -\frac{S}{\varepsilon} \cos\theta \to A = 0, \qquad (63)
$$

where $\varepsilon = \frac{U}{fL}$.

The frictional BL vanishes to the lowest order. Also, u_I must satisfy the no-slip BC at $z = 0$, giving us the solution for u_I:

$$
u_I = U_\infty \frac{2}{\sqrt{\pi}} \int_0^{\zeta/\sqrt{\mu_{diff} t}} \exp\left(-\varphi^2\right) d\varphi \qquad (64)
$$

Next order BL solution still has $V_e = 0 \Rightarrow A = 0$.

BL contribution to buoyancy flux yields:

$$
B = -\frac{\delta\alpha}{2\sigma S \sin\theta} \left[\frac{S}{\varepsilon}\cos\theta + \frac{\delta U_\infty}{\sin\theta \sqrt{\pi \mu_{diff} t}}\right] \qquad (65)
$$

The long time solution in BL is obtained as $t \to \infty$:

$$b_e = -E^{1/2}\frac{S}{\varepsilon\alpha}\cos\theta\exp\left(-\alpha\zeta\right)\cos\left(\alpha\zeta\right), \tag{66}$$

which is the steady state solution (in non-dimensional form) already attained.
Hence it is possible to consider arbitrary interior flows but, at least with the simple physics here, the boundary layer control eventually expunges the along isobath flow and yields an asymptotically weak frictional boundary layer. This, in one sense resolves the conundrum posed by the steady boundary layer solution in which the interior flow and the cross-shelf flow depended only on the stratification and the vertical thermal diffusion coefficient. Nevertheless, the solution presented here eventually approaches that very constrained solution.

References

P. MacCready Garrett, C. and P. B. Rhines. Boundary mixing and arrested ekman layers: Rotating stratified flow near a sloping boundary. *Annu. Rev. Fluid Mech.*, 25:291–323, 1993.

P. MacCready and P. B. Rhines. Slippery bottom boundary layers on a slope. *J. Phys. Ocean.*, 23: 5–22, 1993.