Oceanographic Applications

Lecture 7

Tsunami, seen from top floor of hotel in Sri Lanka 2004

C. Chapman

Fig. 1. View of the ocean when the water had reached its highest level at about 10:12 A.M. local time. This picture was taken 7 minutes after water was at its lowest level. The view is from the top floor of the beachside Triton Hotel, looking seaward over the reception area and across the swimming pool. Photo courtesy of Chris Chapman. See the Eos *Electronic Supplement for additional photos: http://www.agu.org/eos_elec.000929e2.html.*

Oceanographic Applications

This lecture:

- Review derivation of KdV and KP
- Tsunami of 2004
- Kadomtsev-Petviashvili (KP) equation: theory and experiment

Review of theory for shallow water

1. Governing equations (no surface tension)

$$\partial_t \eta + \nabla\phi \cdot \nabla\eta = \partial_z \phi$$
$$\partial_t \phi + \tfrac{1}{2} |\nabla\phi|^2 + g\eta = 0$$

on $z = \eta(x,y,t)$,

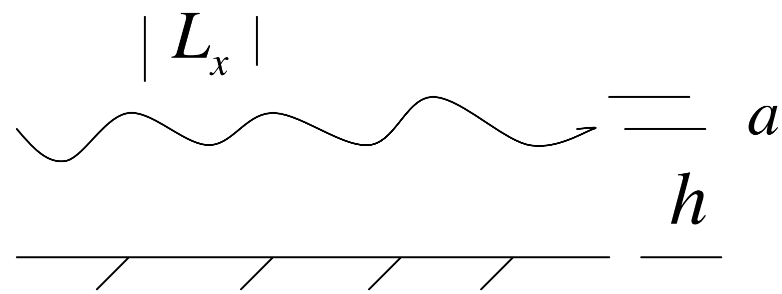
$$\nabla^2 \phi = 0$$

$-h < z < \eta(x,y,t)$,

$$\partial_n \phi = 0$$

on $z = -h$.

Review of theory for shallow water



2. To derive KP (or KdV)

Assume

- Small amplitude waves $a << h$
- Shallow water (long waves) $h << L_x$
- Nearly 1-D motion $L_x << L_y$
- All small effects balance

$\varepsilon << 1,\qquad \frac{a}{h} = O(\varepsilon),\quad (\frac{h}{L_x})^2 = O(\varepsilon),\quad (\frac{L_x}{L_y})^2 = O(\varepsilon).$

At leading order:

Wave equation in 1-D:

$$\partial_t^2 \eta = c^2 \partial_x^2 \eta \quad \text{with} \quad c^2 = gh$$

➔ $$\eta = \varepsilon h[f(x-ct;y,\varepsilon t) + F(x+ct;y,\varepsilon t)] + O(\varepsilon^2)$$

At leading order:

Wave equation in 1-D:

$$\partial_t^2 \eta = c^2 \partial_x^2 \eta \quad \text{with} \quad c^2 = gh$$

➔ $$\eta = \varepsilon h [f(x-ct;\varepsilon y,\varepsilon t) + F(x+ct;\varepsilon y,\varepsilon t)] + O(\varepsilon^2)$$

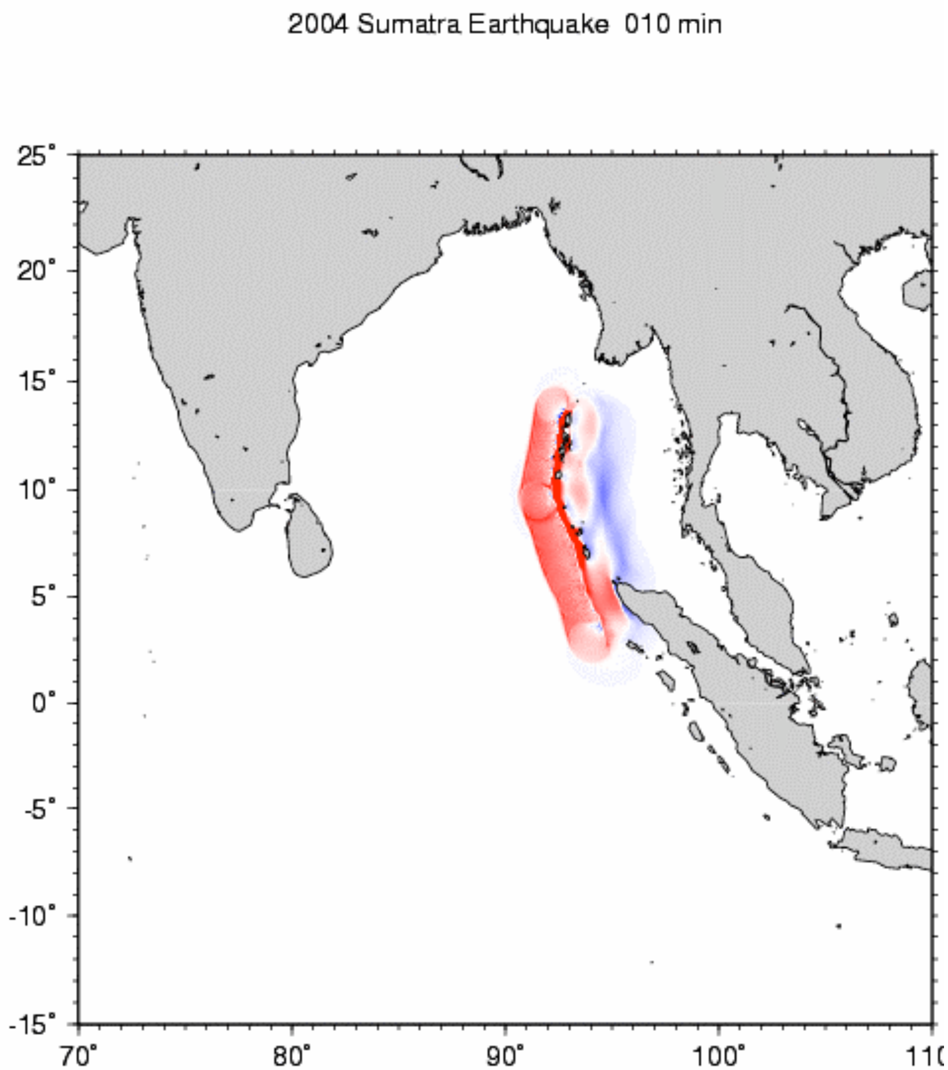
At next order, $f(\xi,\zeta,\tau)$ satisfies either

$$\partial_\tau f + f\partial_\xi f + \partial_\xi^3 f = 0 \qquad \text{KdV}$$

or

$$\partial_\xi (\partial_\tau f + f\partial_\xi f + \partial_\xi^3 f) + \partial_\zeta^2 f = 0 \qquad \text{KP}$$

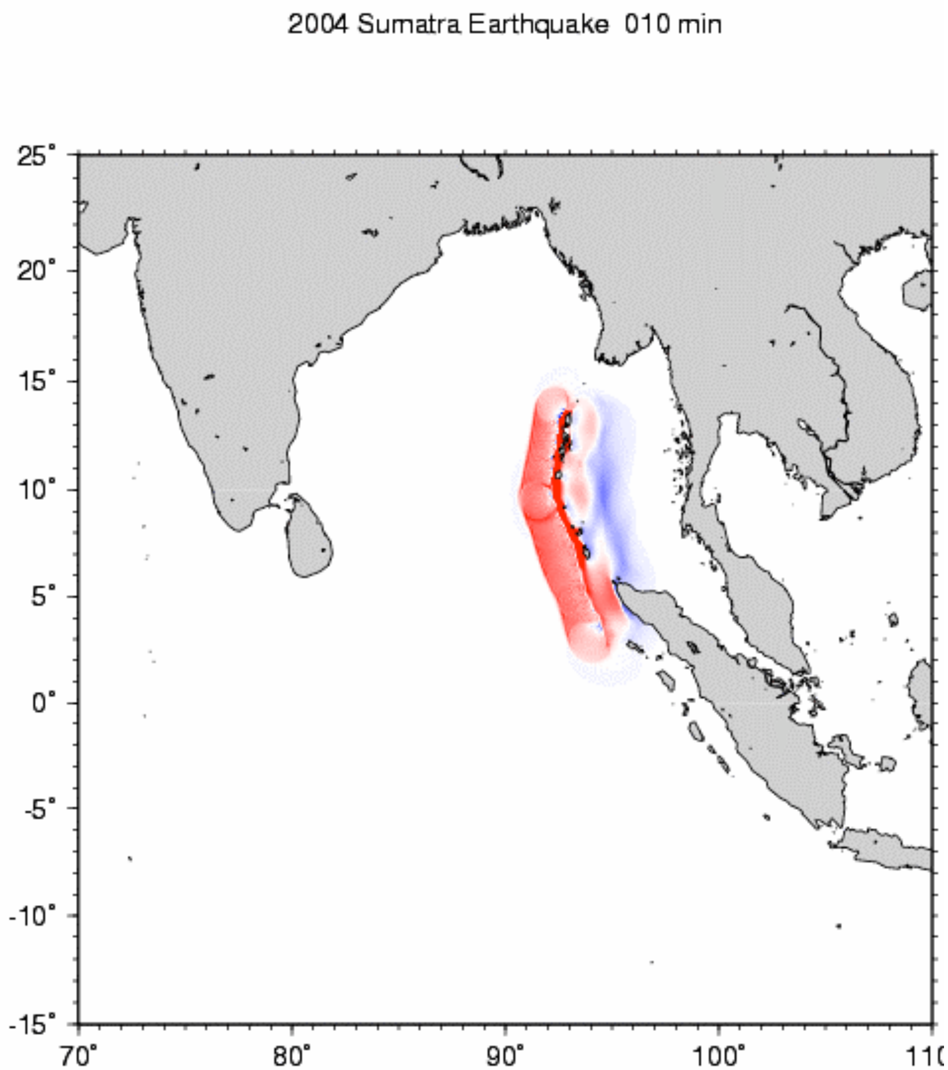
Tsunami of Dec. 26, 2004



Kenji Satake, Japan
http://staff.aist.go.jp/kenji.satake/

or, see
Steven Ward, US
http://www.es.ucsc.edu/~ward

Tsunami of 2004



Approximate Length scales: Indian Ocean

h = 3.5 km

For tsunami:

a = 1-2 m

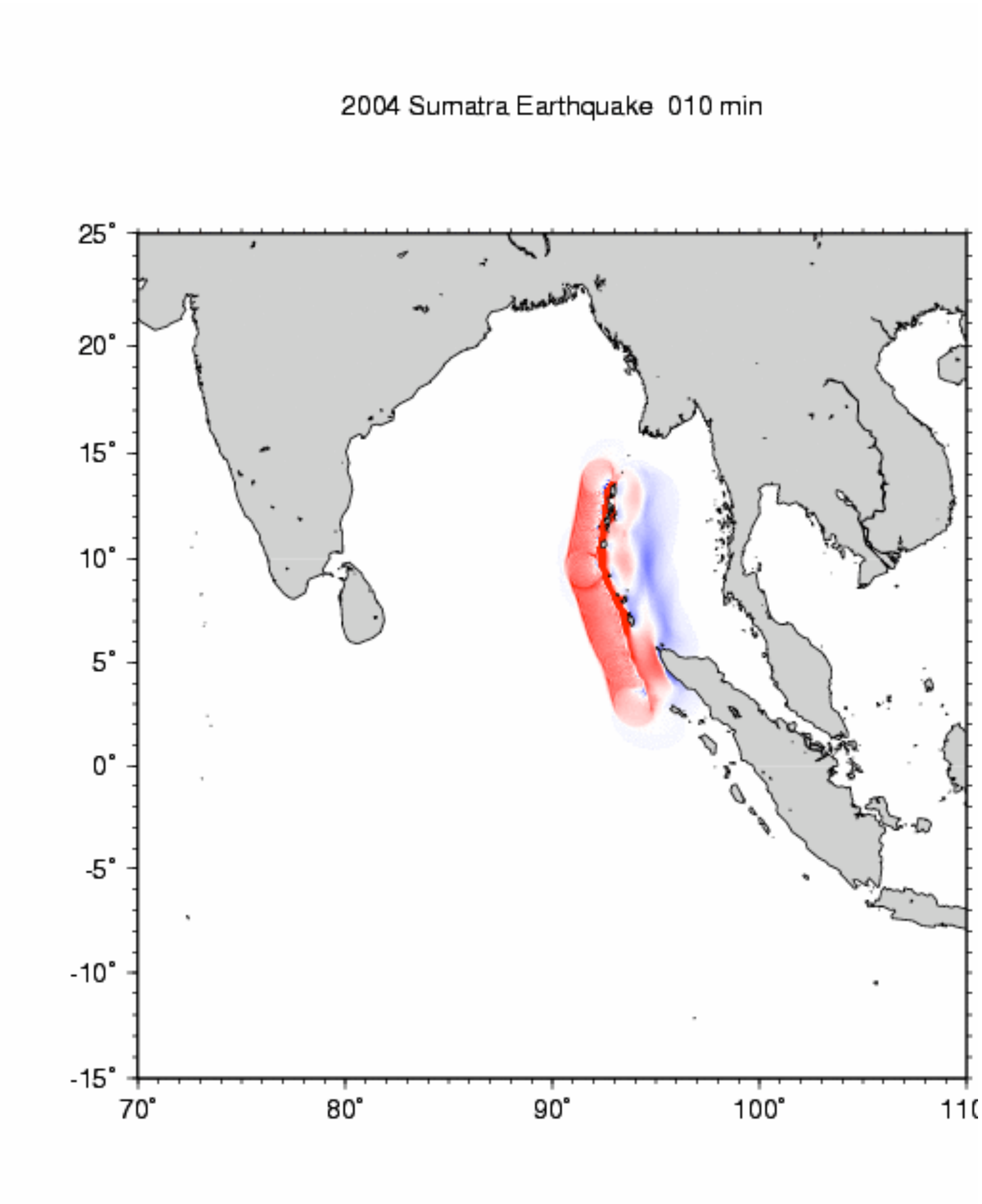
L_x = 100 km

L_y = 1000 km

c = 650 km/hr (wave speed)

u = 200 m/hr (water speed)

Tsunami of 2004



Kenji Satake, Japan

http://staff.aist.go.jp/kenji.satake/

Tsunami of 2004

Q: Was the tsunami a soliton?

A: No

KdV (or KP) dynamics occurs on a long time scale.

For tsunami, typical horizontal length = 100 km

Distance across the Bay of Bengal = 1500 km

Too short for KdV dynamics to develop.

Tsunami of 2004

Q: Can a tsunami become a soliton?

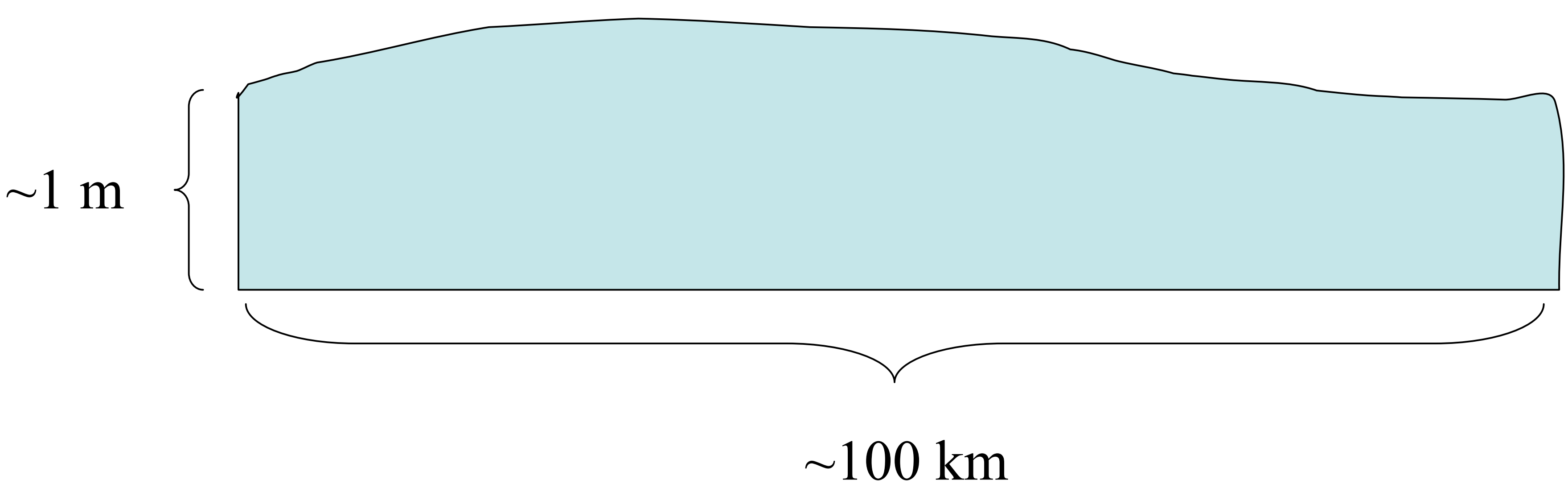
A: Perhaps

The largest earthquake ever recorded occurred off Chile in May, 1960.

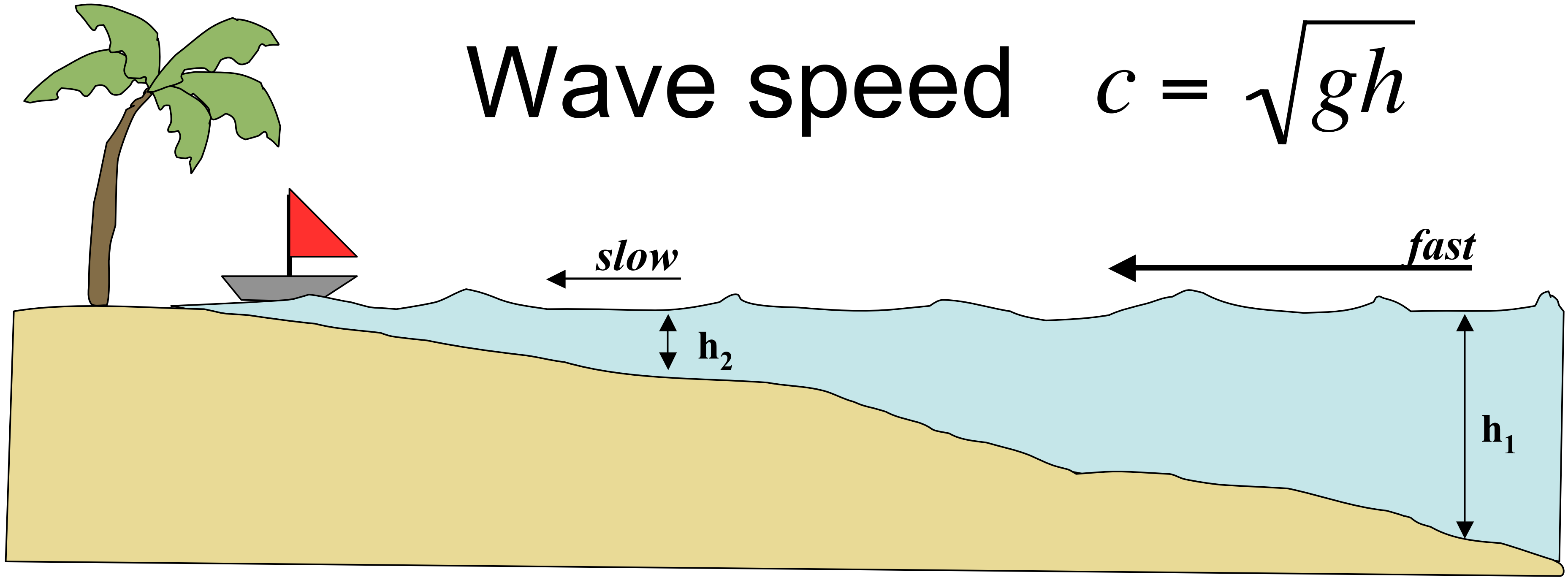
After 15 hours, it killed 61 people in Hawaii.

After 22 hours, it killed 197 people in Japan.

Volume of water displaced
(per width of shoreline)



Wave speed $c = \sqrt{gh}$

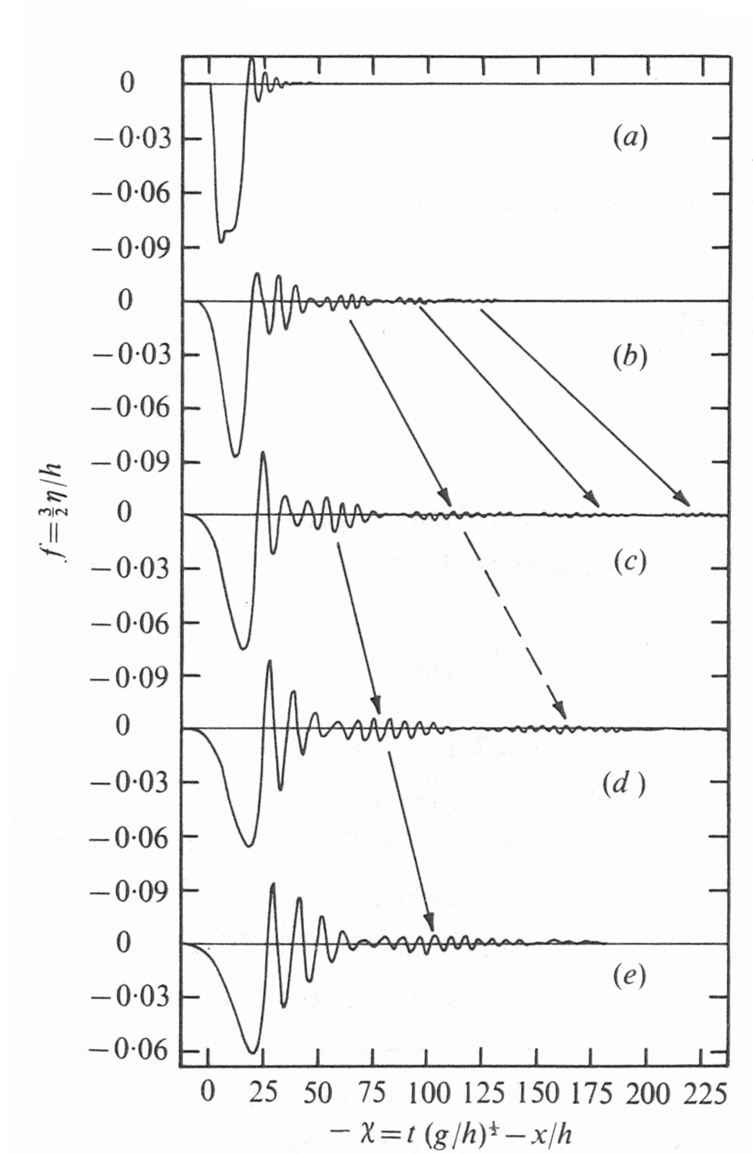


- Open ocean
  - c = 650 km/hr
  - *A wave 100 km long passes by in about 9 minutes*

- Near shore
  - Front of wave slows as it approaches the shore
  - Back of the wave is still in deep water
  - Consequence: *Wave compresses horizontally and grows vertically*

What happened in Thailand?

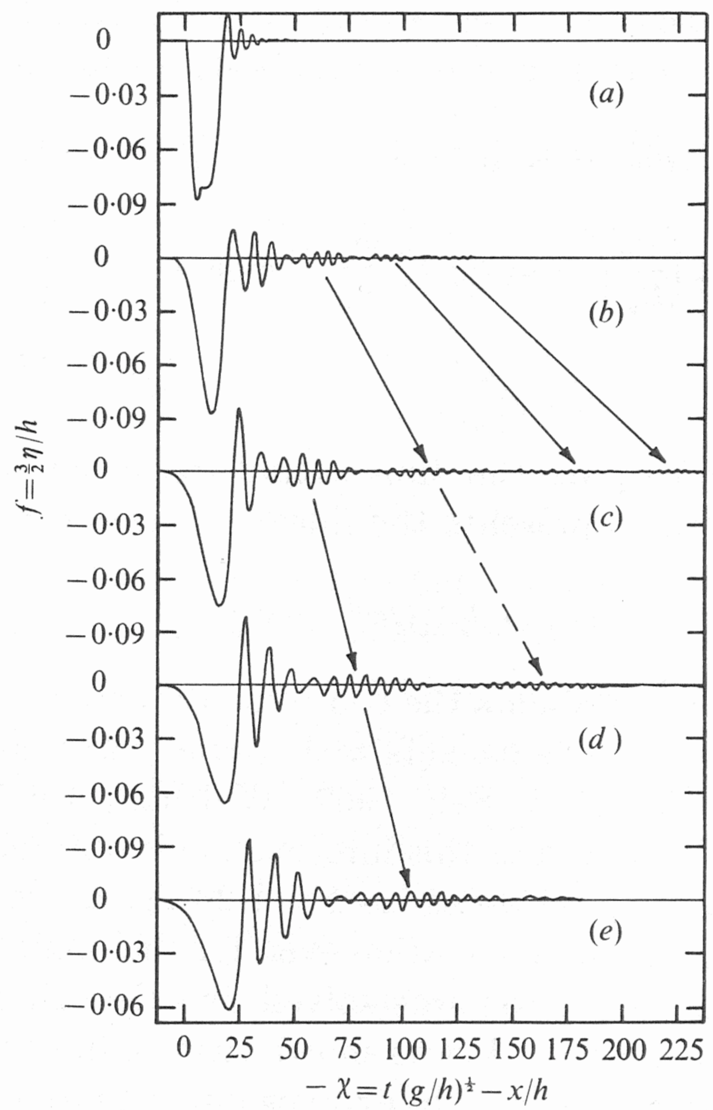
Recall J. Hammack's experiment on a wave of depression.



What happened in Thailand?

Recall J. Hammack's experiment on a wave of depression.

The first wave that hit Thailand was negative (wave of depression).



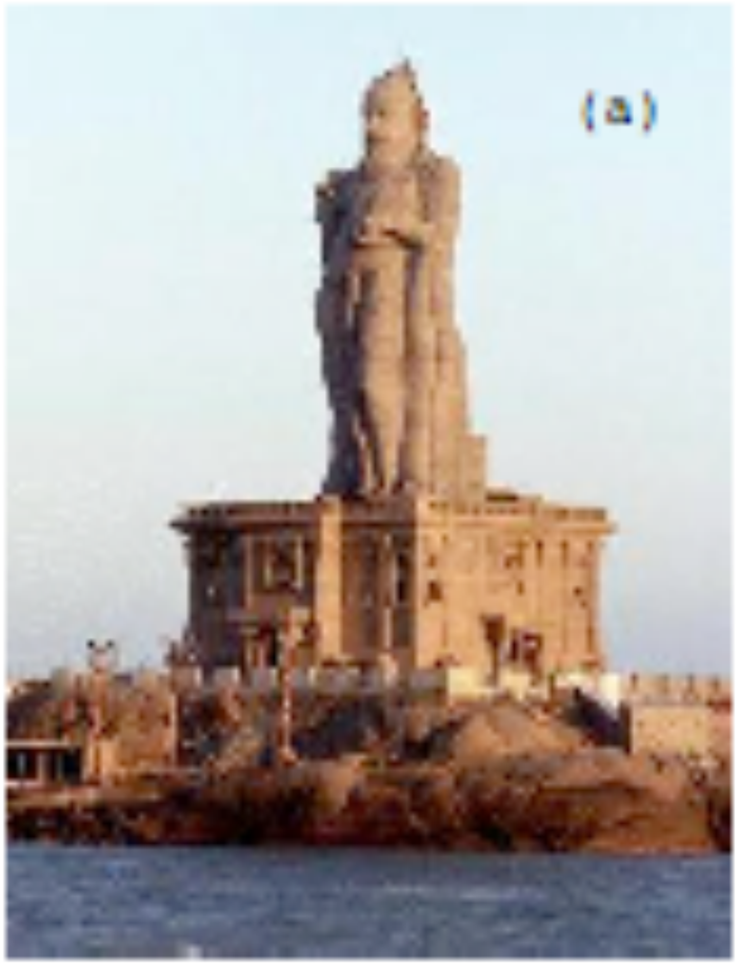
What happened in Thailand?

Front of 2004 tsunami reaches the shore of Thailand

. The tsunami of 26 December 2004 approaching Hat Ray Leah beach on the Krabi coast, Thailand. (Copyright Scanpix

Note two breaking wave fronts (photo from Constantin & Johnson, 2008)

The tsunami reaches land



Statue of Thiruvalluvar (ancient Tamil poet)
at southern tip of India - statue 133 ft tall
(thanks to M. Lakshmanan & www.bhoomikaindia.org)

Simple, crude predictions (for tsunami warning system)

- A tsunami can be generated by a thrust fault, a normal fault or a landslide (all under water).
- A strike-slip fault (by itself) will not generate a tsunami.
- A crucial quantity for estimating the size of a tsunami is the volume of water displaced by the underwater seismic event.

More simple predictions (for tsunami warning system)

- The time required for the tsunami to propagate from x_1 to x_2 along a fixed path is approximately

$$T = \int_{x_1}^{x_2} \frac{ds}{\sqrt{g \cdot h(s)}}$$

- The detailed dynamics of a tsunami near shore seem to be poorly understood.

Other waves in shallow water

- Most ocean surface waves are caused by storms and winds.
- Travel thousands of kms, over several days
- Oscillate, approximately periodically
- Long waves travel faster than short waves

Other waves in shallow water

Objective:

Find the natural structure(s) of oscillatory ocean waves, including those in shallow water

Oscillatory waves in shallow water

- Simplest model:
  All waves travel with speed = $\sqrt{gh}$

- For long waves of moderate amplitude, all traveling in approximately the same direction in water of uniform depth, a better approximation is:

$$\partial_\xi (\partial_\tau f + f\partial_\xi f + \partial_\xi^3 f) + \partial_\zeta^2 f = 0$$

(Kadomtsev & Petviashvili, 1970)

Oscillatory waves in shallow water

Miracle: The KP equation is completely integrable.

It admits an infinite family of periodic or quasi-periodic solutions of (ξ,ζ,τ). All of these have the form

$$f(\xi,\zeta,\tau) = 12\partial_\xi^2 \log \Theta$$

where Θ is a Riemann theta function of genus g

(g = integer). The genus is the number of independent phases in the solution.

References on quasiperiodic KP solutions

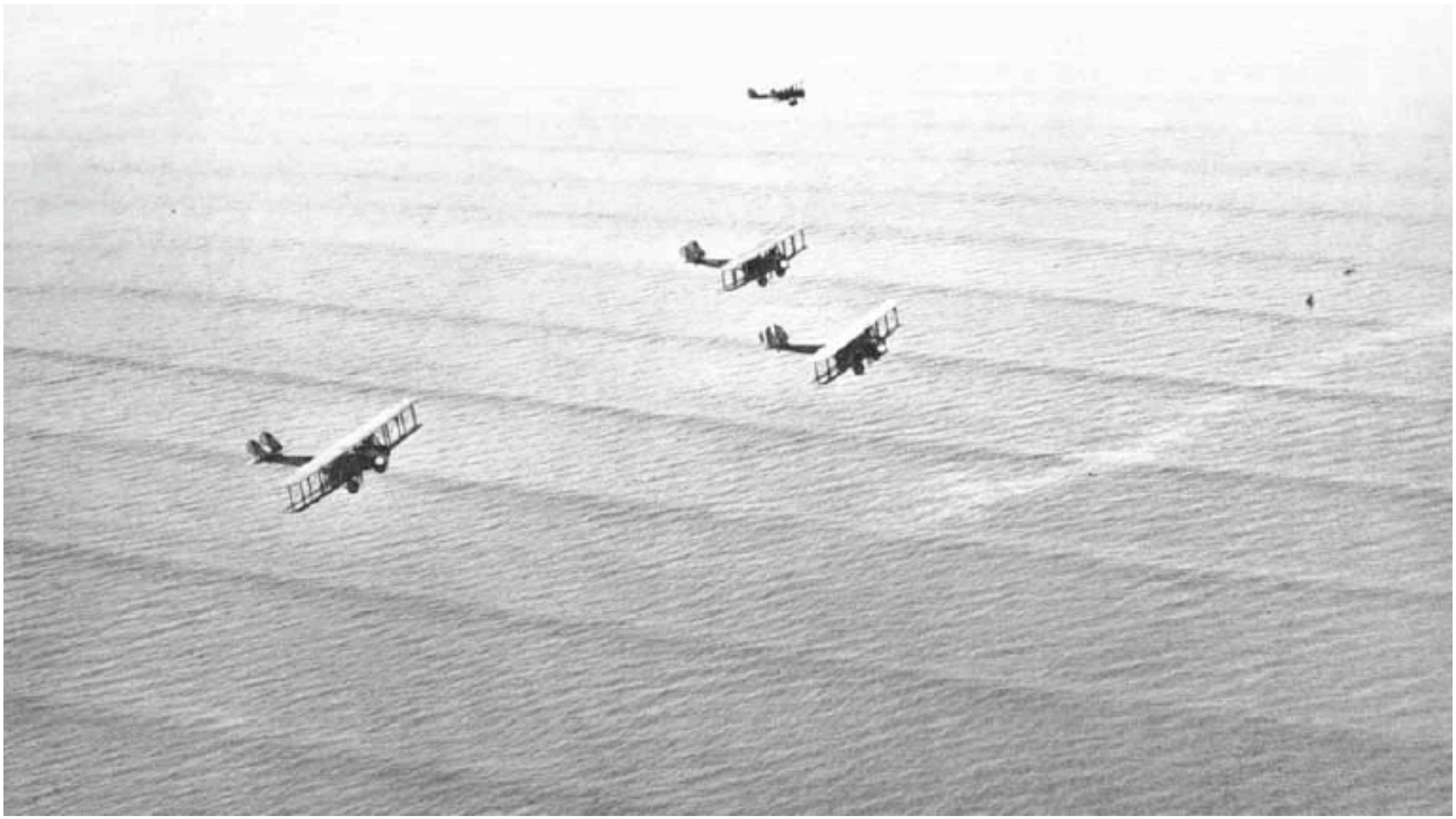
Krichever (1976, 1977a,b, 1989)
Dubrovin (1981)
Bobenko & Bordag (1989)
Belokolos, Bobenko, Enol'skii, Its & Matveev (1994)
Dubrovin,Flickinger & Segur (1997)
Deconinck & Segur (1998)
Feldman, Krörrer & Trubowitz (2003)
Deconinck, Heil, Bobenko, van Hoeij, Schmies (2004)

http://www.amath.washington.edu/~bernard/kp.html

References on exotic multi-soliton KP solutions

G. Biondini, S. Chakravarty, Y. Kodama together or alone, since 2007 or so

KP solution of genus 1 = KdV cnoidal wave

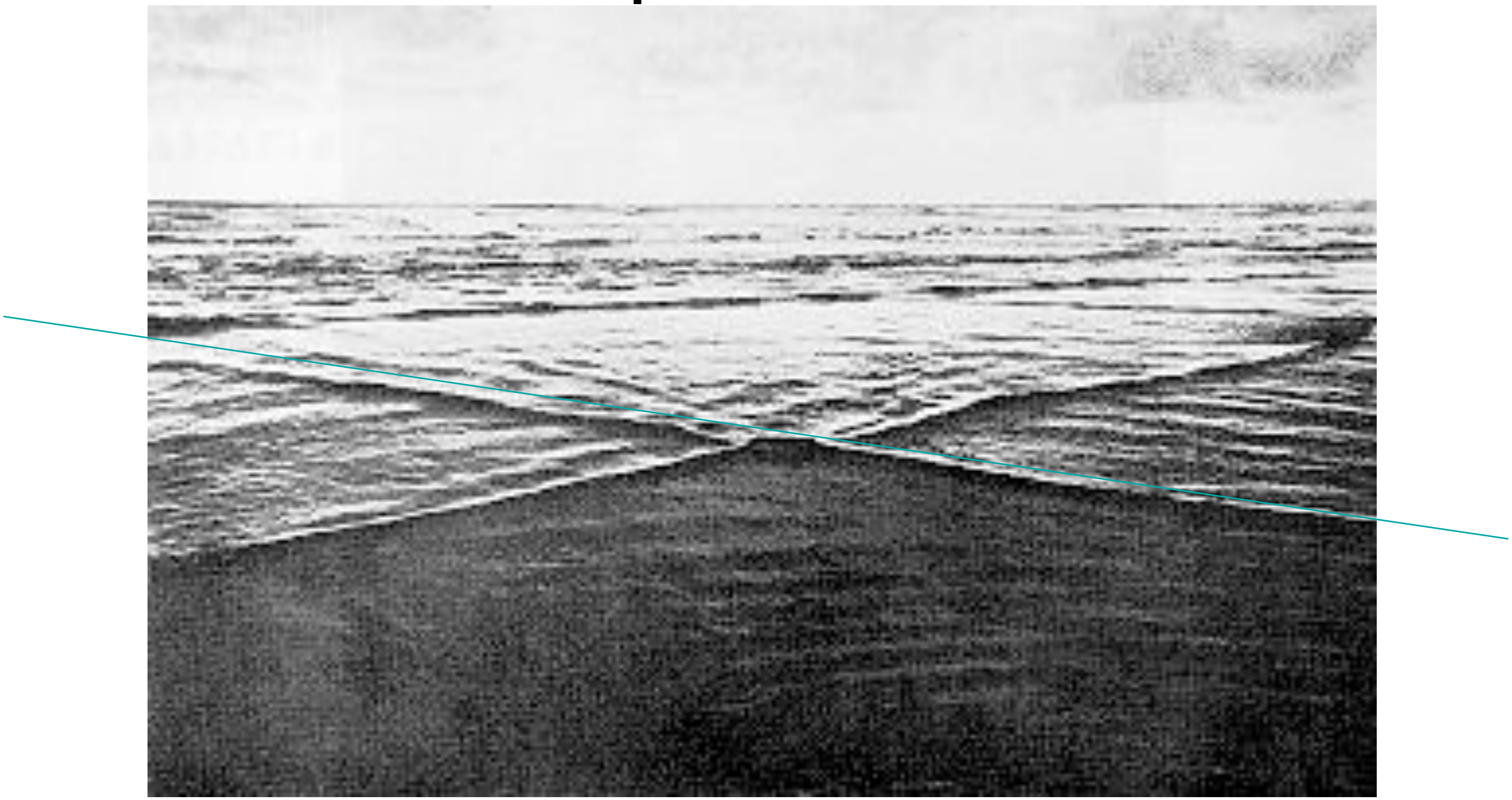


(National Geographic, 1933)

Oblique interaction of plane waves in shallow water

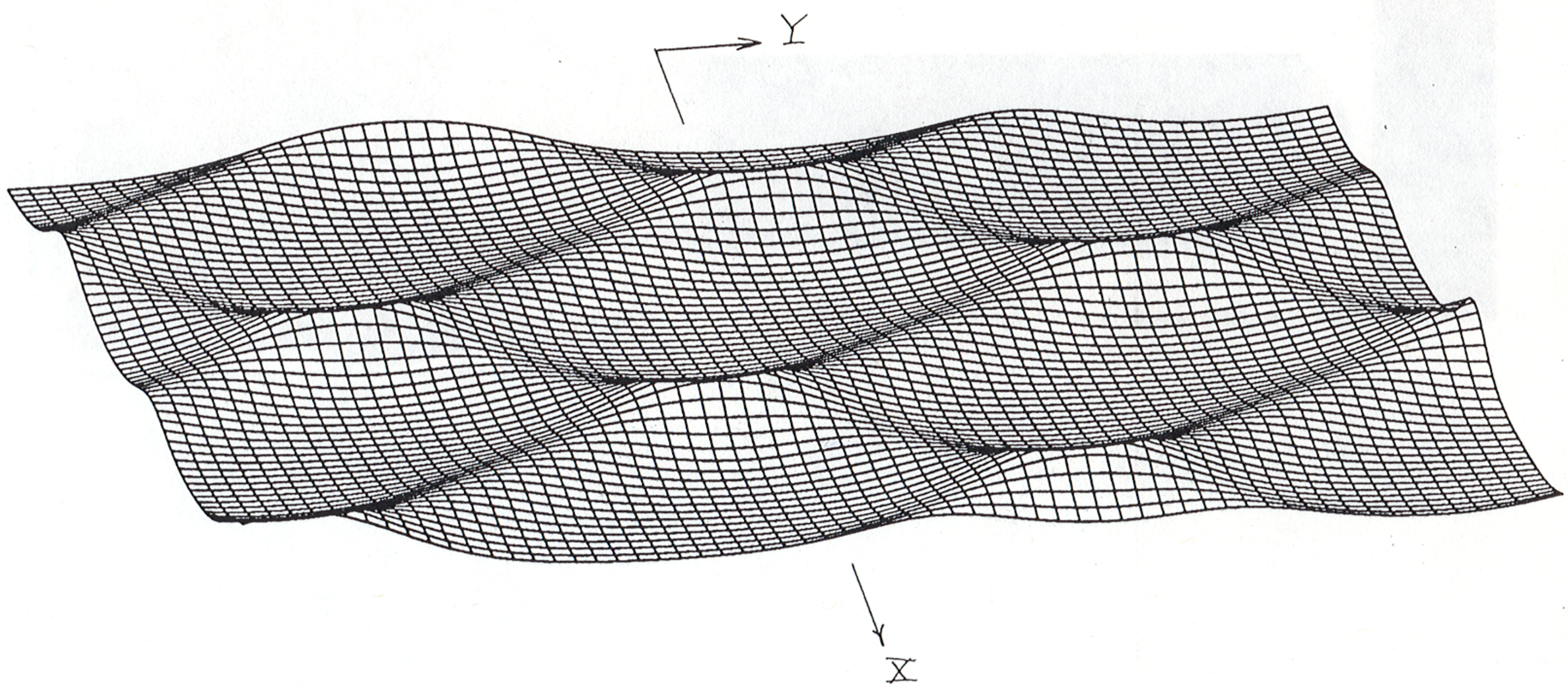
Photo due to T. Toedtemeier

Interaction of 2 KP solitons, with phase shift



Interpretation #1

Two-phase solution of KP
(genus 2)



Interpretation #2

Two-phase Waves in Laboratory Experiments

Hammack, McCallister, Scheffner, Segur, 1995

(Periodic tile is a hexagon for KP solution of genus 2)

Two-dimensional periodic waves in shallow water. Part 2 97

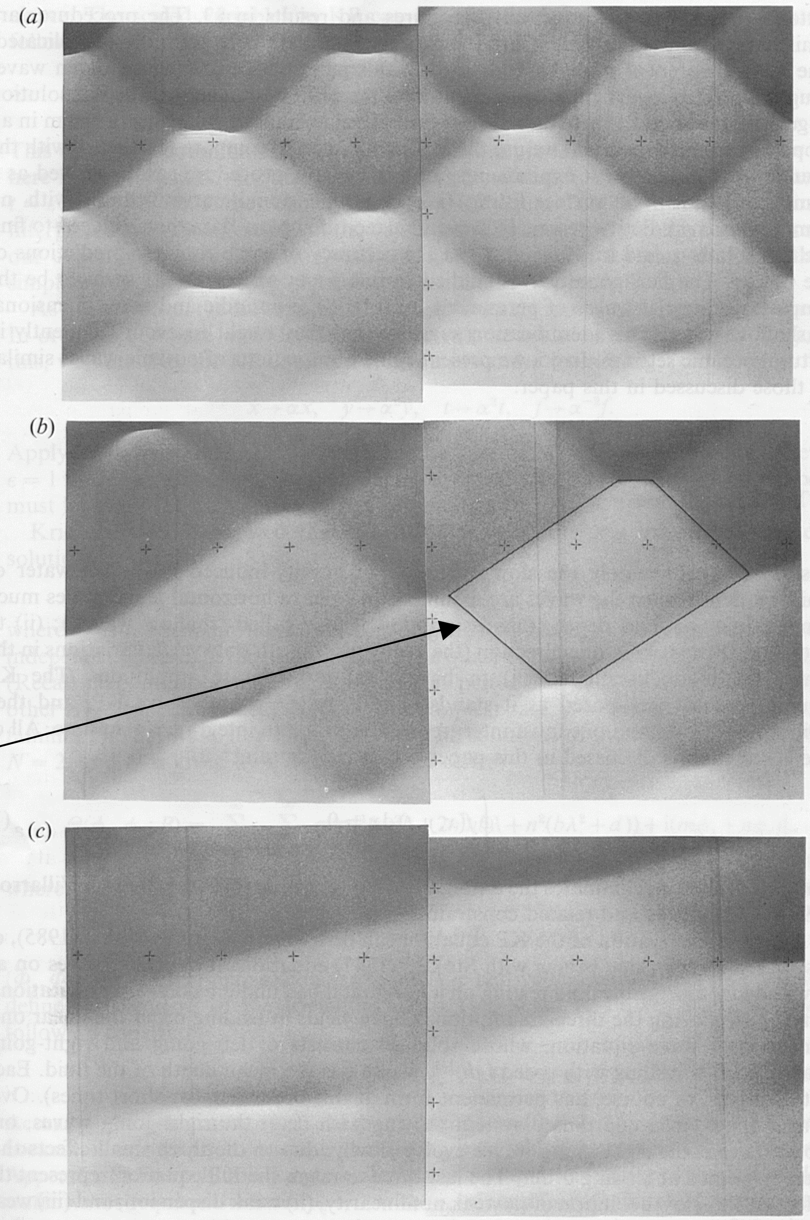
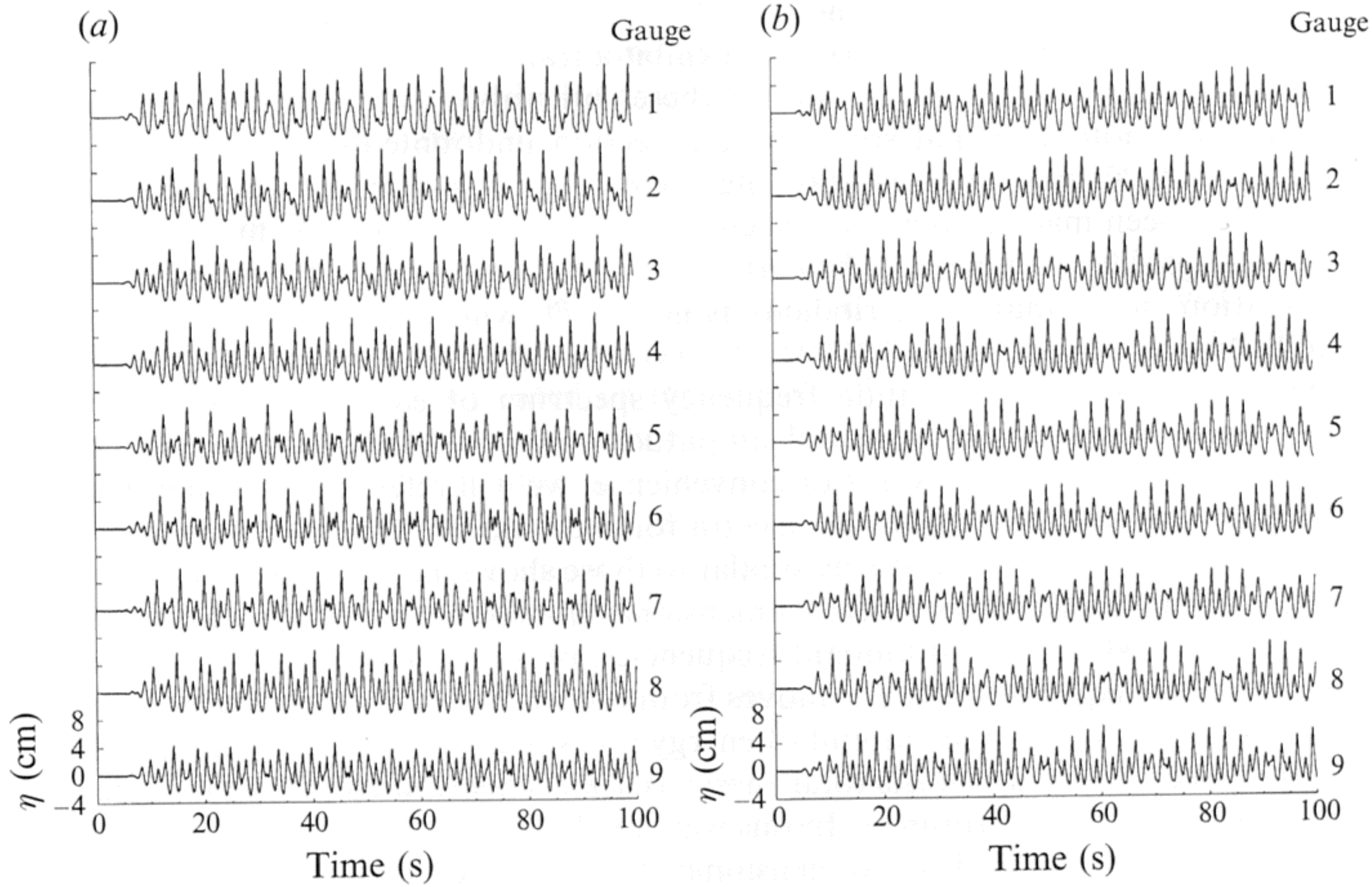


FIGURE 1. Mosaic of two overhead photographs, showing surface patterns of waves in shallow water. Experiment: (*a*) KP38.50204; (*b*) KP400203-300303; (*c*) KP220203-110403. Each of these wave patterns has a basic hexagonal template; one such hexagon is drawn in figure 1(*b*).

Two-phase waves:



Time series, measured at 9 locations across tank, for 2 of the wave patterns of nearly permanent form on previous slide

Two-phase waves:

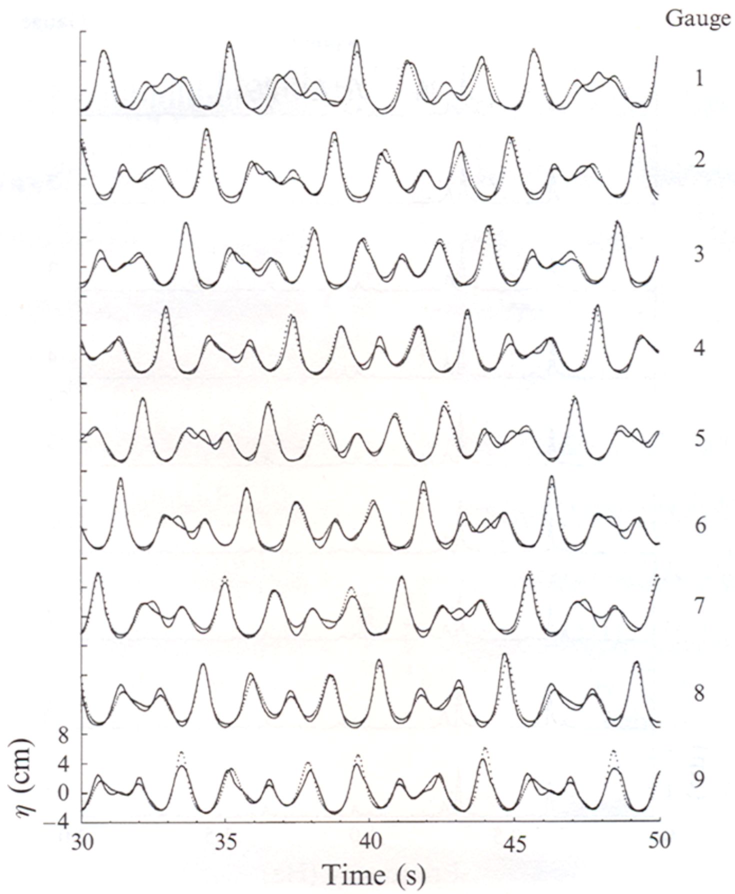
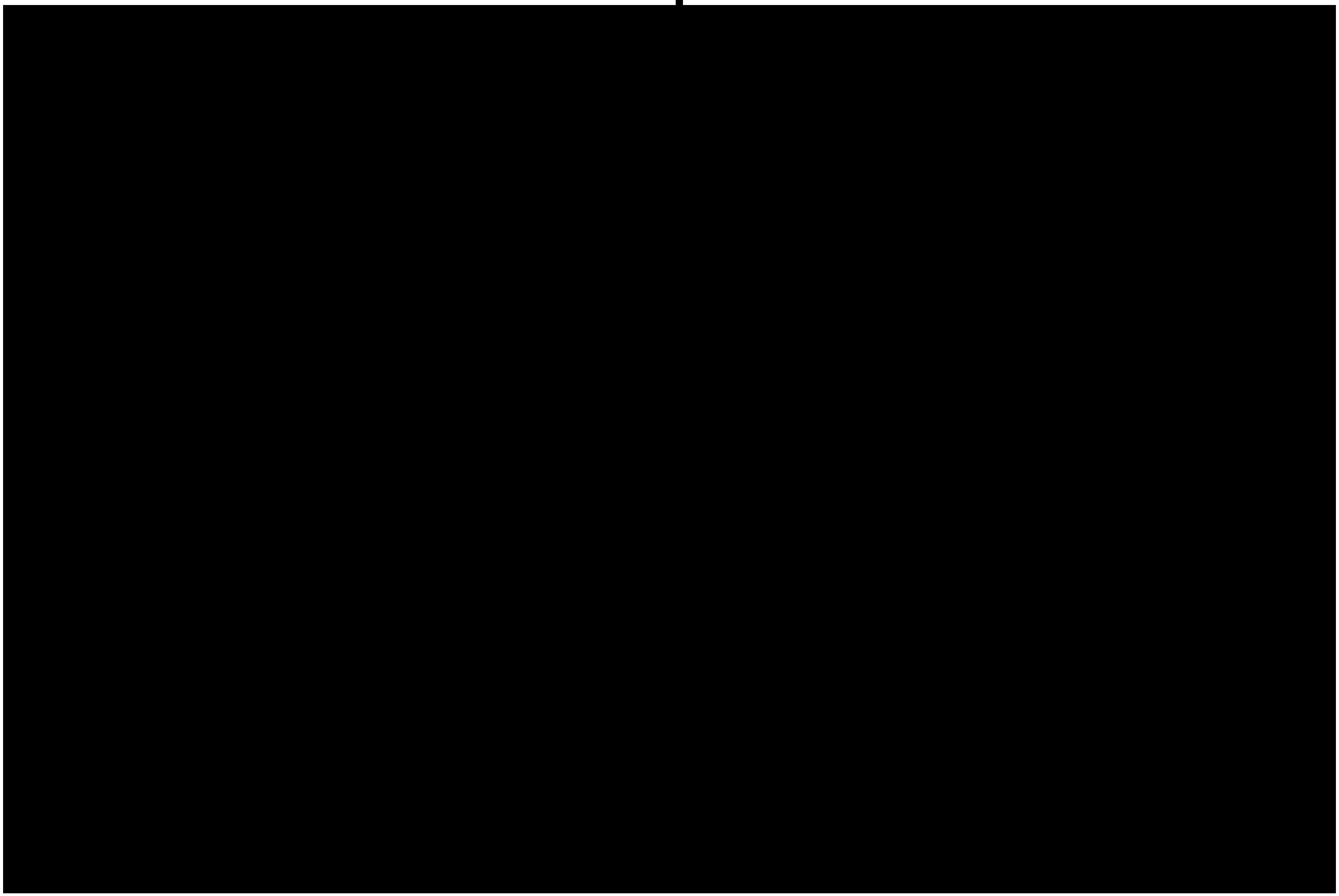


FIGURE 5. Detailed comparison of measured wave records (——) with the best KP solution (……) at the same locations, for 20 s of data from experiment KP400203-300303. The comparison began 30 s into the experiment.

Comparison of measured time-series (——) and
KP theory (---), for first wave pattern in previous slide

Video of two-phase waves



(Hammack, Scheffner, Segur, 1989) - see also lecture 20

Q: Is KP theory needed for two-phase, periodic wave patterns of permanent form?

Recall:

- KP is an approximate theory, which should be accurate in a particular limit.
- An approximate theory can provide hypotheses for more complete theories.

Q: Is KP theory needed for two-phase, periodic wave patterns of permanent form?

- Craig & Nicholls (2000) consider the full water-wave equations (with gravity and surface tension, but no viscosity)
- They prove that these equations admit travelling waves of permanent form with two-dimensional, periodic surface patterns, in water of **any** depth.

Q: Is KP theory needed for two-phase, periodic wave patterns of permanent form?

- Craig & Nicholls (2000) consider the full water-wave equations (with gravity and surface tension, but no viscosity)
- They prove that these equations admit travelling waves of permanent form with two-dimensional, periodic surface patterns, in water of **any** depth.

Shallow water : the periodic tile is a hexagon,

as predicted by KP.

Deep water : the periodic tile is a rectangle.

KP does not apply in deep water, but see Lecture 14.

Two-phase, periodic wave patterns of permanent form

- Craig and Nicholls' work starts with Zakharov's (1968) Hamiltonian formulation of the water wave equations.
- Their mathematical procedure is essentially bifurcation theory, but with a *2-D* bifurcation parameter.

  (Track solutions on a 2-D surface, instead of a 1-D curve.)
- Their method leads naturally to a numerical method to compute these solutions.

  (Craig & Nicholls, 2002)

Two-phase, periodic wave patterns of permanent form

- Earlier work by Reeder & Shinbrot (1981) & Sun (1993), on waves with surface tension, obtained partial results because of small divisors.
- Craig & Nicholls' method fails for pure gravity waves (no surface tension) because of small divisors.

Two-phase, periodic wave patterns of permanent form

- Earlier work by Reeder & Shinbrot (1981) & Sun (1993), on waves with surface tension, obtained partial results because of small divisors.
- Craig & Nicholls' method fails for pure gravity waves (no surface tension) because of small divisors.
- Iooss & Plotnikov (2008) prove existence of three-dimensional wave patterns of permanent form on deep water, for pure gravity waves (no surface tension).

  ("Small divisor problem in the theory of 3-dimensional…")

Open questions for KP theory and/or 3-D waves in shallow water

- Stability of 3-D waves of permanent form (*i.e*, with 2-D surface patterns) is open:
  - for the full equations of water waves;
  - for the KP equation.
- In addition to its solutions of genus 2, KP admits other periodic solutions of permanent form, with genus > 2. These have not been explored.
- Almost all KP solutions with genus > 2 are time-dependent in any Galilean coordinate system. These describe energy-sharing among nonlinear wave modes. Untested.

Open questions for KP theory and/or 3-D waves in shallow water

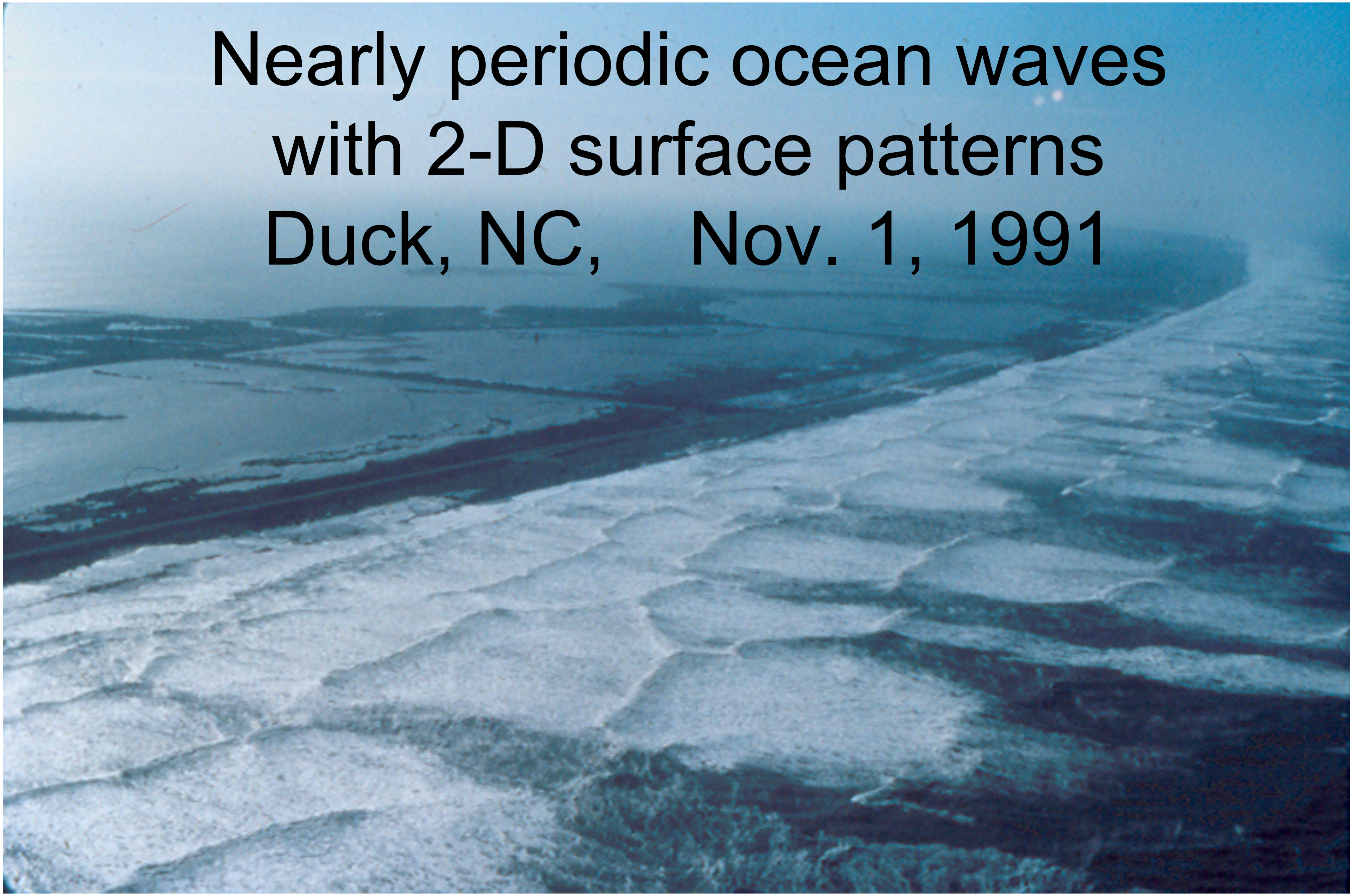
What is the effect of variable depth near shore?

- A tsunami is easy to predict away from shore.

  Near shore, variable depth, wave reflection, refraction and breaking all can be important.

  An adequate model is needed.
- Rip currents, edge waves and trapped waves all occur near shore (see Lecture 20).
- Sediment transport has great practical importance.

Is there a good way to make use of Harry Yeh's tank at Oregon State U?


Nearly periodic ocean waves
with 2-D surface patterns
Duck, NC, Nov. 1, 1991

Hurricane Katrina - Sept. 2006

The damaging part of the hurricane was the storm surge, which travels with speed $\sqrt{gh}$ and carries mass