

# Lecture 3: Convective Heat Transfer I

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## 1 Introduction

In the first lecture, we discussed radiative transfer in the climate system. Here, we will delve into the convective heat transfer within the climate system, and how this leads to the radiative-dry convection equilibrium as an extension of the radiative equilibrium that we saw in Lecture 1.

## 2 The Buoyancy Equation

From the first lecture, we have that the buoyancy,  $B$ , of a fluid particle is given by the formula (stemming from Archimedes' Principle)

$$B = g \frac{\rho_e - \rho_b}{\rho_b}, \quad (1)$$

where subscripts  $e$  and  $b$  denote that the variable pertains to the environment and the sample fluid parcel, respectively. For convenience, we make two variable substitutions. First, we switch from using density to specific volume  $\alpha$  using the relation  $\alpha = 1/\rho$ , which yields

$$B = g \frac{\alpha_b - \alpha_e}{\alpha_e} = g \frac{\delta\alpha}{\alpha_e}. \quad (2)$$

Specific volume, however, is not conserved under adiabatic compression, but the entropy  $s$  is conserved. Hence comes our second substitution from  $\alpha$  to  $s$ . To make this substitution, we go through a few steps as follows:

$$\delta\alpha = \left(\frac{\partial\alpha}{\partial p}\right)_s \delta p + \left(\frac{\partial\alpha}{\partial s}\right)_p \delta s.$$

Setting the first term on the right side to zero, we are left with

$$\delta\alpha = \left(\frac{\partial\alpha}{\partial s}\right)_p \delta s.$$

We can then employ a Maxwell Relation

$$\left(\frac{\delta\alpha}{\delta s}\right)_p = \left(\frac{\delta T}{\delta p}\right)_s. \quad (3)$$

(The Maxwell Relations are a set of equations stating the relation between derivatives of thermodynamic variables. For a more in-depth discussion and derivation of the Maxwell Relations, refer to a standard Thermodynamics textbook.)

The term on the right can be plugged into the above equation for buoyancy (2):

$$B = \frac{g}{\alpha} \left( \frac{\delta T}{\delta p} \right)_s \delta s.$$

Assuming a hydrostatic environment, we can replace  $p$  with vertical height  $z$  according to  $\alpha dp = -g dz$ , giving the final equation

$$B = - \left( \frac{\delta T}{\delta z} \right)_s \delta s = \Gamma \delta s, \tag{4}$$

where  $\Gamma$  is the adiabatic lapse rate.

### 3 Stability

One of the primary questions in this lecture pertains to the instability of the atmosphere to convection. We can consider a parcel of air that we displace upwards. If the buoyancy of the parcel is also upwards, then the parcel is unstable and will continue to accelerate upwards. If, however, the buoyancy is downwards, then the parcel will accelerate back toward its original position. Relating this now to entropy, we see that if entropy decreases with height, then a particle displaced upwards will have a higher entropy than its surrounding and will thus be unstable. If entropy is constant in height, then the atmosphere is neutrally stable.

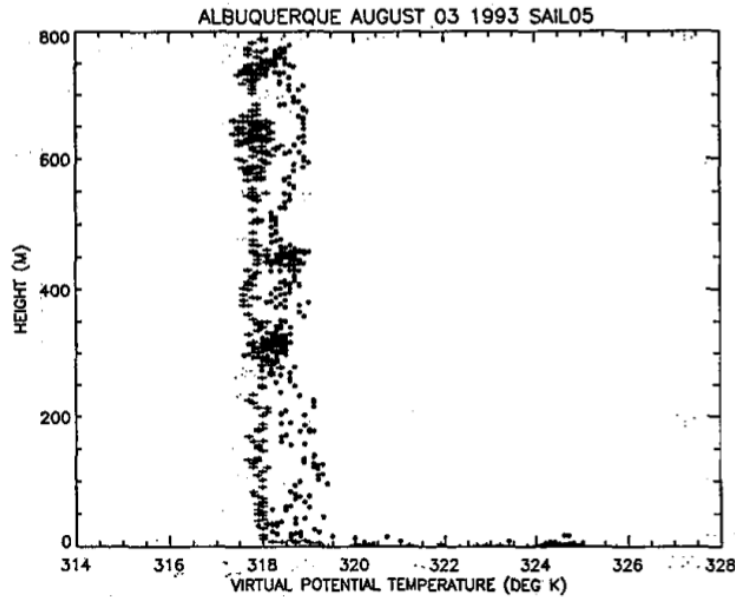


Figure 1: The profile of virtual potential temperature, which can be considered to be entropy for our purposes.

Figure 1 shows measurements taken from a model airplane of the entropy (the x-axis shows the virtual potential temperature, which behaves essentially like entropy) in the

atmosphere up to around 800m. It is clear that the entropy profile is nearly constant, with the exception of a few meters above the surface. This constant entropy layer implies the existence of a convecting layer, which can account for the mismatch in temperatures between the ground and the layer of air just above the ground in the radiative equilibrium model. Convection acts on a small enough time scale that it is able to destabilize the atmosphere, yielding a constant entropy layer.

## 4 The Prandtl Problem

(For a detailed description of the Prandtl problem discussed below, see Prandtl L 1925 Z. Angew. Math. Mech. 5 136.)

We consider now the Prandtl Problem, as shown in Figure 2. There is a rough bottom at constant temperature, which we set to be  $z=0$ . The fluid, which is subject to gravity, is cooled such that the vertical integral of cooling over the whole depth is constant. In mathematical terms, this can be written

$$\int_0^{\infty} \dot{Q} dz = \text{const.} \quad (5)$$

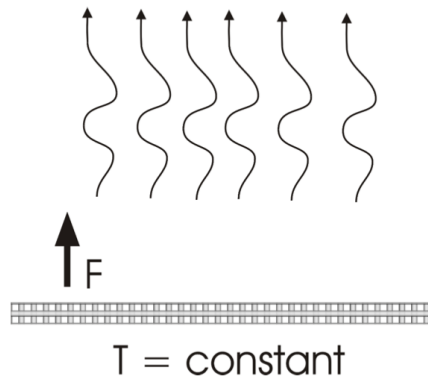


Figure 2: Schematic of the Prandtl problem.

There is a flux of heat from the lower boundary, which is the convective (buoyancy) flux  $F$ .  $F$  should be constant, and can be written as an average defined by

$$F = \overline{w'B'}, \quad (6)$$

where  $w'$  is the perturbation vertical velocity and  $B'$  the perturbation buoyancy.

To find a scale for the velocity in terms of  $F$  and  $z$ , we perform dimensional analysis. The following are the relevant dimensions in the problem:

$$F \sim \text{length}^2/\text{time}^3$$

$$z \sim \text{length}$$

The desired scaling for  $q \sim \text{length}/\text{time}$  can be achieved by

$$q \sim (Fz)^{\frac{1}{3}}. \quad (7)$$

Similarly, we can find a scaling for the buoyancy (which has units of length/time<sup>2</sup>) in terms of  $F$  and  $z$ :

$$B' \sim \left(\frac{F^2}{z}\right)^{\frac{1}{3}}. \quad (8)$$

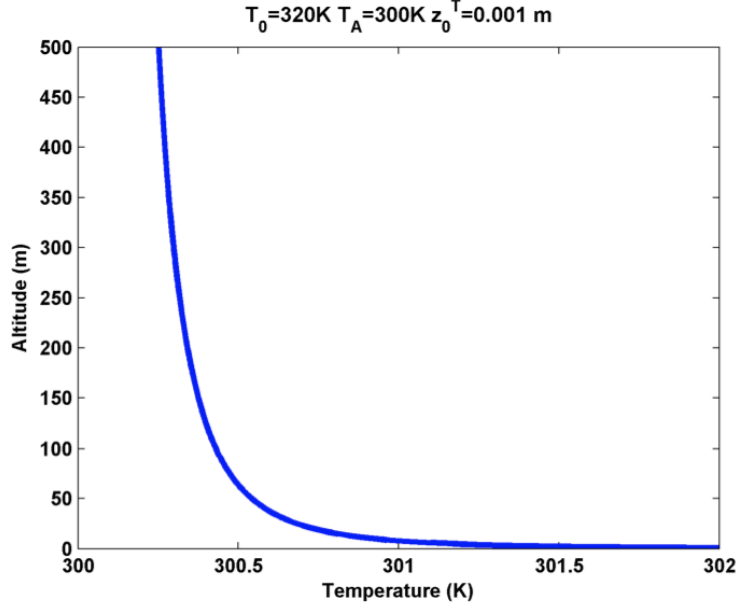


Figure 3: The vertical temperature profile.

We can then express the buoyancy  $B$  in terms of the average buoyancy  $B_0$  and the perturbation buoyancy  $B'$ :

$$B \sim B_0 - cF^{\frac{2}{3}} \left[ (z_0^T)^{-\frac{1}{3}} - z^{-\frac{1}{3}} \right], \quad (9)$$

where we define  $z_0^T$  to be the thermal roughness scale. This dependence is shown in figure 3, where temperature is plotted on the x-axis in lieu of buoyancy, but they are simply related by the coefficient of thermal expansion and thus have qualitatively similar behavior.

The radiative-dry convective equilibrium is shown in figure 4, with the pure radiative equilibrium plotted for comparison. From the graph, a few observations can be made. First, the temperature in the radiative-convective equilibrium in the troposphere and near the surface is significantly larger than in the purely radiative state. Why do we see this temperature increase in the troposphere and surface? This is because we assumed a constant relative humidity (a big assumption!), which implies more water vapor, hence a greater greenhouse effect, and thus a warmer temperature. However, there remains an important question - why is this graph so different than what we observe in the real world? This will be answered in the next section!

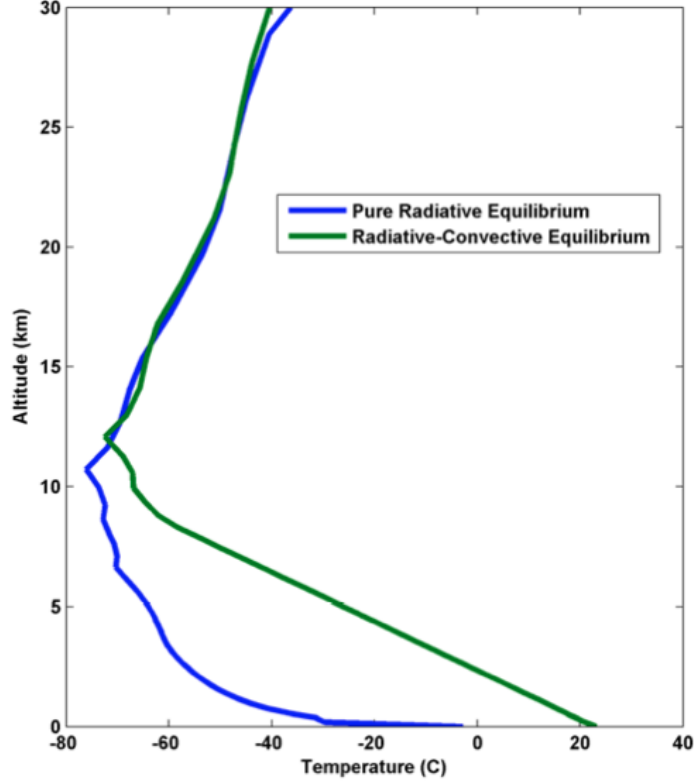


Figure 4: Full calculation of radiative-dry convective equilibrium.

## 5 Radiative - Dry Convective Model

In figure 5 is shown a simple model, very much like the one discussed in the first lecture. There are two layers, both opaque, but we are now assuming that, in addition to radiative fluxes, there are convective fluxes, denoted by  $F$ . By assumption (i.e. because we are forcing convective neutrality), we write

$$\begin{aligned} T_1 &= T_2 + \Delta T \\ T_S &= T_1 + \Delta T = T_2 + 2\Delta T. \end{aligned}$$

We can still write  $\sigma T_e^4 = \sigma T_2^4$ , yielding the temperature of the second layer  $T_2 = T_e$ . This dictates the temperatures of the other layers:

$$\begin{aligned} T_1 &= T_e + \Delta T \\ T_S &= T_e + \Delta T. \end{aligned}$$

At the surface we can write the equation

$$\sigma T_e^4 + \sigma T_1^4 = \sigma T_S^4 + F_S.$$

This can be rearranged to solve for the surface convective flux:

$$F_S = \sigma(-T_S^4 + T_e^4 + T_1^4) = \sigma T_e^4[1 + (1+x)^4 - (1+2x)^4], \quad (10)$$

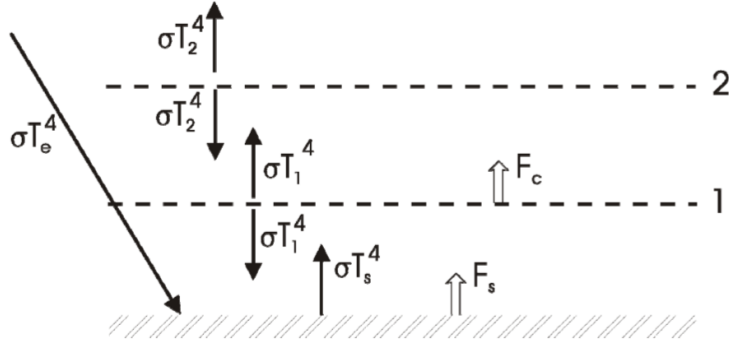


Figure 5: A simple radiative-convective energy balance model.

where  $x$  is defined to be  $x = \frac{\Delta T}{T_e}$ .

Similarly, the equation for the second layer can be written

$$2\sigma T_e^4 = \sigma T_1^4 + F_C,$$

and then solving for the convective flux

$$F_C = \sigma T_e^4 [2 - (1 + x)^4]. \tag{11}$$

The convective fluxes are needed to maintain a constant lapse rate. This leads into the topic introduced below, and discussed in detail in Lecture 5 about moist convection.

## 6 Introduction to Moist Convection

Moist convection is important for a number of reasons, and it will shape the equilibrium curve into something that looks more similar to the real-world picture.

Water in the atmosphere is responsible for a significant amount of heating due to phase changes (discussed below), and is considered one of the most important greenhouse gasses. Moist convection also plays a key role in stratiform cloudiness, and thus the planet's albedo and long-wave trapping.

### 6.1 Water Variables

The following are the variables which will be used in subsequent lectures.

$q = \frac{M_{water}}{M_{air}}$	specific humidity
$e$	vapor pressure (partial pressure of water vapor)
$e^* = 6.112hPa e^{\frac{17.67(T-273)}{T-30}}$	saturation vapor pressure
$H = \frac{e}{e^*}$	relative humidity
$q^*$	saturation specific humidity

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The ideal gas law will be used, where  $R^*$  and  $\bar{m}$  are the universal gas constant and the molecular weights of the constituents, respectively:

$$p = \rho \frac{R^* T}{\bar{m}},$$

which can be rewritten in terms of the vapor pressure

$$e = \rho_v \frac{R^* T}{\bar{m}_v},$$

with  $\bar{m}_v$  referring to the mass of the vapor particles. The specific humidity is the ratio of water vapor density to total density:

$$q = \frac{\rho_v}{\rho} = \frac{m_v}{\bar{m}} \frac{e}{p},$$

and thus the saturation specific humidity is expressed as

$$q^* = \frac{m_v}{\bar{m}} \frac{e^*}{p}. \tag{12}$$

In Lecture 5, which continues the discussion of convective heat transfer, we will use these quantities and equations defined above to study moisture in the atmosphere, which will lead us to the Radiative-Moist Convective Equilibrium (RCE).