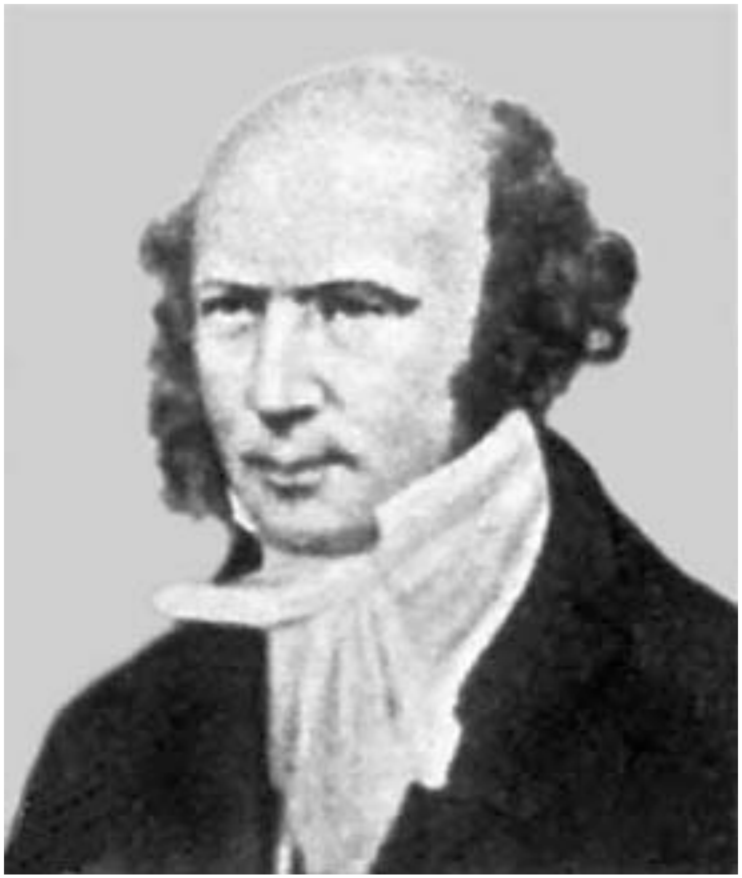
Hamiltonian formulation: water waves

Lecture 3



Wm. R. Hamilton

V.E. Zakharov

Hamiltonian formulation: water waves

This lecture:

A. Rapid review of Hamiltonian machinery (see also extra notes)
B. Hamiltonian formulation of water waves
   - Zakharov, 1967, 1968
   [Lagrangian formulation - Luke, 1967]
C. Some consequences of Hamiltonian structure

A. Review of Hamiltonian systems

1. Example: nonlinear oscillator

$$\boxed{\ddot{\theta} + \omega^2\theta + \alpha\theta^3 = 0,} \qquad \omega^2 > 0, \quad \dot{\theta} = \frac{d\theta}{dt}.$$

A. Review of Hamiltonian systems

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a) Find an energy integral

$$\dot{\theta} \cdot (eq'n) \Rightarrow \qquad E = \frac{1}{2}(\dot{\theta})^2 + \frac{\omega^2}{2}(\theta)^2 + \frac{\alpha}{4}(\theta)^4 = const.$$

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Define $\quad q = \theta(t), \quad p = \dot{\theta}(t)$

equivalent:

$$\boxed{\begin{aligned} \dot{q} &= p, \\ \dot{p} &= -\omega^2 q - \alpha q^3 \end{aligned}}$$

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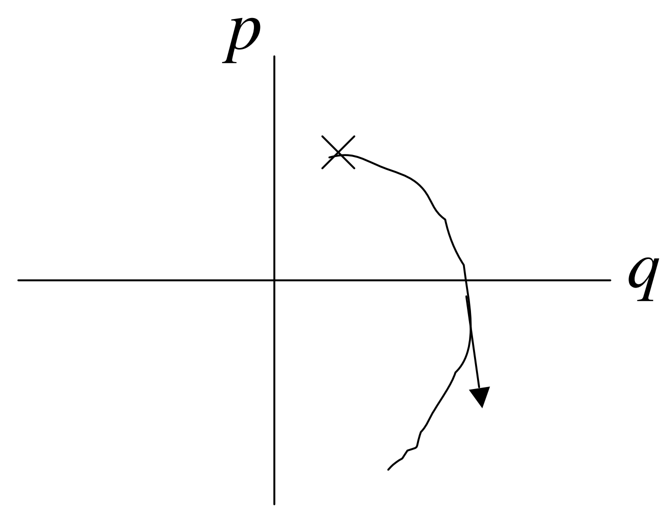

Review of Hamiltonian systems

2. Definition: A system of 2*N* first-order ODEs is <u>Hamiltonian</u> if there exist *N* pairs of coordinates on the phase space,

$$\{p_j(t), q_j(t)\}, \qquad j = 1,2,\ldots,N,$$

and a real-valued Hamiltonian function,

$$H(\vec{p}(t), \vec{q}(t), t),$$

such that the original equations are equivalent to

$$\boxed{\dot{q}_j = \frac{dq_j}{dt} = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}, \quad j = 1,2,\ldots,N.}$$

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$$\dot{q} = p = \frac{\partial H}{\partial p} \quad ✔ \qquad \dot{p} = -\omega^2 q - \alpha q^3 = -\frac{\partial H}{\partial q} \quad ✔$$

Review of Hamiltonian systems

3. Comments

a) Not every system of $2N$ first-order ODEs is Hamiltonian.

b) An essential property of a Hamiltonian system: the flow preserves volume in phase space. (The volume of a "ball" of initial data is preserved.)

c) H is often the physical energy, but not necessarily.

d) H is often a constant of the motion, but not necessarily.

Review of Hamiltonian systems

4. Plausibility argument for volume-preserving flows.

- Start with *M* first-order ODEs

$$\frac{dx_j}{dt} = v_j(\vec{x},t), \quad j = 1,2,\ldots,M.$$

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$$\frac{dx_j}{dt} = v_j(\vec{x},t), \quad j = 1,2,\ldots,M.$$

- Imagine a "fluid" that fills the *M*-dimensional phase space.

➔ $\{x_1(t), x_2(t), \ldots, x_M(t)\}$ are the coordinates of a fluid particle,

➔ $\{v_1(t), v_2(t), \ldots, v_M(t)\}$ are the components of fluid velocity.

Review of Hamiltonian systems

4. Plausibility argument for volume-preserving flows.

(See p. 69 of Arnold's "Classical Mechanics" for a real proof)

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- The fluid is "incompressible", so volume is preserved if

$$\boxed{\nabla \cdot \vec{v} = \sum_{j=1}^{M} \frac{\partial v_j}{\partial x_j} = 0}$$

Review of Hamiltonian systems

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$$\begin{aligned}
\nabla \cdot \vec{v} &= \sum_{j=1}^{N} \frac{\partial}{\partial q_j}\left(\frac{dq_j}{dt}\right) + \sum_{j=1}^{N} \frac{\partial}{\partial p_j}\left(\frac{dp_j}{dt}\right) \\
&= \sum_{j=1}^{N} \frac{\partial}{\partial q_j}\left(\frac{\partial H}{\partial p_j}\right) + \sum_{j=1}^{N} \frac{\partial}{\partial p_j}\left(-\frac{\partial H}{\partial q_j}\right) \\
&= 0.
\end{aligned}$$

Review of Hamiltonian systems

5. Hamiltonian PDEs

Example: nonlinear wave equation, periodic b.c.

$$\boxed{\partial_t^{\,2}\theta = c^2\partial_x^{\,2}\theta - \omega^2\theta - \alpha\theta^3}$$

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b) Choose "conjugate variables"

$$p(x,t) = \partial_t\theta(x,t), \quad q(x,t) = \theta(x,t).$$

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c) Guess: $$H(p,q,t) = \int [\frac{1}{2}p^2 + \frac{c^2}{2}(\partial_x q)^2 + \frac{\omega^2}{2}q^2 + \frac{\alpha}{4}q^4]dx$$

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this defines variational derivative: $\dfrac{\delta H}{\delta p}$

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Q: What is $\frac{\delta H}{\delta p}$? Why?

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$$\Rightarrow H(p,q+\delta q,t) - H(p,q,t) = \int [(-c^2\partial_x^2 q + \omega^2 q + \alpha q^3)\delta q + O((\delta q)^2)]dx$$

$$\frac{\delta H}{\delta q}$$

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End of lightning tour of Hamiltonian systems

B. Inviscid water waves

Recall:

$$\partial_t \eta + \nabla\phi \cdot \nabla\eta = \partial_z \phi,$$ on $z = \eta(x,y,t)$

$$\partial_t \phi + \frac{1}{2}|\nabla\phi|^2 + g\eta = \frac{\sigma}{\rho}\nabla\cdot\{\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\},$$ on $z = \eta(x,y,t)$

$$\nabla^2\phi = 0$$ $-h(x,y) < z < \eta(x,y,t)$

$$\partial_n \phi = 0$$ on $z = -h(x,y)$

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A. (Zakharov): on $z = \eta(x,y,t)$

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Q: Where does t-evolution occur?

A. (Zakharov): on $z = \eta(x,y,t)$

➔ Propose conjugate variables:

$$\boxed{\eta(x,y,t), \quad \psi(x,y,t) = \phi(x,y,z,t)\,|_{z=\eta}}$$

Water waves as Hamiltonian system

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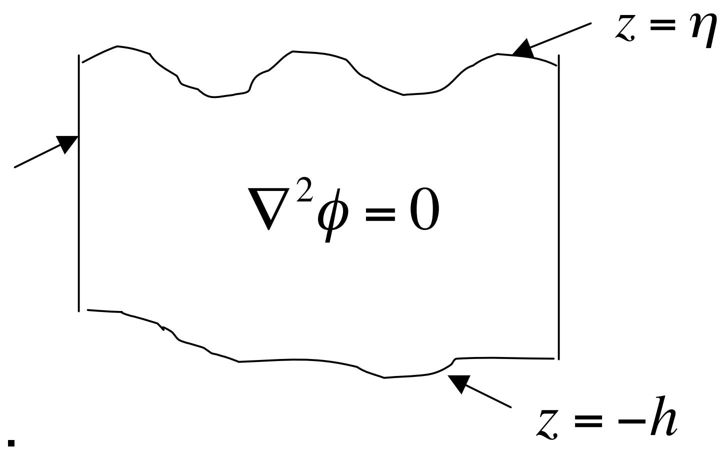
Plausibility check:



Suppose at some fixed t,

$\{\eta(x,y,t),\ \psi = \phi(x,y,t)|_{z=\eta}\}$ are given.

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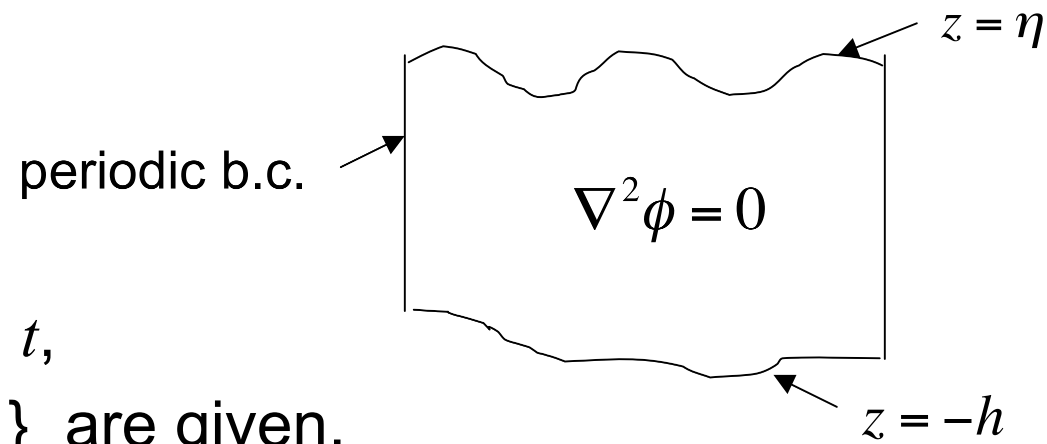

Suppose at some fixed t,

$\{\eta(x,y,t),\ \psi = \phi(x,y,t)|_{z=\eta}\}$ are given.

Then $\phi(x,y,z,t)$ is determined uniquely in domain.

[We need a procedure to find $\phi(x,y,z,t)$ from $\{\eta,\ \psi\}$].

Result: At any fixed time, $\{\eta,\ \psi\}$ determine the entire solution.

Water waves as Hamiltonian system

Proposed conjugate variables:

$$\eta(x,y,t), \quad \psi(x,y,t) = \phi(x,y,z,t)\,|_{z=\eta}$$

Q: What is $H(\eta,\psi)$?

A: Physical energy (from HW #1):

$$\boxed{H = \iint_R \left[\frac{1}{2}\int_{-h}^{\eta} |\nabla\phi|^2\, dz + \frac{1}{2} g\eta^2 + \frac{\sigma}{\rho}\left(\sqrt{1+|\nabla\eta|^2} - 1\right)\right] dxdy}$$

kinetic energy potential energy

Water waves as Hamiltonian system

Claim (Zakharov, 1968):

Let R be a fixed region in x-y plane. Let $h(x,y)$ be continuous and differentiable on R. Define

$$H(\eta,\psi) = \iint_R [\frac{1}{2}\int_{-h}^{\eta} |\nabla\phi|^2\, dz + \frac{1}{2} g\eta^2 + \frac{\sigma}{\rho}(\sqrt{1+|\nabla\eta|^2} - 1)]dxdy$$

We need to show that

$$\boxed{\partial_t \eta = \frac{\delta H}{\delta \psi}, \quad \partial_t \psi = -\frac{\delta H}{\delta \eta}}$$

are equivalent to the two boundary conditions on $z = \eta(x,y,t)$.

Water waves as Hamiltonian system

Step 1: Rewrite 2 eq'ns on $z = \eta$ in terms of $\{\eta, \psi\}$ and normal velocity on $z = \eta$.

- Define $F(x,y,z,t) = z - \eta(x,y,t)$, so $F = 0$ on $z = \eta$.

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- Define $F(x,y,z,t) = z - \eta(x,y,t)$, so $F = 0$ on $z = \eta$.
- unit normal vector on $z = \eta$:

$$\vec{n} = \frac{\nabla F}{|\nabla F|} = \frac{\{-\partial_x \eta, -\partial_y \eta, 1\}}{\sqrt{1 + |\nabla \eta|^2}} .$$

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- Normal component of velocity on $z = \eta$:

$$\partial_n \phi = \nabla \phi \cdot \hat{n} = \frac{-\nabla \phi \cdot \nabla \eta + \partial_z \phi}{\sqrt{1 + |\nabla \eta|^2}}$$

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- Eq'n #1 on $z = \eta$:

$$\partial_t \eta + \nabla \phi \cdot \nabla \eta = \partial_z \phi \qquad \Leftrightarrow \qquad \boxed{\partial_t \eta = \sqrt{1 + |\nabla \eta|^2}\, \partial_n \phi.}$$

Water waves as Hamiltonian system

<u>Step 2</u>: Rewrite 2nd eq'n on $z = \eta$:

- $$\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + g\eta - \frac{\sigma}{\rho} \nabla \cdot \{ \frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}} \} = 0$$

Water waves as Hamiltonian system

<u>Step 2</u>: Rewrite 2nd eq'n on $z = \eta$:

- $$\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + g\eta - \frac{\sigma}{\rho} \nabla \cdot \{ \frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}} \} = 0$$
- But $$\psi(x,y,t) = \phi(x,y,z,t) |_{z=\eta(x,y,t)}$$

➔ $$\partial_t \psi = \partial_t \phi |_{z=\eta} + \partial_z \phi |_{z=\eta} \, \partial_t \eta$$ (chain rule)

Water waves as Hamiltonian system

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$$= \partial_t \phi|_{z=\eta} + \partial_z \phi|_{z=\eta}\ \{\partial_z \phi - \nabla \phi \cdot \nabla \eta\}|_{z=\eta}$$

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- $$\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + g\eta - \frac{\sigma}{\rho}\nabla\cdot\{\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\} = 0$$
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➔ $$\partial_t\psi = \partial_t\phi|_{z=\eta} + \partial_z\phi|_{z=\eta}\ \partial_t\eta$$ (chain rule)

$$= \partial_t\phi|_{z=\eta} + \partial_z\phi|_{z=\eta}\ \{\partial_z\phi - \nabla\phi\cdot\nabla\eta\}|_{z=\eta}$$

- Eq'n #2 on $z = \eta$:

$$\boxed{\partial_t\psi + \frac{1}{2}[(\partial_x\phi)^2 + (\partial_y\phi)^2 - (\partial_z\phi)^2] + (\partial_z\phi)\nabla\phi\cdot\nabla\eta + g\eta - \frac{\sigma}{\rho}\nabla\cdot\{\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\} = 0}$$

Water waves as Hamiltonian system

The test:

$$H = \iint_R \Big[\underbrace{\frac{1}{2}\int_{-h}^{\eta} |\nabla\phi|^2\, dz}_{H_{\text{kin}}} + \underbrace{\frac{1}{2} g\eta^2 + \frac{\sigma}{\rho}\left(\sqrt{1+|\nabla\eta|^2} - 1\right)}_{H_{\text{pot}}}\Big] dxdy$$

Q: $\partial_t \eta = \dfrac{\delta H}{\delta \psi}$? $\qquad \partial_t \psi = -\dfrac{\delta H}{\delta \eta}$?

(check this) (see Zakharov's paper)

Water waves as Hamiltonian system

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H_{kin} H_{pot}

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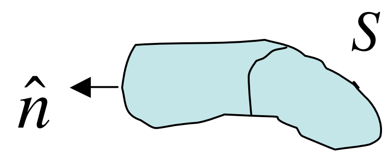
1) $\dfrac{\delta H_{pot}}{\delta \psi} = 0$ (easy)

2) $\dfrac{\delta H_{kin}}{\delta \psi}$ (not so easy)

Water waves as Hamiltonian system

<u>The test</u> (continued)



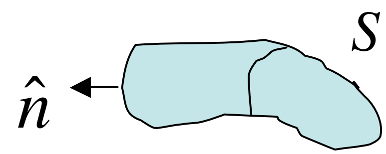
Recall divergence theorem:

Let *S* be a piecewise smooth, closed, oriented, 2-D surface with outward normal $\hat{n}$. Let $\vec{F}$ be a continuously differentiable vector field defined on *S* and its interior, *V*. Then

$$\oiint_S [\vec{F} \cdot \hat{n}]\, ds = \iiint_V [\nabla \cdot \vec{F}]\, dv$$

Water waves as Hamiltonian system

<u>The test</u> (continued)



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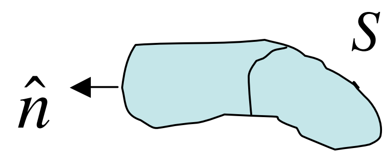
➔ $$\nabla\cdot\vec{F} = \frac{1}{2}[|\nabla\phi|^2 + \cancel{\phi\nabla^2\phi}]$$

Water waves as Hamiltonian system

<u>The test</u> (continued)



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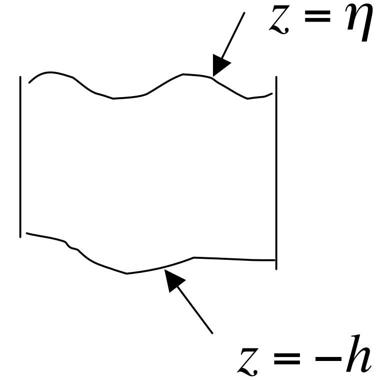
Water waves as Hamiltonian system

<u>The test</u> (continued)



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Water waves as Hamiltonian system

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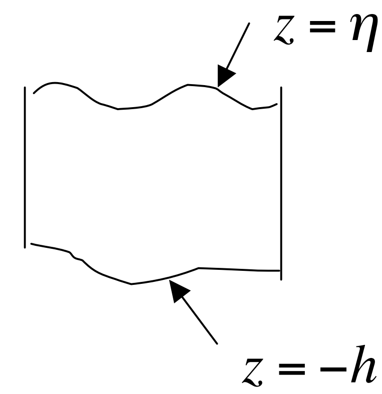

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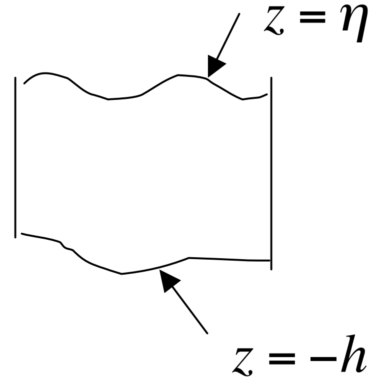
Water waves as Hamiltonian system

The test (continued)



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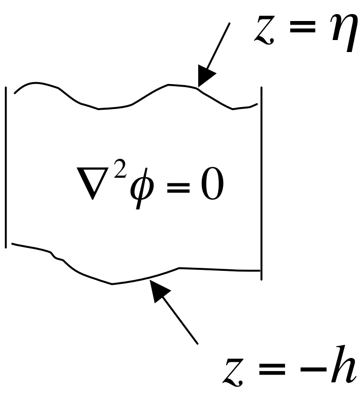
Last step: Relate $\partial_n\phi|_{z=\eta}$ to ψ

Water waves as Hamiltonian system

<u>Last step</u>: Relate $\partial_n \phi \,|_{z=\eta}$ to ψ



Dirchlet-to-Neumann map:

Water waves as Hamiltonian system

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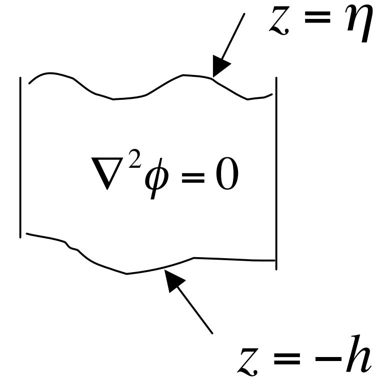

Dirchlet-to-Neumann map:

There is $G(x,y;\mu,\nu)$, symmetric Green's f'n

$$\partial_n \phi(x,y,z,t)|_{z=\eta} = \iint_{\substack{free\\surface}} [\psi(\mu,\nu,t)G(x,y;\mu,\nu)]ds$$

$$= \iint_R [\psi(\mu,\nu,t)G(x,y;\mu,\nu)]\sqrt{1+|\nabla\eta|^2}\,d\mu d\nu$$

Water waves as Hamiltonian system

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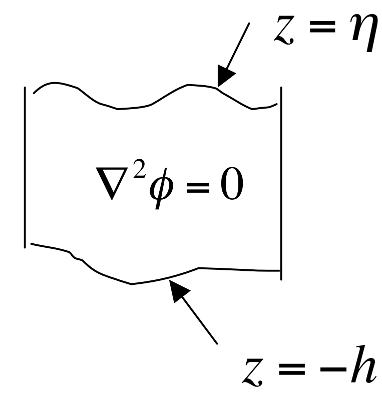

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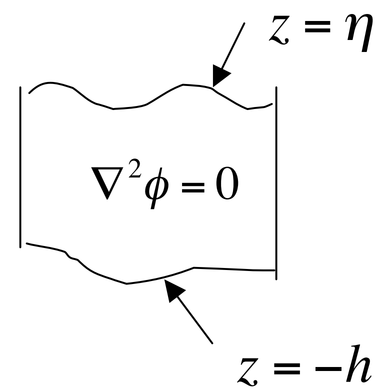
Substitute into $H_{\text{kin:}}$

$$H_{kin} = \frac{1}{2} \iint_R dxdy \sqrt{1+|\nabla\eta|^2} \iint_R d\mu d\nu \sqrt{1+|\nabla\eta|^2}\, \psi(x,y,t)\psi(\mu,\nu,t) G(x,y;\mu,\nu)$$

Water waves as Hamiltonian system

<u>Last step</u>: Relate $\partial_n \phi \,|_{z=\eta}$ to ψ



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Finally! Vary ψ, hold η fixed.

Water waves as Hamiltonian system

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Vary ψ, hold η fixed

Water waves as Hamiltonian system

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Vary ψ, hold η fixed

$$\delta H_{kin} = \frac{1}{2}\iint_R dxdy\sqrt{\ldots}\iint_R d\mu d\nu\sqrt{\ldots}[\delta\psi(x,y)\psi(\mu,\nu)+\psi(x,y)\delta\psi(\mu,\nu)]G(\ldots)$$

Water waves as Hamiltonian system

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But G is symmetric ➔

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➔ $$\delta H_{kin} = \iint_R dxdy\sqrt{1+|\nabla\eta|^2}\delta\psi(x,y)\partial_n\phi\,|_{z=\eta}$$

Water waves as Hamiltonian system

$$H_{kin} = \frac{1}{2}\iint_R dxdy\sqrt{1+|\nabla\eta|^2}\iint_R d\mu d\nu\sqrt{1+|\nabla\eta|^2}\psi(x,y,t)\psi(\mu,\nu,t)G(x,y;\mu,\nu)$$

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➔ $$\delta H_{kin} = \iint_R dxdy\sqrt{1+|\nabla\eta|^2}\,\delta\psi(x,y)\partial_n\phi|_{z=\eta}$$

➔ $$\boxed{\frac{\delta H}{\delta\psi} = \sqrt{1+|\nabla\eta|^2}\,\partial_n\phi|_{z=\eta} = \partial_t\eta}$$ ✔

Water waves as Hamiltonian system

Conclusion: Zakharov is correct!

- The equations of inviscid,irrotational water waves are Hamiltonian.
- Conjugate variables are $\{\eta, \psi\}$.
- The Hamiltonian is the physical energy.

C. So what?

Q: What does Hamiltonian structure buy?

A: Volume-preserving flow ➔

- Asymptotic stability is impossible

  neutral stability is only choice
- "attractors" and "repellers" are impossible
- Symplectic integrators: numerical integrators that preserve volume in phase space
- For water waves, (η, ψ) **are** good variables
- Complete integrability

C. So what?

Q: What is complete integrability?

1. Need to define Poisson bracket for correct statement.
2. If a system of $2N$ first-order ODEs is Hamiltonian, and if one finds N (**not** $2N$) constants of the motion, in involution relative to the Poisson bracket, then the motion is confined to an N-dimensional submanifold of $2N$ dim. phase space.

- If this manifold is compact, it is a <u>torus</u>.
- The N action variables are constants of the motion.
- N angle variables are coordinates on the torus.
- All of soliton theory fits into this framework.

Next lecture:

The (completely integrable)

Korteweg-de Vries equation

as an approximate model

of waves of moderate amplitude

in shallow water.