The shallow water equations

Lecture 8

(photo due to Clark Little /SWNS)

The shallow water equations

This lecture:

1) Derive the shallow water equations
2) Their mathematical structure
3) Some consequences
4) Some open questions

Derive shallow water equations

1. Commonly used in large-scale ocean models
2. Start with Euler's equations, no surface tension

$$p = 0, \qquad \frac{D\eta}{Dt} = \partial_t \eta + \vec{u} \cdot \nabla \eta = w, \qquad \text{on } z = \eta(x,y,t)$$

$$\frac{D\vec{u}}{Dt} + \frac{1}{\rho} \nabla p + g\hat{z} = 0, \qquad \nabla \cdot \vec{u} = 0, \qquad -h(x,y) < z < \eta(x,y,t)$$

$$\vec{u} \cdot \nabla(z + h(x,y)) = 0 = w + \vec{u} \cdot \nabla h(x,y), \qquad \text{on } z = -h(x,y)$$

Derive shallow water equations

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2. Start with Euler's equations, no surface tension

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$$\vec{u} \cdot \nabla(z + h(x,y)) = \quad 0 = w + \vec{u} \cdot \nabla h(x,y), \qquad \text{on } z = -h(x,y)$$

3. Step one: global mass conservation

Derive shallow water equations

$$\nabla \cdot \vec{u} = 0, \qquad -h(x,y) < z < \eta(x,y,t)$$

Derive shallow water equations

$$\nabla \cdot \vec{u} = 0, \qquad -h(x,y) < z < \eta(x,y,t)$$

$$\int_{-h}^{\eta} [\nabla \cdot \vec{u}] dz = \int_{-h}^{\eta} [\partial_x u + \partial_y v + \partial_z w] dz = 0$$

Derive shallow water equations

$$\nabla \cdot \vec{u} = 0, \qquad -h(x,y) < z < \eta(x,y,t)$$

$$\int_{-h}^{\eta} [\nabla \cdot \vec{u}] dz = \int_{-h}^{\eta} [\partial_x u + \partial_y v + \partial_z w] dz = 0$$

$$\Rightarrow \quad \partial_x \int_{-h}^{\eta} [u] dz \quad -u|_{z=\eta}\, \partial_x \eta \quad +u|_{-h}\, \partial_x(-h)$$

$$+\partial_y \int_{-h}^{\eta} [v] dz \quad -v|_{\eta}\, \partial_y \eta \quad +v|_{-h}\, \partial_y(-h)$$

$$+w|_{\eta} \quad -w|_{-h} \quad = 0.$$

Derive shallow water equations

$$\nabla \cdot \vec{u} = 0, \qquad -h(x,y) < z < \eta(x,y,t)$$

$$\int_{-h}^{\eta} [\nabla \cdot \vec{u}] dz = \int_{-h}^{\eta} [\partial_x u + \partial_y v + \partial_z w] dz = 0$$

$$\partial_x \int_{-h}^{\eta} [u] dz \quad -u|_{z=\eta}\, \partial_x \eta \quad +u|_{-h}\, \partial_x(-h)$$

$$+\partial_y \int_{-h}^{\eta} [v] dz \quad -v|_{\eta}\, \partial_y \eta \quad +v|_{-h}\, \partial_y(-h)$$

$$+w|_{\eta} \quad -w|_{-h} \quad = 0.$$

Derive shallow water equations

$$\nabla \cdot \vec{u} = 0, \qquad -h(x,y) < z < \eta(x,y,t)$$

$$\int_{-h}^{\eta} [\nabla \cdot \vec{u}] dz = \int_{-h}^{\eta} [\partial_x u + \partial_y v + \partial_z w] dz = 0$$

$$\partial_x \int_{-h}^{\eta} [u] dz \quad -u \,|_{z=\eta}\; \partial_x \eta$$

$$+\partial_y \int_{-h}^{\eta} [v] dz \;\; -v \,|_{\eta}\; \partial_y \eta$$

$$+w \,|_{\eta} \qquad = 0.$$

Derive shallow water equations

$$\nabla \cdot \vec{u} = 0, \qquad -h(x,y) < z < \eta(x,y,t)$$

$$\int_{-h}^{\eta} [\nabla \cdot \vec{u}] dz = \quad \int_{-h}^{\eta} [\partial_x u + \partial_y v + \partial_z w] dz = 0$$

$$\partial_x \int_{-h}^{\eta} [u] dz \quad - u \,|_{z=\eta}\, \partial_x \eta$$

$$+ \partial_y \int_{-h}^{\eta} [v] dz \; - v \,|_{\eta}\, \partial_y \eta$$

$$+ w \,|_{\eta} \qquad = 0.$$

➔ $$\boxed{\partial_t \eta + \partial_x \int_{-h}^{\eta} [u] dz + \partial_y \int_{-h}^{\eta} [v] dz = 0.}$$ (exact)

Derive shallow water equations

Summary so far (no approximations):

- Conservation of mass:

$$\partial_t \eta + \partial_x \int_{-h}^{\eta} [u] dz + \partial_y \int_{-h}^{\eta} [v] dz = 0.$$

- 3 momentum eq'ns:

$$\frac{D\vec{u}}{Dt} + \frac{1}{\rho} \nabla p + g\hat{z} = 0, \qquad -h(x,y) < z < \eta(x,y,t)$$

Derive shallow water equations

Step 2: Assume long waves, but **not** small amplitudes.

$$\frac{D\vec{u}}{Dt} + \frac{1}{\rho}\nabla p + g\hat{z} = 0, \qquad -h(x,y) < z < \eta(x,y,t)$$

- Neglect vertical accelerations

$$\cancel{\frac{Dw}{Dt}} + \frac{1}{\rho}\partial_z p + g = 0,$$

➔ Hydrostatic pressure

$$p(x,y,z,t) = g\rho \cdot \{\eta(x,y,t) - z\}$$

Derive shallow water equations

Hydrostatic pressure in fluid

$$p(x,y,z,t) = g\rho\{\eta(x,y,t) - z\}$$

$$\partial_t u + u\partial_t u + v\partial_y u + w\partial_z u + g\partial_x \eta = 0,$$

$$\partial_t v + u\partial_t v + v\partial_y v + w\partial_z v + g\partial_y \eta = 0.$$

- Assume no vertical variation in (u, v)

$$\partial_t u + u\partial_t u + v\partial_y u + \cancel{w\partial_z u} + g\partial_x \eta = 0,$$

$$\partial_t v + u\partial_t v + v\partial_y v + \cancel{w\partial_z v} + g\partial_y \eta = 0.$$

$$\partial_t \eta + \partial_x \int_{-h}^{\eta} [u] dz + \partial_y \int_{-h}^{\eta} [v] dz = 0.$$

Shallow water equations

$$\partial_t \eta + \partial_x \{(u)(\eta + h)\} + \partial_y \{(v)(\eta + h)\} = 0,$$
$$\partial_t u + u\partial_x u + v\partial_y u + g\partial_x \eta = 0,$$
$$\partial_t v + u\partial_x v + v\partial_y v + g\partial_y \eta = 0.$$

- $h(x,y)$: known bottom topography
- Allows for variable depth in natural way
- Want: $\{\eta(x,y,t), u(x,y,t), v(x,y,t)\}$
- Similar to equations for gas dynamics in 2-D
- Common variation: include (Earth's) rotation

Q: What is mathematical structure of eq'ns?

Math structure #1: Hyperbolic PDEs

$$\partial_t \eta + \partial_x \{(u)(\eta + h)\} + \partial_y \{(v)(\eta + h)\} = 0,$$
$$\partial_t u + u\partial_x u + v\partial_y u + g\partial_x \eta = 0,$$
$$\partial_t v + u\partial_x v + v\partial_y v + g\partial_y \eta = 0.$$

$$\partial_t \begin{pmatrix} \eta \\ u \\ v \end{pmatrix} + \begin{bmatrix} u & \eta + h & 0 \\ g & u & 0 \\ 0 & 0 & u \end{bmatrix} \partial_x \begin{pmatrix} \eta \\ u \\ v \end{pmatrix} + \begin{bmatrix} v & \eta + h & 0 \\ 0 & v & 0 \\ g & 0 & v \end{bmatrix} \partial_y \begin{pmatrix} \eta \\ u \\ v \end{pmatrix} = -\begin{pmatrix} u\partial_x h + v\partial_y h \\ 0 \\ 0 \end{pmatrix}$$

Math structure #1: Hyperbolic PDEs

$$\partial_t \eta + \partial_x \{(u)(\eta + h)\} + \partial_y \{(v)(\eta + h)\} = 0,$$
$$\partial_t u + u\partial_x u + v\partial_y u + g\partial_x \eta = 0,$$
$$\partial_t v + u\partial_x v + v\partial_y v + g\partial_y \eta = 0.$$

$$\partial_t \begin{pmatrix} \eta \\ u \\ v \end{pmatrix} + \begin{bmatrix} u & \eta + h & 0 \\ g & u & 0 \\ 0 & 0 & u \end{bmatrix} \partial_x \begin{pmatrix} \eta \\ u \\ v \end{pmatrix} + \begin{bmatrix} v & \eta + h & 0 \\ 0 & v & 0 \\ g & 0 & v \end{bmatrix} \partial_y \begin{pmatrix} \eta \\ u \\ v \end{pmatrix} = -\begin{pmatrix} u\partial_x h + v\partial_y h \\ 0 \\ 0 \end{pmatrix}$$

- No dispersion! Solutions are non-dispersive.
- Eq'ns admit discontinuous (weak) solutions, that approximate breaking waves [How do waves break?]
- CLAWPACK (www.clawpack.org) developed by Randy Leveque (U of W) and others

Math structure #2 (if $v \equiv 0, \partial_y \equiv 0$):

Method of characteristics (Stoker, 1948)

$$\partial_t(\eta + h) + u\partial_x(\eta + h) + (\eta + h)\partial_x u = 0,$$
$$\partial_t u + u\partial_x u + g\partial_x(\eta + h) = g\partial_x h.$$

Math structure #2 (if $v \equiv 0, \partial_y \equiv 0$): Method of characteristics (Stoker)

$$\begin{pmatrix} g \\ 1 \end{pmatrix} \cdot \boxed{\begin{aligned} \partial_t(\eta + h) + u\partial_x(\eta + h) + (\eta + h)\partial_x u = 0, \\ \partial_t u + u\partial_x u + g\partial_x(\eta + h) = g\partial_x h. \end{aligned}}$$

- Define $c^2(x,y,t) = g \cdot (\eta + h)$

Math structure #2 (if $v \equiv 0, \partial_y \equiv 0$):

Method of characteristics (Stoker)

$$\begin{pmatrix} g \\ 1 \end{pmatrix} \cdot \quad \boxed{\begin{aligned} \partial_t(\eta + h) + u\partial_x(\eta + h) + (\eta + h)\partial_x u = 0, \\ \partial_t u + u\partial_x u + g\partial_x(\eta + h) = g\partial_x h. \end{aligned}}$$

- Define $c^2(x,y,t) = g \cdot (\eta + h)$

$$c \cdot [\partial_t(2c) + u\partial_x(2c) + (c)\partial_x u] = 0,$$

$$\partial_t u + u\partial_x u + (c)\partial_x(2c) = g\partial_x h,$$

$$\Rightarrow \quad \boxed{\begin{aligned} \partial_t(u + 2c) + u\partial_x(u + 2c) + (c)\partial_x(u + 2c) = g\partial_x h, \\ \partial_t(u - 2c) + u\partial_x(u - 2c) - (c)\partial_x(u - 2c) = g\partial_x h. \end{aligned}}$$

Math structure #2 (if $v \equiv 0, \partial_y \equiv 0$):
Method of characteristics (Stoker)

$$\partial_t(u+2c) + u\partial_x(u+2c) + (c)\partial_x(u+2c) = g\partial_x h,$$
$$\partial_t(u-2c) + u\partial_x(u-2c) - (c)\partial_x(u-2c) = g\partial_x h.$$

- Along curves in the *x-t* plane defined by

Along
$$\frac{dx}{dt} = u + c, \qquad \frac{d(u+2c)}{dt} = g\frac{dh}{dx}.$$
$$\frac{dx}{dt} = u - c, \qquad \frac{d(u-2c)}{dt} = g\frac{dh}{dx}.$$

- If h = *const* or $h(x) = mx + b$, ➔ Riemann invariants
- Apparently this method does **not** generalize to $\{x,y,t\}$.

Math structure #3: Linearize eq'ns

$$\partial_t \eta + \partial_x\{(u)(\eta + h)\} + \partial_y\{(v)(\eta + h)\} = 0,$$
$$\partial_t u + u\partial_x u + v\partial_y u + g\partial_x \eta = 0,$$
$$\partial_t v + u\partial_x v + v\partial_y v + g\partial_y \eta = 0.$$

$$\partial_t \eta + \partial_x (uh) + \partial_y (vh) = 0,$$
$$\partial_t u + g\partial_x \eta = 0,$$
$$\partial_t v + g\partial_y \eta = 0.$$

Math structure #3: Linearize eq'ns

$$\partial_t \eta + \partial_x\{(u)(\eta + h)\} + \partial_y\{(v)(\eta + h)\} = 0,$$
$$\partial_t u + u\partial_x u + v\partial_y u + g\partial_x \eta = 0,$$
$$\partial_t v + u\partial_x v + v\partial_y v + g\partial_y \eta = 0.$$

$$\begin{pmatrix}\sqrt{g}\\ \sqrt{h}\\ \sqrt{h}\end{pmatrix}\cdot \quad \begin{aligned} \partial_t \eta + \partial_x(uh) + \partial_y(vh) &= 0,\\ \partial_t u + g\partial_x \eta &= 0,\\ \partial_t v + g\partial_y \eta &= 0.\end{aligned}$$

(expect $c(x,y) = \sqrt{gh}$)

➔

$$\partial_t(\eta\sqrt{g}) + \partial_x(u\sqrt{h}\cdot\sqrt{gh}) + \partial_y(v\sqrt{h}\cdot\sqrt{gh}) = 0,$$
$$\partial_t(u\sqrt{h}) + \sqrt{gh}\cdot\partial_x(\eta\sqrt{g}) = 0,$$
$$\partial_t(v\sqrt{h}) + \sqrt{gh}\cdot\partial_y(\eta\sqrt{g}) = 0.$$

Good form for linearized equations: $h = h(x,y)$

Structure #3: Linearized eq'ns

$$\partial_t(\eta\sqrt{g}) + \partial_x(u\sqrt{h}\cdot\sqrt{gh}) + \partial_y(v\sqrt{h}\cdot\sqrt{gh}) = 0,$$
$$\partial_t(u\sqrt{h}) + \sqrt{gh}\cdot\partial_x(\eta\sqrt{g}) = 0,$$
$$\partial_t(v\sqrt{h}) + \sqrt{gh}\cdot\partial_y(\eta\sqrt{g}) = 0.$$

- Consequence (a): Eliminate $\{u\sqrt{h}, v\sqrt{h}\}$

Structure #3: Linearized eq'ns

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- Consequence (a): Eliminate $\{u\sqrt{h}, v\sqrt{h}\}$

➔ $$\partial_t^{\,2}(\eta\sqrt{g}) = \nabla\cdot\{gh\cdot\nabla(\eta\sqrt{g})\}$$

Linear wave equation in 2-D, with variable speed ($c^2 = gh$).
(Come back to use this.)

Structure #3: Linearized eq'ns

$$\partial_t(\eta\sqrt{g}) + \partial_x(u\sqrt{h}\cdot\sqrt{gh}) + \partial_y(v\sqrt{h}\cdot\sqrt{gh}) = 0,$$
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- Consequence (a): Eliminate $\{u\sqrt{h}, v\sqrt{h}\}$

$$\partial_t^{\,2}(\eta\sqrt{g}) = \nabla\cdot\{gh\cdot\nabla(\eta\sqrt{g})\}$$

Linear wave equation in 2-D, with variable speed ($c^2 = gh$).

- Consequence (b): Wave eq'n drops one *t*-derivative.

Define vorticity: $\omega(x,y,t) = \partial_y u - \partial_x v$

➔ $$\partial_t \omega = 0$$

Applications: Linearized wave eq'n

$$\partial_t^{\,2}(\eta\sqrt{g}) = \nabla \cdot \{gh \cdot \nabla(\eta\sqrt{g})\}$$

- Tsunamis
  - A good model for the propagation of a tsunami across the open ocean (away from shore, where the wave compresses horizontally, and grows vertically).
  - In the open ocean, the local speed of propagation, in any direction, is $c = \sqrt{gh(x,y)}$

Applications: Linearized wave eq'n

$$\partial_t^2(\eta\sqrt{g}) = \nabla \cdot \{gh \cdot \nabla(\eta\sqrt{g})\}$$

- Tsunamis
  - A good model for the propagation of a tsunami across the open ocean (away from shore, where the wave compresses horizontally, and grows vertically).
  - In the open ocean, the local speed of propagation, in any direction, is $c = \sqrt{gh(x,y)}$
- Wave shoaling (see lecture 20)

Math structure #4: Track vorticity in 2-D

$$\partial_t \eta + \partial_x\{(u)(\eta + h)\} + \partial_y\{(v)(\eta + h)\} = 0, \quad (1)$$

$$\partial_t u + u\partial_x u + v\partial_y u + g\partial_x \eta = 0, \quad (2)$$

$$\partial_t v + u\partial_x v + v\partial_y v + g\partial_y \eta = 0. \quad (3)$$

- Define $\omega = \partial_y u - \partial_x v$
- Compute, from (2), (3)

$$\partial_t(\omega) + u\partial_x(\omega) + v\partial_y(\omega) + (\omega)(\partial_x u + \partial_y v) = 0 \quad (4)$$

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$$\partial_t(\omega) + u\partial_x(\omega) + v\partial_y(\omega) + (\omega)(\partial_x u + \partial_y v) = 0 \quad (4)$$

➔

$$\partial_t \omega + \partial_x(u\omega) + \partial_y(v\omega) = 0 \quad (5)$$

➔ Total (integrated) vorticity is conserved.

Math structure #4: Track vorticity in 2-D

$$\partial_t \eta + \partial_x\{(u)(\eta + h)\} + \partial_y\{(v)(\eta + h)\} = 0, \quad (1)$$

$$\partial_t u + u\partial_x u + v\partial_y u + g\partial_x \eta = 0, \quad (2)$$

$$\partial_t v + u\partial_x v + v\partial_y v + g\partial_y \eta = 0. \quad (3)$$

- Define $\omega = \partial_y u - \partial_x v$
- Compute, from (2), (3)

$$\partial_t(\omega) + u\partial_x(\omega) + v\partial_y(\omega) + (\omega)(\partial_x u + \partial_y v) = 0 \quad (4)$$

$$\partial_t \omega + \partial_x(u\omega) + \partial_y(v\omega) = 0 \quad (5)$$

Total (integrated) vorticity is conserved.

- From (1)

$$\partial_t(\eta + h) + u\partial_x(\eta + h) + v\partial_y(\eta + h) + (\eta + h)(\partial_x u + \partial_y v) = 0 \quad (6)$$

Math structure #4: Track vorticity in 2-D

$$\partial_t\eta + \partial_x\{(u)(\eta+h)\} + \partial_y\{(v)(\eta+h)\} = 0, \quad (1)$$

$$\partial_t u + u\partial_x u + v\partial_y u + g\partial_x\eta = 0, \quad (2)$$

$$\partial_t v + u\partial_x v + v\partial_y v + g\partial_y\eta = 0. \quad (3)$$

$$\partial_t(\omega) + u\partial_x(\omega) + v\partial_y(\omega) + (\omega)(\partial_x u + \partial_y v) = 0 \quad (4)$$

$$\partial_t(\eta+h) + u\partial_x(\eta+h) + v\partial_y(\eta+h) + (\eta+h)(\partial_x u + \partial_y v) = 0 \quad (6)$$

➔

$$\partial_t\left(\frac{\omega}{\eta+h}\right) + u\partial_x\left(\frac{\omega}{\eta+h}\right) + v\partial_y\left(\frac{\omega}{\eta+h}\right) = 0 \quad (7)$$

Math structure #4: Track vorticity in 2-D

$$\partial_t \eta + \partial_x\{(u)(\eta + h)\} + \partial_y\{(v)(\eta + h)\} = 0, \quad (1)$$

$$\partial_t u + u\partial_x u + v\partial_y u + g\partial_x \eta = 0, \quad (2)$$

$$\partial_t v + u\partial_x v + v\partial_y v + g\partial_y \eta = 0. \quad (3)$$

$$\partial_t(\omega) + u\partial_x(\omega) + v\partial_y(\omega) + (\omega)(\partial_x u + \partial_y v) = 0 \quad (4)$$

$$\partial_t(\eta + h) + u\partial_x(\eta + h) + v\partial_y(\eta + h) + (\eta + h)(\partial_x u + \partial_y v) = 0 \quad (6)$$

$$\partial_t\left(\frac{\omega}{\eta + h}\right) + u\partial_x\left(\frac{\omega}{\eta + h}\right) + v\partial_y\left(\frac{\omega}{\eta + h}\right) = 0 \quad (7)$$

- $\left(\frac{\omega}{\eta + h}\right)$ is called the "potential vorticity"
- (7) is called "Ertel's theorem" (1942)
- Potential vorticity is carried (without change) by each "fluid particle" (*i.e.*, by each water column)

Math structure #4: Track vorticity in 2-D

$$\partial_t \eta + \partial_x\{(u)(\eta + h)\} + \partial_y\{(v)(\eta + h)\} = 0, \quad (1)$$

$$\partial_t u + u\partial_x u + v\partial_y u + g\partial_x \eta = 0, \quad (2)$$

$$\partial_t v + u\partial_x v + v\partial_y v + g\partial_y \eta = 0. \quad (3)$$

$$\partial_t\left(\frac{\omega}{\eta + h}\right) + u\partial_x\left(\frac{\omega}{\eta + h}\right) + v\partial_y\left(\frac{\omega}{\eta + h}\right) = 0 \quad (7)$$

- Let $G(\zeta)$ be any differentiable function

Then $\partial_t G(\varsigma) = \frac{dG}{d\varsigma} \cdot \partial_t \zeta$ etc.

➔ $$\partial_t\left\{G\left(\frac{\omega}{\eta + h}\right)\right\} + u\partial_x\left\{G\left(\frac{\omega}{\eta + h}\right)\right\} + v\partial_y\left\{G\left(\frac{\omega}{\eta + h}\right)\right\} = 0 \quad (8)$$

Any smooth function of $\left(\frac{\omega}{\eta + h}\right)$ is carried by each water column

Math structure #4: Track vorticity in 2-D

$$\partial_t \eta + \partial_x \{(u)(\eta + h)\} + \partial_y \{(v)(\eta + h)\} = 0, \quad (1)$$

$$\partial_t u + u\partial_x u + v\partial_y u + g\partial_x \eta = 0, \quad (2)$$

$$\partial_t v + u\partial_x v + v\partial_y v + g\partial_y \eta = 0. \quad (3)$$

$$\partial_t \omega + \partial_x (u\omega) + \partial_y (v\omega) = 0 \quad (5)$$

$$\partial_t \{G(\frac{\omega}{\eta + h})\} + u\partial_x \{G(\frac{\omega}{\eta + h})\} + v\partial_y \{G(\frac{\omega}{\eta + h})\} = 0 \quad (8)$$

Math structure #4: Track vorticity in 2-D

$$\partial_t \eta + \partial_x \{(u)(\eta + h)\} + \partial_y \{(v)(\eta + h)\} = 0, \quad (1)$$

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$$\partial_t v + u \partial_x v + v \partial_y v + g \partial_y \eta = 0. \quad (3)$$

$$\partial_t \omega + \partial_x (u\omega) + \partial_y (v\omega) = 0 \quad (5)$$

$$\partial_t \{G(\frac{\omega}{\eta + h})\} + u\partial_x \{G(\frac{\omega}{\eta + h})\} + v\partial_y \{G(\frac{\omega}{\eta + h})\} = 0 \quad (8)$$

$$\Rightarrow \quad \partial_t \{\omega \cdot G(\frac{\omega}{\eta + h})\} + \partial_x \{u\omega \cdot G(\frac{\omega}{\eta + h})\} + \partial_y \{v\omega \cdot G(\frac{\omega}{\eta + h})\} = 0 \quad (9)$$

Infinitely many conservation laws!

Math structure #4: Track vorticity in 2-D

$$\partial_t \eta + \partial_x\{(u)(\eta + h)\} + \partial_y\{(v)(\eta + h)\} = 0, \quad (1)$$

$$\partial_t u + u\partial_x u + v\partial_y u + g\partial_x \eta = 0, \quad (2)$$

$$\partial_t v + u\partial_x v + v\partial_y v + g\partial_y \eta = 0. \quad (3)$$

$$\partial_t\{\omega \cdot G(\frac{\omega}{\eta + h})\} + u\partial_x\{\omega \cdot G(\frac{\omega}{\eta + h})\} + v\partial_y\{\omega \cdot G(\frac{\omega}{\eta + h})\} = 0 \quad (9)$$

The potential vorticity is **very** constrained.

Q: Can we split the motion from these equations into 2 pieces:

- 2 "irrotational" pressure waves, with speed $\sqrt{g(\eta + h)}$
- a rotational wave, constrained by (9)?

Last topic: How do waves break?

$$\partial_t \eta + \partial_x\{(u)(\eta + h)\} + \partial_y\{(v)(\eta + h)\} = 0, \quad (1)$$

$$\partial_t u + u\partial_x u + v\partial_y u + g\partial_x \eta = 0, \quad (2)$$

$$\partial_t v + u\partial_x v + v\partial_y v + g\partial_y \eta = 0. \quad (3)$$

The shallow water equations are hyperbolic, so waves can break in shallow water, and they do.

Q: What mathematical model(s) of wave breaking should be added to these equations to describe wave breaking in shallow water?

- dissipative "shock waves" ?
- dispersive "collisionless shocks" ?
- other ? (please specify)

How do waves break in shallow water?

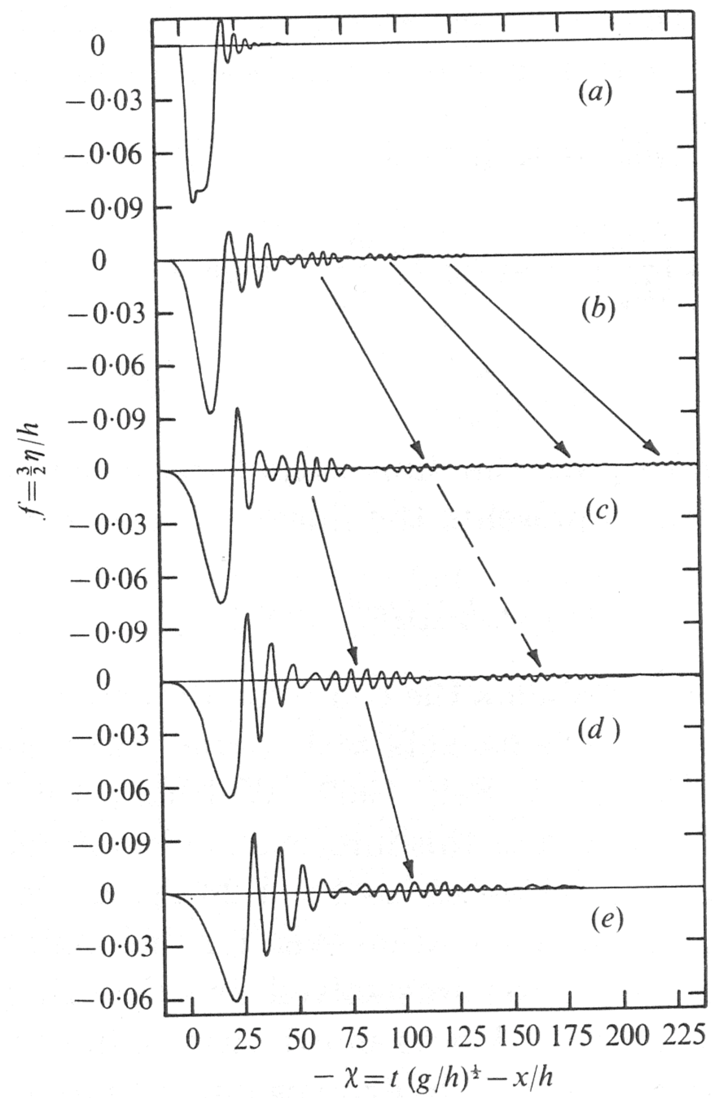
Choice 1: A “plunging breaker” - dissipative
(CLAWPACK probably uses this)

How do waves break in shallow water?

Recall Hammack's experiments in shallow water

"Undular bore"

- dispersive



How do waves break in shallow water?

Front of 2004 tsunami reaches the shore of Thailand

. The tsunami of 26 December 2004 approaching Hat Ray Leah beach on the Krabi coast, Thailand. (Copyright Scanpix

Note two breaking wave fronts

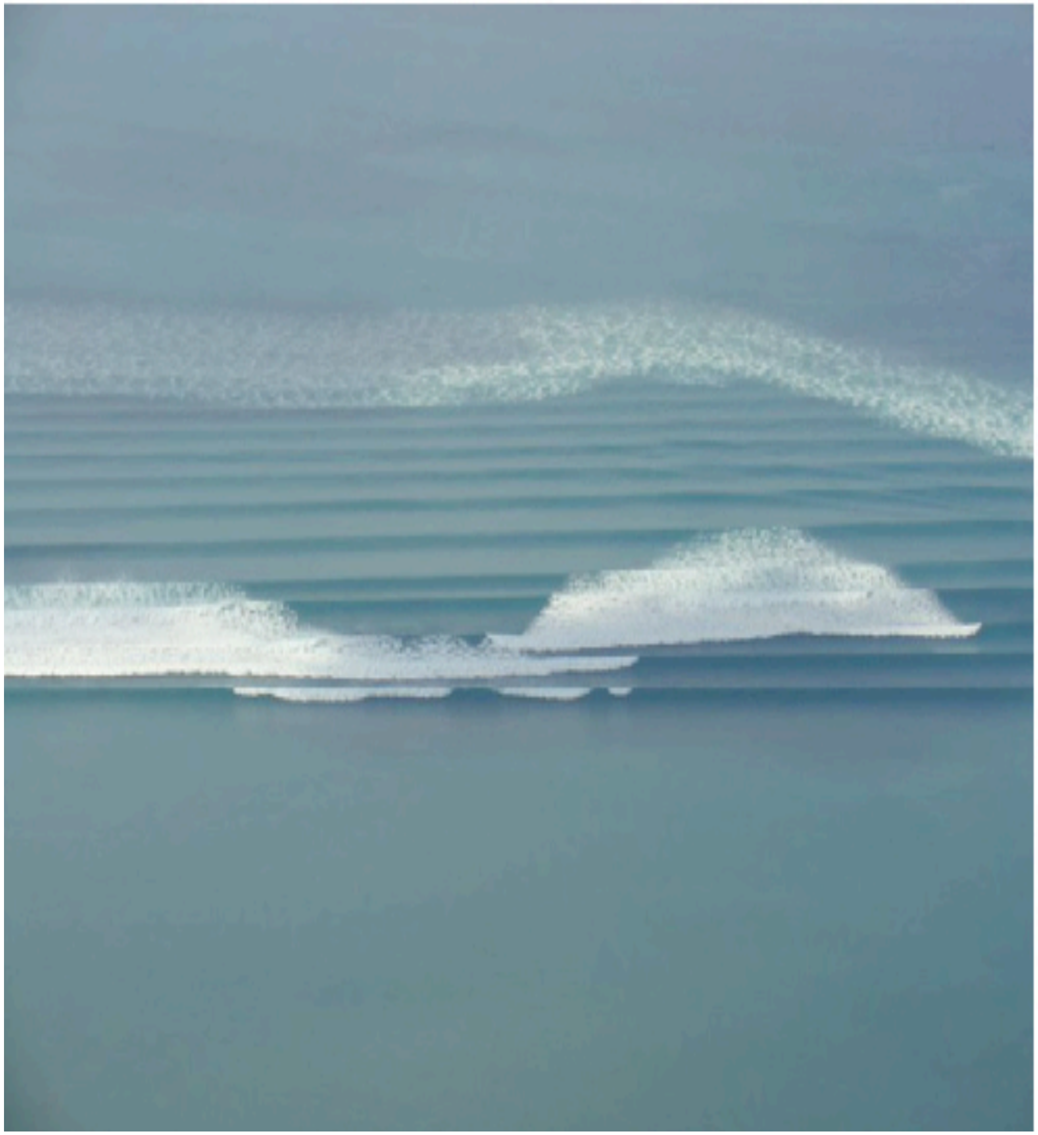
(photos from Constantin & Johnson, 2008)

How do waves break in shallow water?

Front of 2004 tsunami reaches the shore of Thailand

Train of oscillatory waves behind front

(Constantin & Johnson)



How do waves break in shallow water?

Photo due to
Clark Little, SWNS

Summary:

The “shallow water equations” are similar to the equations of gas dynamics in 2-D. But breaking water waves seem to be more complicated than ordinary shock waves in gas dynamics.

Q: How to model wave breaking properly?

Thank you for your attention

(photo due to Clark Little, SWNS)

Wave shoaling in shallow water

Q: Why do waves crests in shallow water often line up parallel to the beach?

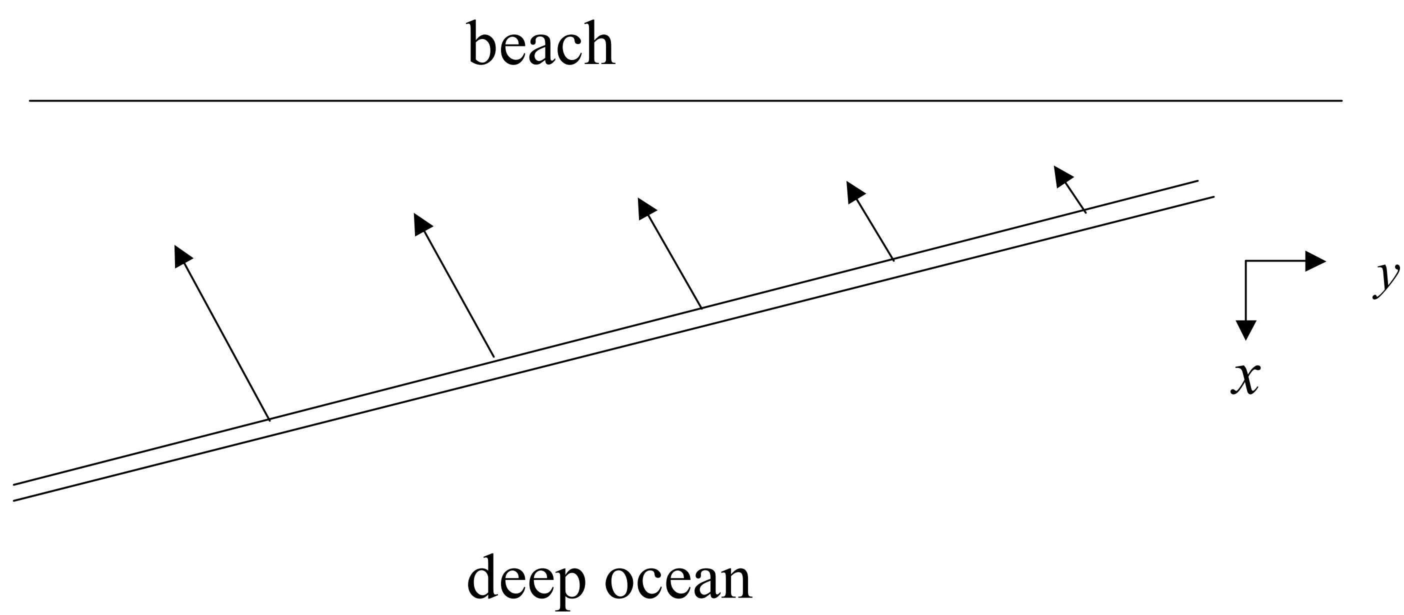
Lima, Peru 2004

Duck, NC 1991

Wave shoaling in shallow water

$$\partial_t^2 \eta = \nabla \cdot \{ g h(x,y) \nabla \eta \}$$

If $h = h(x)$ near shore, then $c(x) = \sqrt{g h(x)}$



Wave shoaling in shallow water

Q: Why do waves crests in shallow water often line up parallel to the beach?

Jones Beach
Long Island, NY