Boundary Layers: Homogeneous Ocean Circulation

Lecture 7 by Angel Ruiz-Angulo

The first explanation for the western intensification of the wind-driven ocean circulation was provided by Henry Stommel (1948). The following chapter considers that work and subsequent developments in the context of boundary layer theory.

1 The homogeneous model

Midlatitude ocean circulation can be approached by using *boundary layer theory*. We begin by idealizing the ocean basin as a box with irregular bottom and filled up with homogeneous water. At the top of the box, the ocean surface, the wind flows only on the zonal direction, x, but varies on the meridional direction, y. This imposed wind stress results in the surface Ekman layer, which drives subsurface ocean waters via vertical *Ekman pumping*. Figure 1 shows the idealized ocean basin for this model.

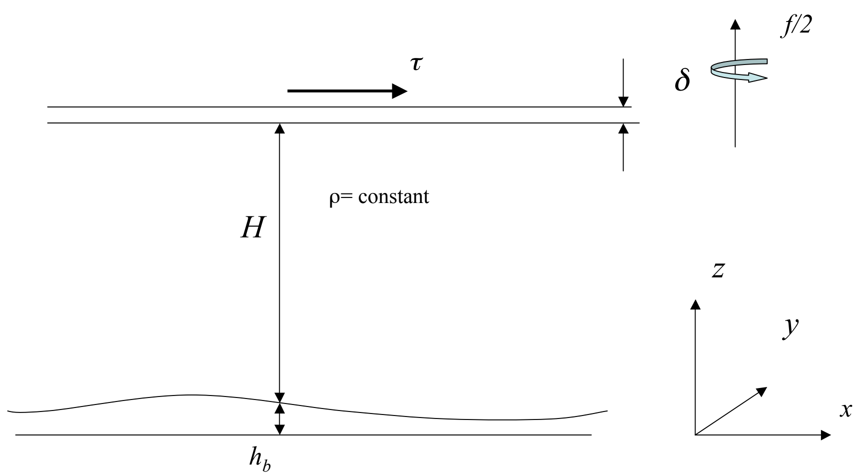

Figure 1: Proposed model for wind driven flows, allowing the bottom to have some variations on the topography

The Ekman pumping results in a vertical velocity, w_e, which is proportional to the curl of the wind stress (See lecture 1)

$$w_{top} = w_e = \hat{k} \cdot \nabla \times \left(\frac{\vec{\tau}}{\rho f} \right), \text{ at } \quad z = H + h_b \tag{1}$$

The contribution from the bottom Ekman pumping is given by

$$w_{bottom} = \frac{\delta}{2}\zeta + \vec{u}\cdot\nabla h_b = \frac{\delta}{2}\left[v_x - u_y\right] + uh_{bx} + vh_{by}, \quad \text{at} \quad z = h_b \tag{2}$$

In addition to the component due to the interior relative vorticity, ζ, the magnitude of w_{bottom} is affected by the interaction of the velocity with the topography. Note that the first term on the RHS corresponds to the classical *flat bottom solution*.

Assume that the interior (the fluid away from the bottom and top Ekman layers) is a homogeneous geostrophic flow over a non-uniform bottom. We now introduce the *beta approximation*. On a spherical planet, the Coriolis parameter is $f = 2\Omega sin\theta$. By expanding around a reference latitude, θ_0, and keeping the first order term we find the parameters for a Cartesian framework called $\beta-plane$:

$$f = \underbrace{2\ \Omega\ sin\theta_0}_{\text{reference Coriolis parameter}} + \underbrace{\frac{2\ \Omega\ cos\theta_0}{R}}_{\beta \text{ parameter}} \quad y \quad + \quad .... \tag{3}$$

Thus,

$$f = f_0 + \beta_0 y$$

In general, β is expressed as:

$$\beta = \frac{\partial f}{\partial y} = \frac{2\Omega cos\theta}{R}$$

We assume $\beta_0\ y\ << f_0$, which is called he $\beta-$plane approximation. Physically we are working on a cartesian plane tangent to the sphere at the reference latitude θ_0.

1.1 Equations of motion

By taking the curl of the N-S equations we can writhe the governing equations for the model in terms of vorticity

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\zeta + \beta v = f\frac{\partial w}{\partial z} + A\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\zeta \tag{4}$$

If u and v are independent of z (thermal wind and constant density) then w must be a linear combination of z. The upper and lower limits of the vertical velocity are given by the top Ekman layer (Eq. 1) and the bottom Ekman layer (Eq. 2), therefore:

$$\frac{\partial w}{\partial z} = \frac{w_{top} - w_{bottom}}{H}$$

Applying this approximation and integrating vertically Eq. 4 then becomes

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\zeta + \beta v + \frac{f_0\vec{u}}{H} = \frac{f_0 w_e}{H} - \frac{f_0\delta}{2H}\zeta + A\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\zeta \tag{5}$$

Since the interior remains in geostrophic balance horizontally we can introduce the *geostrophic stream function*:

$$\psi = \frac{p}{\rho f_0} \quad \text{where} \quad u = -\frac{\partial\psi}{\partial y} \quad \text{and} \quad v = \frac{\partial\psi}{\partial x}$$

The global variables are scaled with:

Velocity:	U
Length:	L
Potential, ψ:	UL
Time:	$(\beta_0 L)^{-1}$

Let us choose U such that it balances the input of vorticity by the wind with the advection of planetary vorticity

$$U = \frac{\tau_0}{\rho H_0 L \beta_0}$$

Similarly, the Ekman pumping scales as,

$$W_e = \frac{\tau_0}{\rho f_0 L}$$

Finally, scaling Eq. 5 results in:

$$\frac{\partial}{\partial t}\nabla^2\psi + \delta_I^2 J(\psi, \nabla^2\psi + \psi_x + \eta J(\psi, h_b) = w_e - \delta_s \nabla^2\psi + \delta_m^3 \nabla^4\psi \tag{6}$$

$$\delta_I = \frac{(U/\beta_0)^{1/2}}{L}, \quad \eta = \frac{f_0 \Delta h_b}{H_0 \beta_0 L}, \quad \delta_s = \frac{f_0 \delta}{2 H_0 \beta L}, \quad \delta_m = \frac{(A/\beta)^{1/3}}{L}$$

Where, δ_I is the inertial scale , η is the relative strength of the bottom topography to $\beta-$effect, δ_s is the Stommel boundary layer scale and δ_m corresponds to the Munk's boundary layer scale.

2 The singular perturbation problem

Assume that all the boundary layers are small compared to the length of the basin, L, i.e. $\delta_i/L << 1$, $\delta_s/L << 1$ and $\delta_m/L << 1$. Considering that the bottom is flat in the interior and ignoring the inertial and friction terms, the governing equation is:

$$\psi_x = w_e(x, y) \tag{7}$$

This is the Sverdrup relation. The solution to this equation cannot satisfy no-normal flow at both boundaries.There are two solutions based on the boundary conditions, either $\psi(x = 0) = 0$ at the western boundary or $\psi(x = x_e) = 0$ at the eastern boundary, where $x = x_e$ corresponds to the eastern boundary. Hence, the two possible interior solutions are:

1) Satisfying $\psi(x = 0) = 0$, no normal flow on the western boundary:

$$\psi = \int_0^x w_e(x', y)dx' \tag{8}$$

Using a similar wind stress distribution as Stommel Stommel (1948), the solution to Eq. 8 in the basin is shown in Figure 2. The solutions are:

$$\psi_1 = -x sin(\pi y), \quad u_1 = -\frac{\partial \psi_1}{\partial y} = x\pi cos(\pi y) \quad \text{and} \quad v_1 = -sin(\pi y)$$

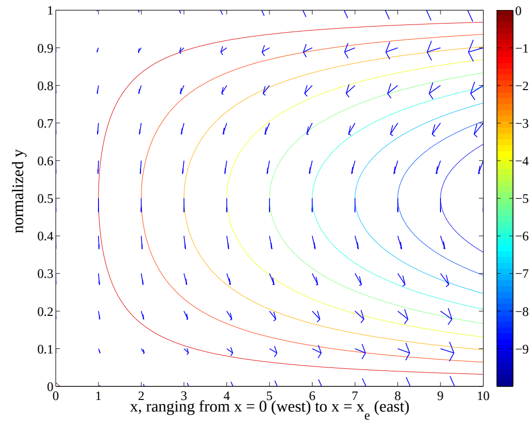

Figure 2: Streamlines and the velocity field inside the basin model corresponding to the solution of the Eq. 8.

2) The other potential solution is no normal flow on the eastern boundary, $\psi(x = x_e) = 0$.

$$\psi = -\int_x^{x_e} w_e(x', y)dx' \tag{9}$$

Using the same wind stress as before, the following geostrophic potential satisfies the boundary conditions for Eq. 9. The corresponding solution in the basin model domain is shown in Figure 3.

$$\psi_2 = (x_e - x)sin(\pi y), \quad u_2 = -\frac{\partial \psi_1}{\partial y} = -(x_e - x)\pi cos(\pi y) \quad \text{and} \quad v_2 = -sin(\pi y)$$

2.1 An Integral constraint

By taking a steady and closed streamline from the interior of the basin and integrating over the closed contour, C, the Eq. 6 results in

$$\oint_C \frac{\partial \vec{u}}{\partial t} \cdot d\vec{s} + \cancelto{0}{\oint_C \vec{u}\left[\delta_I^2 \zeta + y + \eta h_b\right] \cdot \hat{n} ds} = \oint_C \vec{\tau} \cdot d\vec{s} - \delta_s \oint_C \vec{u} \cdot d\vec{s} + \delta_m^3 \oint_C \nabla \zeta \cdot \hat{n} ds \tag{10}$$

The left hand side of the Eq. 10 is equal to zero for a steady closed streamline, the temporal term vanishes and since there is no flux across any steady closed streamline, the second term vanishes as well, therefore

$$0 = \oint_C \vec{\tau} \cdot d\vec{s} - \delta_s \oint_C \vec{u} \cdot d\vec{s} + \delta_m^3 \oint_C \nabla \zeta \cdot \hat{n} ds \tag{11}$$

From Eq. 11 it is possible to observe that the circulation (net input of vorticity) on each streamline should be balanced by either diffusion in the interior or friction at the bottom. The

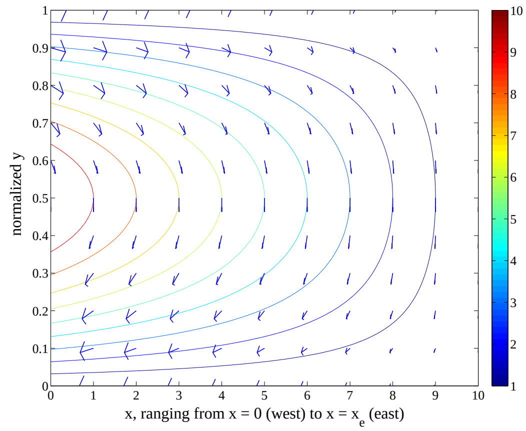

Figure 3: Streamlines and velocity field inside the basin model corresponding to the solution of the Eq. 9, i.e., $\psi(x = xe) = 0$.

Munk layer δ_m includes, in principle, the unresolved eddies within A_H. The explicit flux from eddies, if known, is included in the flux vorticity integral by an additional flux term, i.e.

$$0 = \oint_C \vec{\tau} \cdot d\vec{s} - \delta_s \oint_C \vec{u} \cdot d\vec{s} + \delta_m^3 \oint_C \nabla\zeta \cdot \hat{n} ds - \delta_I^2 \oint_C \overline{\vec{u}'\zeta'} \cdot \hat{n} ds \tag{12}$$

Since the basin model is itself a streamline, this last term should be zero for the streamline coincident with the boundary (no normal flow through the boundaries).

2.2 The Energy constraint

Intuitively, by looking at wind stress distribution shown in Figure 4, the natural (comfortable) solution to the Sverdrup expression, Eq. 7, corresponds to the one that satisfies no flow at the eastern boundary, Eq. 9. Additionally, this solution compares well with the observations. In order to prove the validity of this intuitive choice we look at the energetics of the fluid flow for a steady circulation in a rectangular ocean basin on the $\beta-$plane. The system of equations needed to solve the energetics is governed by the simplified vorticity equation (Eq. 4) and the following boundary conditions for the given domain **D**. It is

$$\psi\Big|_{\partial D} = 0, \qquad \text{and} \qquad \nabla\psi\Big|_{\partial D} = 0 \qquad if \quad \delta_m \neq 0$$

where,

$$D = [0 \ \leq \ x \ \leq x_e] \ \times \ [0 \ \leq \ y \ \leq 1].$$

The energy equation is obtained by multiplying Equation 4 by ψ, integrating over the whole basin, **D**, and applying the boundary conditions. Finally, the result is:

$$\langle w_e \ \psi \rangle = -\delta_s \ \langle |\nabla\psi|^2 \rangle - \delta_m^3 \ \langle |\nabla^2\psi|^2 \rangle \tag{13}$$

where,

$$\langle \text{ f } \rangle \equiv \int\int_D \text{f } dxdy$$

The equilibrium has been reached, the forcing term, $\langle w_e \psi \rangle$ is balanced by the dissipation terms. The condition to satisfy this balance is that w_e and ψ must be negatively correlated, and this favors the circulation of Figure 3

3 The linear boundary layer problem

We now explore another simplification of the governing equation (Eq. 6) where the amplitudes of the relative motion are small, i.e.

$$\delta_I << \delta_s \quad \text{and} \quad \delta_I << \delta_m.$$

The resulting equation is a the linear boundary layer problem:

$$\psi_x = w_e - \delta_s \nabla^2 \psi + \delta_m^3 \nabla^4 \psi \tag{14}$$

The proposed *interior solution* for this problem is ψ_I.

$$\psi_I(x, y) = -\int_x^{x_e} w_e(x', y)dx' + \Psi(y). \tag{15}$$

Note that the limits of integration make no distinction between the eastern and western boundaries, so no intensification is expected in the interior (temporary ignorance!!).

3.1 The Stommel Model

For the interior of the linear boundary layer we need to manipulate Equation 14. By assuming small variations in the flow along the boundary layer and large variations across the boundary layer flow, we can now keep only the x derivatives. Furthermore, scaling by $x = \delta\xi$ results in $\delta w_e << 1$, which can be neglected. We now integrate once over η so that

$$\underbrace{\phi}_{a} = -\underbrace{\left(\frac{\delta_s}{\delta}\right)\frac{\partial \phi}{\partial \xi}}_{b} + \underbrace{\left(\frac{\delta_m^3}{\delta^3}\right)\frac{\partial^3 \phi}{\partial \xi^3}}_{c}. \tag{16}$$

Assuming that $\delta_s >> \delta_m$ and $\delta \sim \delta_s$ allows us to ignore the term c. Since this is the highest-order derivative in the equation this becomes a singular perturbation problem. Stommel's model for the boundary layer problem is recovered (Stommel (1948)).

$$\frac{\partial \phi}{\partial \xi} + \phi = 0. \tag{17}$$

It has the following solution,

$$\phi = A(y)e^{-\xi}.$$

No normal flow condition is necessary at one of the boundaries; as before, we can apply this condition on either the eastern boundary or the western boundary. For the western boundary, $x = 0$,

$$A = -\psi_I(0, y).$$

Let us define a new boundary layer coordinate, ξ', for the eastern boundary

$$\xi' = \frac{(x_e - x)}{\delta_s} \quad \text{then,} \quad \frac{\partial \phi}{\partial \xi'} - \phi = 0.$$

Then, our new boundary layer equation is:

$$\frac{\partial \phi}{\partial \xi'} - \phi = 0.$$

The corresponding solution is:

$$\phi = A(y)e^{\xi'}.$$

This solution has exponential growth of the BL on the eastern boundary, which is physically not possible since the BL should be finite and should be absorbed smoothly by the interior. Therefore, we keep the first solution, which actually corresponds to the *western intensification* (our temporary ignorance has been removed!). Looking at the general solution for the interior, $\Psi(y) = 0$ on the boundary. Finally, combining our equations for the linear BL (Eqns. 14 and 15) with the valid solution results in

$$\psi(x,y) = \psi_I(x,y) - \psi_I(0,y)e^{-(x/\delta_s)}$$

with,

$$\psi_I(x,y) = -\int_x^{x_e} w_e(x',y)dx'.$$

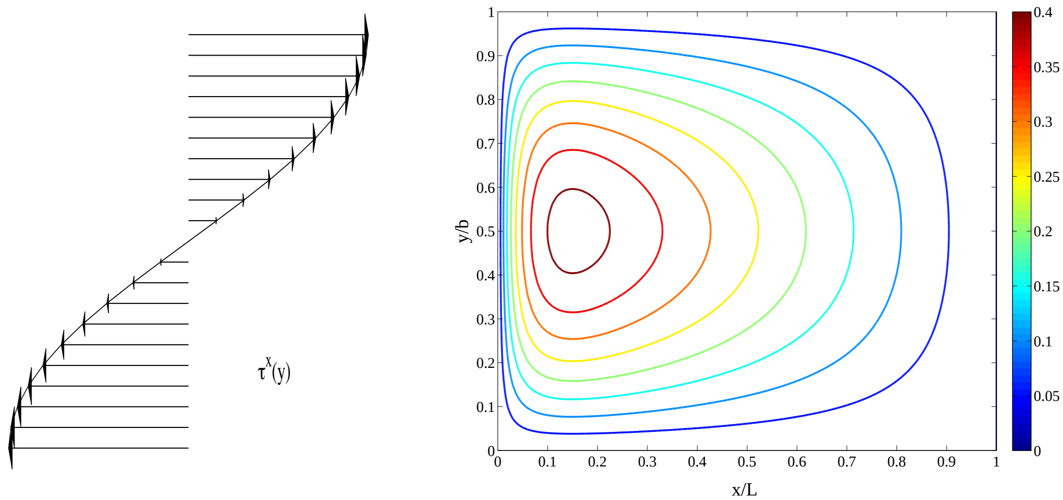

Figure 4: Streamlines corresponding to wind driven circulation in the ocean basin based on Stommel's model. The dimensions of the basin is **L** (west to east) by **b** (south to north), the size of the boundary layer respect to the basin length is $\delta_s/L = 0.05$. Henry Stommel proposed $\tau = -Fcos(\pi y/b)$ (Stommel (1948))
.

The western intensification represented in Figure 4 is controlled by the boundary layer and the $\beta-$effect.

So far, Stommel's model neglects the **no slip** condition on the western boundary. Figure 5 shows the meridional velocity v. Note that the velocity is northward close to the boundary layer and then turns southward as Sverdrup flow for most of the ocean basin extent.

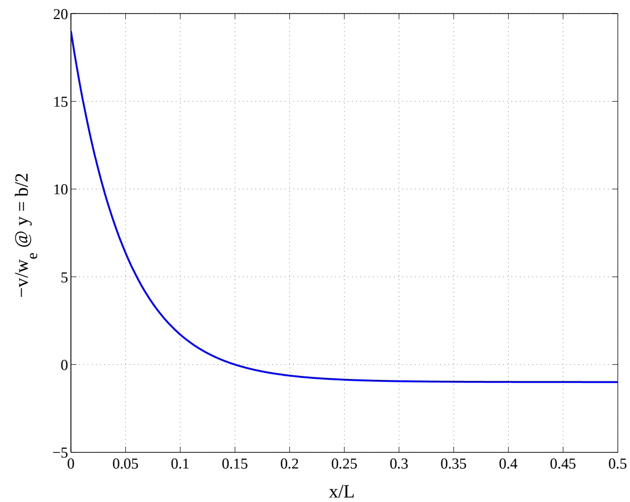

Figure 5: Meridional velocity in the middle of the basin as predicted by Stommel's model (normalized by $w_e(y = b/2)$). Note that the only boundary condition satisfied is zero flow through the western boundary, i.e., $v(x = 0) \neq 0$. As before $\delta_s/L = 0.05$

3.2 The no slip condition and the sublayer

Stommel's model assumption that $\delta_m/\delta \sim 0$ leaves no room to satisfy a *no slip* condition, as a natural consequence the vorticity balance of the whole basin depends on the lateral diffusion term.

In order to satisfy the no slip boundary condition, we now rewrite the Equation 16 with a slightly different scaling, $x = \delta_s \xi$.

$$\underbrace{\phi}_{a} = -\underbrace{\frac{\partial \phi}{\partial \xi}}_{b} + \underbrace{\left(\frac{\delta_m^3}{\delta_s^3}\right)\frac{\partial^3 \phi}{\partial \xi^3}}_{c}. \tag{18}$$

It is necessary now to keep both of the terms; **b** and **c** that we are adding to Stommel's model as an additional sublayer. Defining $\xi = l\ \eta$ as the sublayer scale and balancing the terms **b** and **c** we find

$$l = \left(\frac{\delta_m}{\delta_s}\right)^{3/2}.$$

The thickness of the sublayer inside the Stommel boundary layer is given by,

$$\delta_{sub} = \delta_s\ l \quad = \frac{\delta_m^{3/2}}{\delta_s^{1/2}} \quad = \quad \left[\frac{A}{L^2}\frac{H_0}{\sqrt{2vf}}\right]^{1/2}$$

After scaling Eq. 18 and integrating it once over η, we find the correction function for the sublayer:

$$\chi_{\eta\eta} - \chi = 0, \quad \text{where,} \quad \chi(\eta) = C(y)\ e^{-\eta}.$$

The solution should be bounded, therefore the term proportional to $exp(+\eta)$ automatically goes to zero. Hence, we could rewrite Stommel's solution with the additional sublayer correction function

$$\psi(x, y) = \psi_I(x, y) + A(y)e^{-(x/\delta_s)} + C(y)e^{-(x/\delta_{sub})}.$$

Applying the boundary conditions of no slip, $\psi_x(0, y) = 0$, and no flow at the western boundary, $\psi(0, y) = 0$ allows us to find the function *C(y)* since *A(y)* is already known

$$C = -\frac{\delta_{sub}}{\delta_s}\ A \quad \text{and,} \quad A = -\psi_I(0, y).$$

Finally, the **total solution** for the ocean basin including no slip at the western boundary is given by,

$$\psi(x, y) = \psi_I(x, y) - \psi_I(0, y)\left[e^{-(x/\delta_s)} - \left(\frac{\delta_{sub}}{\delta_s}\right) e^{-(x/\delta_{sub})}\right]. \tag{19}$$

Figure 6 shows the resulting profile for the meridional velocity. Note that the magnitude of v, approaches to zero near the western boundary.

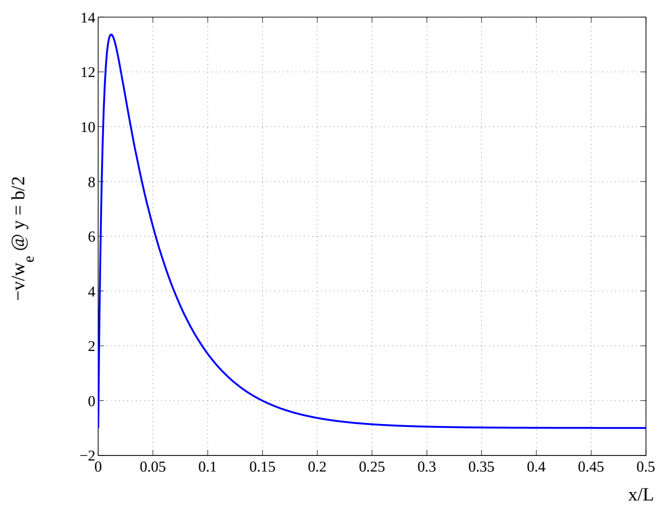

Figure 6: Meridional velocity in the middle of the basin adding no slip condition at the western boundary, i.e., $v(x = 0) = 0$. $\delta_s/L = 0.05$ and $\delta_{sub} = 0.0045$

References

H. Stommel. The westward intensification of wind-driven ocean currents. *Trans. Amer. Geophys. Union*, 29:202–206, 1948.