Lecture 10a: The Hadley Cell

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June 27

In this short lecture we take a look at the general circulation of the atmosphere, and in particular the Hadley cell. Look again at the zonally averaged circulation in the top panel of (lec.2:fig.2). The centre two circulations are the Hadley cells. Deep tropical convection lifts air near at the Intertropical Convergence Zone (ITCZ) near the equator. At the tropopause its vertical motion is inhibited by strong static stability, so it begins a poleward migration which extends as far as some critical latitude ϑ_H. In this lecture we will attempt to explain *why* the Hadley cell terminates at ϑ_H, and not some other latitude. We will outline three possibilities.

1. The Hadley cell is terminated in order to satisfy certain thermodynamic constraints, described in section 1.
2. The Hadley cell is terminated by the onset of baroclinic instability, described in section 2.
3. The Hadley cell is terminated by the effects of the breaking of Rossby waves, described in section 3.

Almost certainly none of these models describes the real Hadley cell in anything other than an approximate way, but this does not mean they are not useful.

1 A Zonally Symmetric Steady Model of the Hadley cell

We begin with a a model of the zonally symmetric circulation – that is, the circulation has no eddies, in fact no variation at all in the zonal direction. A parcel of air moving polewards away from the boundary layer will then conserve its axial angular momentum, as shown in Figure 1. To construct a mathematical model, following Schneider & Lindzen (1977) and Held & Hou (1980), we suppose the following.

1. The circulation is steady.
2. The polewards moving air conserves its axial angular momentum, whereas the zonal flow associated with the near-surface, equatorwards moving flow is frictionally retarded and weak.
3. The circulation is in thermal wind balance.
4. The flow is symmetric about the equator. Seasons can in fact be added to such a model.

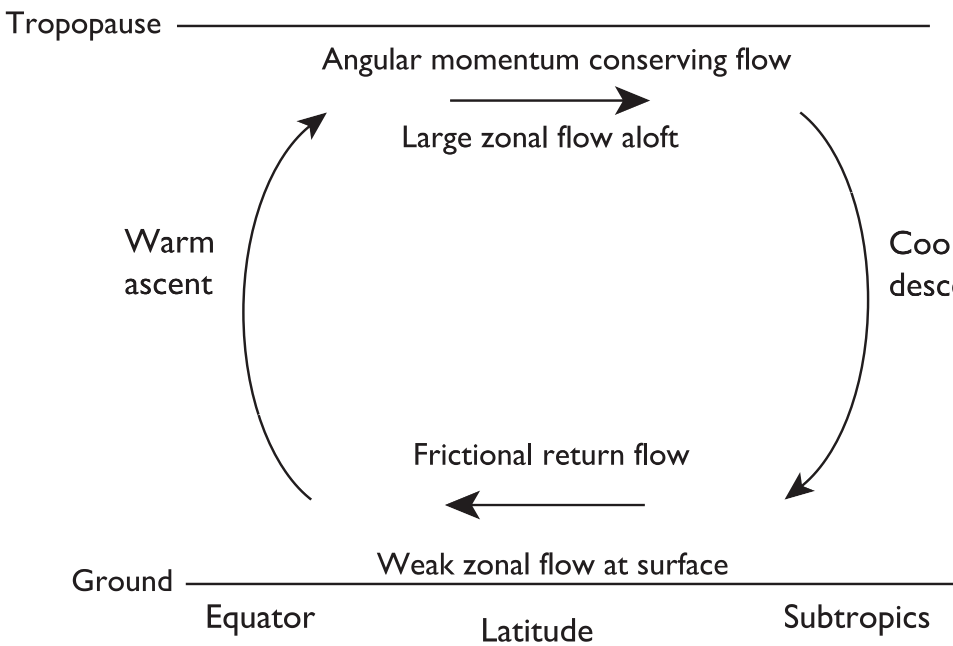

Figure 1: A simple model of the Hadley cell. Rising air near the equator moves polewards near the tropopause, descending in the subtropics and returning near the surface. The polewards moving air conserves its axial angular momentum, leading to a zonal flow that increases away from the equator. By the thermal wind relation the temperature of the air falls as it moves polewards, and to satisfy the thermodynamic budget it sinks in the subtropics. The return flow at the surface is frictionally retarded and small.

1.1 Angular momentum conservation

Momentum equation:

$$\frac{\partial \overline{u}}{\partial t} - (f + \overline{\zeta})\overline{v} + \overline{w}\frac{\partial \overline{u}}{\partial z} = -\frac{1}{a\cos^2\vartheta}\frac{\partial}{\partial \vartheta}(\cos^2\vartheta\overline{u'v'}) - \frac{\partial \overline{u'w'}}{\partial z}, \tag{1}$$

where $\overline{\zeta} = -(a\cos\vartheta)^{-1}\partial_\vartheta(\overline{u}\cos\vartheta)$ and the overbars represent zonal averages. We simplify this to

$$(f + \overline{\zeta})\overline{v} = 0. \tag{2}$$

It is easy to show that this is equivalent to

$$2\Omega\sin\vartheta = \frac{1}{a}\frac{\partial \overline{u}}{\partial \vartheta} - \frac{\overline{u}\tan\vartheta}{a}. \tag{3}$$

and the solution is (c.f. Fig. 2)

$$\overline{u} = \Omega a\frac{\sin^2\vartheta}{\cos\vartheta} \equiv U_M. \tag{4}$$

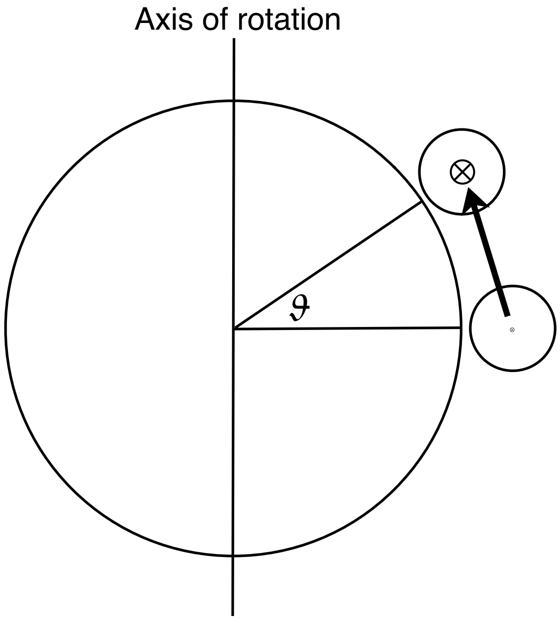

Figure 2: If a ring of air at the equator moves polewards it moves closer to the axis of rotation. If the parcels in the ring conserve their angular momentum their zonal velocity must increase; thus, if $m = (\overline{u} + \Omega a \cos\vartheta) a \cos\vartheta$ is preserved and $\overline{u} = 0$ at $\vartheta = 0$ we recover (4).

Temperature field

Thermal wind balance:

$$2\Omega \sin\vartheta \frac{\partial u}{\partial z} = -\frac{1}{a} \frac{\partial b}{\partial \vartheta}, \tag{5}$$

where $b = g\,\delta\theta/\theta_0$ is the buoyancy and $\delta\theta$ is the deviation of potential temperature from a constant reference value θ_0. (Be reminded that θ is potential temperature, whereas ϑ is latitude.) Vertically integrating from the ground to the height H where the outflow occurs and substituting (4) for u yields

$$\frac{1}{a\theta_0} \frac{\partial \theta}{\partial \vartheta} = -\frac{2\Omega^2 a}{gH} \frac{\sin^3\vartheta}{\cos\vartheta}, \tag{6}$$

where $\theta = H^{-1} \int_0^H \delta\theta \, \mathrm{d}z$ is the vertically averaged potential temperature. If the latitudinal extent of the Hadley cell is not too great we can make the small-angle approximation, and replace $\sin\vartheta$ by ϑ and $\cos\vartheta$ by one, then integrating (6) gives

$$\theta = \theta(0) - \frac{\theta_0 \Omega^2 y^4}{2gHa^2}, \tag{7}$$

where $y = a\vartheta$ and $\theta(0)$ is the potential temperature at the equator, as yet unknown. Away from the equator, the zonal velocity given by (4) increases rapidly polewards and the temperature correspondingly drops. How far polewards is this solution valid? And what determines the value of the integration constant $\theta(0)$? To answer these questions we turn to thermodynamics.

1.2 Thermodynamics

In the above discussion, the temperature field is slaved to the momentum field in that it seems to follow passively from the dynamics of the momentum equation. Nevertheless, the thermodynamic equation must still be satisfied. Let us assume that the thermodynamic forcing can be represented by a Newtonian cooling to some specified radiative equilibrium temperature, θ_E; this is a severe simplification, especially in equatorial regions where the release of heat by condensation is important. The thermodynamic equation is then

$$\frac{\mathrm{D}\theta}{\mathrm{D}t} = \frac{\theta_E - \theta}{\tau}, \tag{8}$$

where τ is a relaxation time scale, perhaps a few weeks. Let us suppose that θ_E falls monotonically from the equator to the pole, and that it increases linearly with height, and a simple representation of this is

$$\frac{\theta_E(\vartheta, z)}{\theta_0} = 1 - \frac{2}{3}\Delta_H P_2(\sin\vartheta) + \Delta_V\left(\frac{z}{H} - \frac{1}{2}\right), \tag{9}$$

where Δ_H and Δ_V are non-dimensional constants that determine the fractional temperature difference between the equator and the pole, and the ground and the top of the fluid, respectively. P_2 is the second Legendre polynomial. At $z = H/2$, or for the vertically averaged field, this approximates to

$$\theta_E = \theta_{E0} - \Delta\theta\left(\frac{y}{a}\right)^2, \tag{10}$$

where θ_{E0} is the equilibrium temperature at the equator, $\Delta\theta$ determines the equator–pole radiative-equilibrium temperature difference, and

$$\theta_{E0} = \theta_0(1 + \Delta_H/3), \qquad \Delta\theta = \theta_0\Delta_H. \tag{11}$$

Now, let us suppose that the solution (7) is valid between the equator and a latitude ϑ_H where $v = 0$, so that within this region the system is essentially closed. Conservation of potential temperature then requires that the solution (7) must satisfy

$$\int_0^{Y_H} \theta\,\mathrm{d}y = \int_0^{Y_H} \theta_E\,\mathrm{d}y, \tag{12}$$

where $Y_H = a\vartheta_H$ is as yet undetermined. Polewards of this, the solution is just $\theta = \theta_E$. Now, we may demand that the solution be continuous at $y = Y_H$ (without temperature continuity the thermal wind would be infinite) and so

$$\theta(Y_H) = \theta_E(Y_H). \tag{13}$$

The constraints (12) and (13) determine the values of the unknowns $\theta(0)$ and Y_H. A little algebra gives

$$Y_H = \left(\frac{5\Delta\theta g H}{3\Omega^2\theta_0}\right)^{1/2}, \tag{14}$$

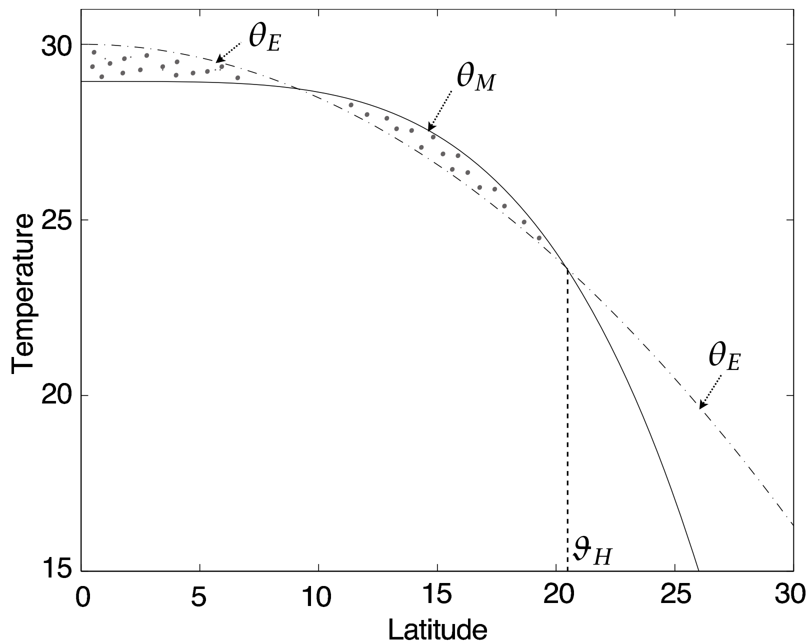

Figure 3: The radiative equilibrium temperature (θ_E, dashed line) and the angular-momentum-conserving solution (θ_M, solid line) as a function of latitude. The two dotted regions have equal areas. The parameters are: $\theta_{EO} = 303\,\mathrm{K}$, $\Delta\theta = 50\,\mathrm{K}$, $\theta_0 = 300\,\mathrm{K}$, $\Omega = 7.272 \times 10^{-5}\,\mathrm{s}^{-1}$, $g = 9.81\,\mathrm{m\ s}^{-2}$, $H = 10\,\mathrm{km}$. These give $R = 0.076$ and $Y_H/a = 0.356$, corresponding to $\vartheta_H = 20.4°$.

and

$$\theta(0) = \theta_{E0} - \left(\frac{5\Delta\theta^2 gH}{18a^2\Omega^2\theta_0}\right). \tag{15}$$

A useful non-dimensional number that parameterizes these solutions is

$$R \equiv \frac{gH\Delta\theta}{\theta_0\Omega^2 a^2} = \frac{gH\Delta_H}{\Omega^2 a^2}, \tag{16}$$

which is the square of the ratio of the speed of shallow water waves to the rotational velocity of the Earth, multiplied by the fractional temperature difference from equator to pole. Typical values for the Earth's atmosphere are a little less than 0.1. In terms of R we have

$$Y_H = a\left(\frac{5}{3}R\right)^{1/2}, \tag{17}$$

and

$$\theta(0) = \theta_{E0} - \left(\frac{5}{18}R\right)\Delta\theta\,. \tag{18}$$

The solution, (7) with $\theta(0)$ given by (18) is plotted in Fig. 3. Perhaps the single most important aspect of the model is that it predicts that the Hadley cell has a *finite* meridional extent, *even for an atmosphere that is completely zonally symmetric.* The baroclinic instability that does occur in mid-latitudes is not necessary for the Hadley cell to terminate

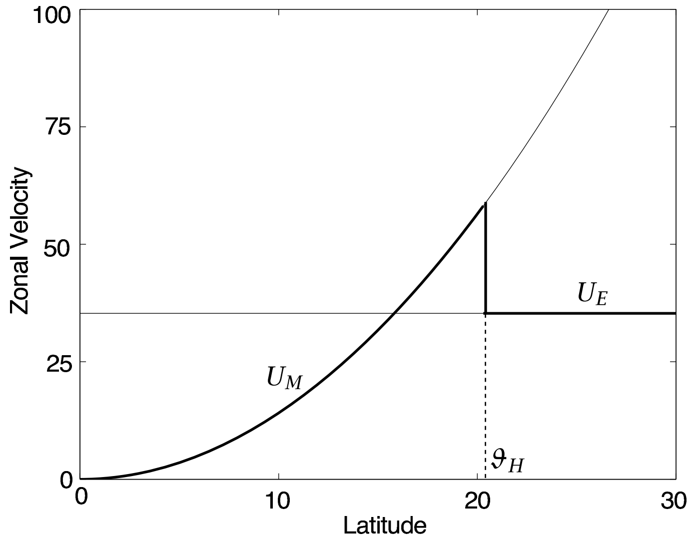

Figure 4: The zonal wind corresponding to the radiative equilibrium temperature (U_E,) and the angular-momentum-conserving solution (U_M) as a function of latitude, given (19) and (20) respectively. The zonal wind (in the model) follows the thick solid line: $u = U_m$ for $\vartheta < \vartheta_H$ ($y < Y_H$), and $u = U_E$ for $\vartheta > \vartheta_H$ ($y > Y_H$), and so has a discontinuity at ϑ_H.

in the subtropics, although it may be an important factor, or even the determining factor, in the real world.

1.3 Zonal wind

The angular-momentum-conserving zonal wind is given by (4), which in the small-angle approximation becomes

$$U_M = \Omega \frac{y^2}{a}. \tag{19}$$

This relation holds for $y < Y_H$. The zonal wind corresponding to the radiative-equilibrium solution is given using thermal wind balance and (10), which leads to

$$U_E = \Omega a R, \tag{20}$$

and this holds polewards of Y_H, or ϑ_H, as sketched in Fig. 4.

2 Baroclinic Instability and Termination of the Hadley Cell

One mechanism that could halt the Hadley cell is baroclinic instability. Having assumed that the surface winds are weak, and knowing the upper level zonal velocity from (4), the shear $\partial U_M / \partial z$ is determined by the height of the tropopause H, which we suppose to be a constant. At some latitude ϑ_C the shear will become baroclinically unstable at which point any assumption of zonal symmetry will break down and the Hadley cell will terminate. What model of baroclinic instability should we use to calculate this? The Eady model has no critical shear — all shears are unstable — but it has no beta-effect and beta is almost certainly important. The Charney model has beta, but it too has no critical shear. However, small shears give rise to shallow, weak instabilities that may not be important. Thus, we are led to the two-level Phillips model of baroclinic instability, because it accounts for the β effect.

In the Phillips model, a flow becomes unstable when it reaches a critical velocity difference between upper and lower levels given by

$$U = U_1 - U_2 = \frac{1}{4}\beta L_d^2 \tag{21}$$

where $L_d = NH/f$ is the baroclinic deformation radius, and on the sphere $\beta = 2\Omega\cos\phi/a$. Both β and the f hiding in L_d make this U grow towards the equator and decay towards the pole.

Now, from the Hadley cell solution

$$U = \Omega a \frac{\sin^2\phi}{\cos\phi} \tag{22}$$

so that, modulo constant factors, the Hadley cell terminates when

$$\frac{\sin^4\phi_c}{\cos^2\phi_c} = \frac{N^2H^2}{\Omega^2 a^2}, \tag{23}$$

or, with a small angle approximation,

$$\vartheta_H \approx \left(\frac{N^2H^2}{\Omega^2 a^2}\right)^{1/4} \sim (NH)^{1/2}. \tag{24}$$

As we discussed previously, both theory and modelling suggest that the tropopause will move higher as Global Warming progresses. This model shows that such an increase in H should be accompanied by a poleward expansion of the Hadley cell, perhaps by 1° – 2° over the 21^{st} century. But perhaps even more significant will be the changes in N^2, which is essentially set by the moist adiabatic lapse rate. A warmer atmosphere will hold more moisture by Clausius-Clapeyron (assuming no major changes in the relative humidity) which reduces the moist adiabatic lapse rate and reduces N^2. Thus the Hadley cell might in fact shrink equatorward based on this reasoning. However, the value of N in (24) should be evaluated at ϑ_H where the baroclinic instability occurs. This is not determined by the moist adiabatic lapse rate, and indeed model results suggest that subtropic static stability may increase with global warming, which would lead to an expansion of the Hadley cell.

3 Effect of Rossby-wave breaking

We will conclude this lecture with an outline of a third model for the extent of the Hadley cell. Recall we had reduced the zonal momentum equation (1) to a balance of two terms; let us now include a third for the momentum balance within the Hadley cell:

$$(f + \overline{\zeta})\overline{v} = -\frac{\partial}{\partial y}\left(\overline{u'v'}\right). \tag{25}$$

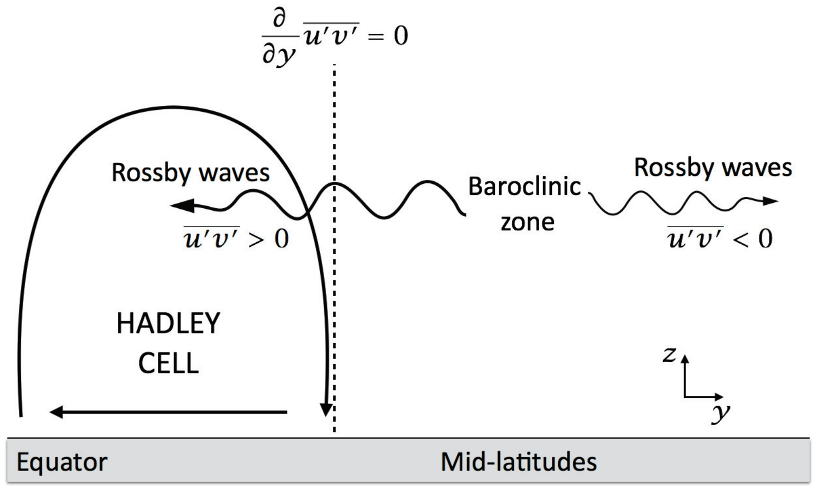

Figure 5: A schematic for the mechanism described in section 3. Rossby waves are generated through baroclinic instability at mid-latitudes, accelerating the flow eastwards: $\partial(\overline{u'v'})/\partial y > 0$. Some propagate equatorwards, and deposit westward momentum, $\partial(\overline{u'v'})/\partial y < 0$, near the critical latitude inside the Hadley cell. At some latitude the Rossby wave momentum flux is neither convergent nor divergent, $\partial(\overline{u'v'})/\partial y = 0$, corresponding to the edge of the Hadley cell.

However, at the edge of the Hadley cell we have $\overline{v} = 0$, and thus $\partial_y \left(\overline{u'v'}\right) = 0$. This is not necessarily the latitude where the flow is baroclinically unstable (Section 2). Rather, baroclinic instability, occurring at some latitude possibly poleward of here, generates Rossby waves; some of these propagate equatorwards and attenuate as they approach a critical latitude where the mean zonal wind matches the Rossby wave's phase speed (see the discussion of lecture 10b). Recalling our previous discussion, angular momentum conservation initiates a situation with weak winds in low-latitudes and strongly eastward winds in mid-latitudes. Thus a Rossby wave generated at mid-latitude has a phase speed somewhat less than the peak eastward wind speed, but certainly still positive for realistic parameters. This Rossby wave, then, will encounter a critical latitude equatorward of which it cannot flow. The wave breaks near this critical latitude and accelerates the zonal wind westward. This acceleration means that the next Rossby wave will encounter its critical latitude slightly more polewards. We thus have a situation in which the Rossby wave momentum flux convergence $\partial(\overline{u'v'})/\partial y$ is positive in the mid-latitudes and negative in the low-latitudes, requiring a zero crossing $\partial(\overline{u'v'})/\partial y = 0$ at some latitude in between, shown schematically in Figure 5. This, as was argued through (25), is the edge of the Hadley cell. Note that this edge is equatorward of where the baroclinic instability occurs (which was taken to be the edge in Section 2). The precise latitude will be established through a feedback between the eastward acceleration by angular momentum conservation and westward acceleration by Rossby wave breaking.

References

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