Lecture 5 - Mathematical Foundations of Stochastic Processes

Charles R. Doering; notes by Yana Bebieva & Giovanni Fantuzzi

June 19, 2015

1 Stratonovich Interpretation of an SDE

There are different ways of interpreting stochastic differential equations (SDE). We know Itō's interpretation that gives the evolution equation for the transition density

$$\frac{\partial}{\partial t}\rho(x,t|y,s) = \frac{\partial}{\partial x_i}\left(-f_i + \frac{1}{2}\frac{\partial}{\partial x_i} g_{ik}g_{jk}\right)\rho = -\frac{\partial}{\partial x_i}J_i.$$

Here, $\vec{J}$ is a probability current vector field and g_{ij} are functions of x and t (i.e. $g = g(x,t)$, but for simplicity we will suppress the arguments).

The transition density satisfies an evolution equation when we differentiate with respect to the initial time, i.e. the Kolmogorov backward equation

$$-\frac{\partial}{\partial s}\rho(x,t|y,s) = \underbrace{\left(f_i\frac{\partial}{\partial y_i} + \frac{1}{2}g_{ik}g_{jk}\frac{\partial}{\partial y_j}\frac{\partial}{\partial y_i}\right)}_{\text{Generator of the process}}\rho.$$

We are still in a white noise limit and we should still get Markov process as the solution of the SDE (i.e. $X(t)$).

Now, we consider two examples where two different kinds of white noise limits give the same answer. Consider the process (in one dimension for simplicity)

$$\dot{X} = f(X) + g(X) \times \text{ (``approximate white noise'')}.$$

White noise has a spectrum that is flat, meaning that correlation function is the delta-function. However if there is a very short time correlation (that we just can not resolve well enough), then this noise can be considered as an approximate white noise (note it is still a real noise). We take a very specific example of approximate white noise as a Gaussian process $g(X)$, that is statistically stationary with a very short correlation time τ. For example, we may consider the Ornstein - Uhlenbeck process $\zeta(t)$, whose SDE is

$$d\zeta(t) = -\frac{1}{\tau}\zeta(t)dt + \frac{1}{\sqrt{\tau}}dW.$$

We pick the amplitude of the noise to be $1/\sqrt{\tau}$, when the relaxation time gets smaller and smaller, the noise influence gets bigger and bigger. The Fokker - Planck equation for $\zeta(t)$ becomes

$$\frac{\partial}{\partial t}\rho(z,t|z_0,t_0) = \frac{\partial}{\partial z}\left(\frac{1}{\tau}z + \frac{1}{2}\frac{1}{\tau}\frac{\partial}{\partial z}\right)\rho = \frac{1}{\tau}\frac{\partial}{\partial z}\left(z + \frac{1}{2}\frac{\partial}{\partial z}\right)\rho,$$

and we see that τ is a time scale for the evolution transition density. The stationary solution of this equation is the Gaussian distribution

$$\rho^{stat}(z) = \frac{1}{\sqrt{\pi}}e^{-z^2} \sim \mathcal{N}(0,1).$$

Moreover, the stationary correlation function is an exponential decay

$$\mathbb{E}(\zeta(t)\zeta(s)) = \frac{1}{2}e^{-\frac{|t-s|}{\tau}},$$

so that the power spectrum of the process is a Lorentzian spectrum

$$S(w) = \frac{\tau}{1+w^2\tau^2}.$$

When τ goes to zero, the spectrum widens but the amplitude decreases to zero. In order to prevent this the amplitude from vanishing, we rescale

$$\zeta(t) \to \frac{1}{\sqrt{\tau}}\zeta(t),$$

so that the spectrum becomes

$$S(w) = \frac{1}{\tau}\frac{\tau}{1+w^2\tau^2}.$$

The amplitude of the spectrum is now 1 when $\tau \to 0$.

The arguments above lead us to conclude that the system of SDEs

$$\frac{dX}{dt} = f(X) + g(X)\frac{1}{\sqrt{\tau}}\zeta(t), \tag{1}$$

$$\frac{d\zeta}{dt} = -\frac{1}{\tau}\zeta + \frac{1}{\sqrt{\tau}}\xi(t), \tag{2}$$

in the limit of short correlation time τ behaves similarly to an SDE for X where the noise is approximately white (note the functions f and g could have explicit time dependence). The combination (1) and (2) is a vector-valued Markov process, and the Fokker - Planck equation for its transitions density is

$$\frac{\partial}{\partial t}\rho(x,z,t|x_0,z_0,t_0) = \left[\frac{\partial}{\partial x}\left(-f - \frac{1}{\sqrt{\tau}}zg\right) + \frac{1}{\tau}\frac{\partial}{\partial z}\left(z + \frac{1}{2}\frac{\partial}{\partial z}\right)\right]\rho \tag{3}$$

We would like to deduce from this an evolution equation for the distribution of X alone, i.e. the marginal (reduced) distribution

$$r(x,t|x_0,t_0) = \int dz\left(\int \rho(x,z,t|x_0,z_0,t_0)\rho^{stat}(z_0)dz_0\right),$$

in the limit $\tau \to 0$.

To this end, let $\epsilon = \sqrt{\tau}$ and rearrange equation (3) by grouping terms of order ϵ^n. We obtain

$$0 = \left(\frac{1}{\epsilon^2} F_0 + \frac{1}{\epsilon} F_1 + F_2 \right) \rho, \tag{4}$$

where

$$\begin{aligned} F_0 &= \frac{\partial}{\partial z}(z + \frac{1}{2}\frac{\partial}{\partial z}) \quad \text{– the Ornstein - Uhlenbeck operator,} \\ F_1 &= -z\frac{\partial}{\partial x} g, \\ F_2 &= -(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} f). \end{aligned}$$

Our ansatz is

$$\rho(x, z, t|x_0, z_0, t_0) = \rho_0 + \epsilon\rho_1 + \epsilon^2\rho_2 + \dots,$$

i.e. the subscript of each ρ_i in the expansion indicates the corresponding power of ϵ. Now we plug this expression for ρ into (4) and group the terms according to the order of ϵ to obtain

$$\mathcal{O}(\epsilon^{-2}) : \qquad 0 = F_0\rho_0 \tag{5a}$$
$$\mathcal{O}\left(\epsilon^{-1}\right) : \qquad 0 = F_0\rho_1 + F_1\rho_0 \tag{5b}$$
$$\mathcal{O}(\epsilon^{0}) : \qquad 0 = F_0\rho_2 + F_1\rho_1 + F_2\rho_0 \tag{5c}$$

We keep in mind that we want to derive from this an evolution equation for the reduced distribution

$$r(x, t|x_0, t_0) = \int dz \left(\int \rho(x, z, t|x_0, z_0, t_0)\rho^{stat}(z_0)dz_0 \right) = r_0 + \epsilon r_1 + \epsilon^2 r_2 + \dots \tag{6}$$

We know all properties of the operator F_0; in particular, we can solve the eigenvalue problem

$$F_0 p_n(z) = -n p_n(z),$$

to find the eigenfunctions

$$p_n(z) = H_n(z)p^{stat}(z) \equiv H_n(z)p_0(z),$$

where the Hermite polynomials H_n are defined as

$$\begin{aligned} H_n(z) &= (-1)^n e^{z^2} \frac{d^n}{dz^n} e^{-z^2}, \\ H_0(z) &= 1, \\ H_1(z) &= 2z, \\ H_2(z) &= 2(2z^2 - 1). \end{aligned}$$

We may compute all Hermite polynomials using the recursion relation

$$zH_n(z) = \frac{1}{2}H_{n+1}(z) + nH_{n-1}(z).$$

Now we are ready to solve (5a) to find

$$0 = F_0\rho_0 \Rightarrow \rho_0(x,z,t) = p_0(z)r_0(x,t)$$

where $p_0(z)$ can be multiplied by any function of x and t (it plays the role of the amplitude of the eigenfunction p_0 at each point in space and time). In this case this function is $r_0(x,t)$ because if we integrate $\rho = \rho_0 + \epsilon\rho_1 + ...$ over z to obtain r, the leading term is exactly r_0.

To order ϵ^{-1}, equation (5b) now reads

$$F_0\rho_1 = -F_1\rho_0 = zp_0(z)\frac{\partial}{\partial x}gr_0(x,t) = \frac{1}{2}p_1(z)\frac{\partial}{\partial x}gr_0(x,t)$$

where we used the recursion relation to replace $zp_0 = p_1/2$. Here, we deal with a linear inhomogeneous differential equation and the general solution of this equation is a particular solution plus a general solution of the homogeneous equation. Therefore

$$\rho_1 = \underbrace{-\frac{1}{2}\frac{\partial}{\partial x}gr_0(x,t)p_1(z)}_{\text{particular solution}} + \underbrace{r_1(x,t)p_0(z)}_{\text{general solution}}.$$

Finally, upon substitution of ρ_0 and ρ_1, equation (5c) becomes

$$F_0\rho_2 = z\frac{\partial}{\partial x}g\left(-\frac{1}{2}\frac{\partial}{\partial x}gr_0(x,t)p_1(z) + r_1(x,t)p_0(z)\right) + \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}f\right)r_0(x,t)p_0(z).$$

Let us now use $zp_1 = 2z^2p_0 = \frac{1}{2}p_2 + p_0$ and $zp_0 = \frac{1}{2}p_1$, to show that

$$\begin{aligned} F_0\rho_2 &= -\left(\frac{1}{4}p_2(z) + \frac{1}{2}p_0(z)\right)\frac{\partial}{\partial x}g\frac{\partial}{\partial x}gr_0(x,t) + \frac{1}{2}p_1(z)\frac{\partial}{\partial x}gr_1(x,t) + (\frac{\partial}{\partial t} + \frac{\partial}{\partial x}f)r_0(x,t)p_0(z) \\ &= p_0(z)\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x}f - \frac{1}{2}\frac{\partial}{\partial x}g\frac{\partial}{\partial x}g\right]r_0(x,t) + p_1(z)\left[\frac{1}{2}\frac{\partial}{\partial x}gr_1(x,t)\right] + p_2(z)\left[-\frac{1}{4}\frac{\partial}{\partial x}g\frac{\partial}{\partial x}gr_0(x,t)\right]. \end{aligned} \tag{7}$$

Again, we need to add a particular solution and the solution of the homogeneous equation, i.e. $\rho_2 = \rho_{part} + r_2(x,t)p_0$. In order for a particular solution for this equation to exist the right hand side should be orthogonal to null space of the operator F_0. We know that $F_0p_1(z) = -p_1(z)$ and $F_0p_2(z) = -2p_2(z)$, but we can not invert F_0 on $p_0(z)$. Thus, in order to solve (7) the coefficient of $p_0(z)$ has to vanish and this gives a condition on $r_0(x,t)$

$$\frac{\partial}{\partial t}r_0(x,t|x_0,t_0) = \frac{\partial}{\partial x}\left(-f + \frac{1}{2}g\frac{\partial}{\partial x}g\right)r_0(x,t)$$

If we find $r_0(x,t)$ that satisfies this, we can get an explicit equation for $r_1(x,t)$, $r_2(x,t)$ etc. So if start with a real noise that is very fast ($\tau \to 0$, or $\epsilon \to 0$ in (6)), equation (1) becomes an SDE with an approximate white noise, $r(x,t) \to r_0(x,t)$ as $r_0(x,t)$ is the leading order term and we conclude that $\rho(x,t|x_0,t_0)$ of the process X satisfies the Fokker - Planck equation (a.k.a. Forward Kolmogorov Equation)

$$\underbrace{\frac{\partial}{\partial t}\rho = \frac{\partial}{\partial x}\left(-f + \frac{1}{2}g\frac{\partial}{\partial x}g\right)\rho}_{\text{Stratonovich Fokker - Planck}}. \tag{8}$$

We remark that Itō's interpretation of the SDE yielded the Fokker - Planck equation

$$\underbrace{\frac{\partial}{\partial t}\rho = \frac{\partial}{\partial x}\left(-f + \frac{1}{2}\frac{\partial}{\partial x}g^2\right)\rho}_{\text{Itō's the Fokker - Planck}}. \tag{9}$$

Thus if g is not a function of x (an additive noise), the transition density satisfying (8) is the same as that obtained from Itō's interpretation.

In the case of multiplicative noise, i.e. $g = g(x,t)$, can rewrite equation (8) as

$$\frac{\partial}{\partial t}\rho = \left(-f - \frac{1}{2}gg' + \frac{\partial}{\partial x}g^2\right)\rho \tag{10}$$

and then this equation is in the Itō form but with the modified drift. So if we let $f \mapsto f + \frac{1}{2}gg'$ in the SDE (9), then equation (10) describes the evolution of X according to the Itō's SDE.

2 Interpretation of an SDE: Itō vs Stratonovich

As remarked at the end of the previous section, in Itō's calculus the solution $X_t = X(t)$ of the SDE

$$dX_t = f(X_t,t)\,dt + g(X_t,t)\,dW_t \tag{11}$$

is the Markov process X_t whose transition density $\rho(x,t|x_0,t_0)$ satisfies the Itō's Fokker-Planck equation

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x}\left(-f + \frac{1}{2}\frac{\partial}{\partial x}g^2\right)\rho \tag{12}$$

(note the convention that an operator acts on all terms to its right). As previously explained, this followed from interpreting (11) as the continuous-time limit of the discrete-time process $X(t+\Delta t) - X(t) = f[X(t),t]\,\Delta t + g[X(t),t]\,\Delta W$, where $\Delta W = W(t+\Delta t) - W(t)$ is the jump of a Wiener process over the time interval Δt.

In the Stratonovich interpretation of (11), instead, the stochastic forcing in the SDE is obtained as the white-noise limit of a coloured stochastic process, such as the Ornstein-Uhlenbeck process. In this case, it is customary to write the SDE as

$$dX_t = f(X_t,t)\,dt + g(X_t,t)\circ dW_t, \tag{13}$$

where the " $\circ$ " sign indicates that dW_t should be interpreted as the white-noise limit of a coloured noise. The difference with Itō's interpretation is that the solution to (13) is the

Markov process X_t whose transition density $\rho(x,t|x_0,t_0)$ satisfies the Stratonovich Fokker-Planck equation

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x}\left(-f + \frac{1}{2} g \frac{\partial}{\partial x} g\right)\rho. \tag{14}$$

Clearly, the differential operators in equations (12) and (14) differ unless $g(x,t)$ is a constant, and the evolution of the PDF ρ depends on which interpretation of the SDE is chosen. However, an Itō SDE can easily be reformulated as a Stratonovich SDE (and vice versa). In fact, the solution of (11) is the same (in the sense of its statistics) as that of the Stratonovich SDE

$$dX_t = \left[f - \frac{1}{2} g \frac{\partial g}{\partial x}\right] dt + g \circ dW_t, \tag{15}$$

since the corresponding Fokker-Planck equation, computed from (14) using the modified drift $f - \frac{1}{2} g \frac{\partial g}{\partial x}$, can be rearranged to obtain (12). Similarly, one can see that the solution of (13) is the same (again, in the sense of its statistics) as the solution of the Itō SDE

$$dX_t = f\,dt + \frac{1}{2} g \frac{\partial g}{\partial x} + g\,dW_t. \tag{16}$$

This is particularly convenient since many physical systems are modelled by a coloured noise with very fast dynamics, which corresponds to a Stratonovich interpretation of the SDE; however, Itō's formulation is easier to implement numerically to simulate the system (for example, via the simple Euler-Maruyama or Milstein discretisation schemes).

Finally, we note that the Stratonovich interpretation maintains the standard rules of calculus for the differential of the random variable $Y_t = F(X_t)$, i.e.

$$dY_t = dF(X_t) = F'(X_t) \circ dX_t. \tag{17}$$

The symbol "$\circ$" indicates that X_t obeys a Stratonovich SDE. Equation (17) can be shown for one-dimensional processes if we assume that F is invertible, with inverse G. Then, the transition density $\rho_Y(y,t|y_0,t_0)$ of Y_t is related to $\rho_X(x,t|x_0,t_0)$ by

$$\rho_Y(y,t|y_0,t_0) = \rho_X[G(y),t|G(y_0),t_0)]G'(y), \tag{18}$$

Moreover, noticing that $G'(\cdot)F'(\cdot) = 1$ and $\frac{\partial}{\partial y} = G' \frac{\partial}{\partial x}$ one obtains

$$\begin{aligned}
\frac{\partial}{\partial t}\rho_Y &= G' \frac{\partial}{\partial t}\rho_X \\
&= \frac{\partial}{\partial x}\left(-f + \frac{1}{2} g \frac{\partial}{\partial x} g\right)\rho_Y \\
&= \frac{\partial}{\partial y}\left(-F'f + \frac{1}{2} F' g \frac{\partial}{\partial x} g \frac{F'}{F'}\right)\rho_Y \\
&= \frac{\partial}{\partial y}\left(-F'f + \frac{1}{2} F' g \frac{\partial}{\partial y} g F'\right)\rho_Y.
\end{aligned} \tag{19}$$

The claimed result follows from the Fokker-Planck equation for a Stratonovich SDE, which implies $dY_t = F'(f\,dt + g \circ dW_t) = F'(X_t) \circ dX_t$.

Example 1. (Stratonovich Logistic Equation) Consider the logistic equation

$$dX_t = (\bar{\mu}X_t - X_t^2)dt + \sigma\, X_t \circ dW_t$$

where $X_t \geq 0$ is the size of a population at time t and $\bar{\mu}$ and σ are given constants. The corresponding Fokker-Planck equation is

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x}\left(x^2 - \bar{\mu}x - \frac{\sigma^2}{2}x + \frac{\sigma^2}{2}\frac{\partial}{\partial x}x^2\right)\rho,$$

and the corresponding stationary distribution can be calculated as

$$\rho^{\text{stat}}(x) = N\, x^{\left(\frac{2\bar{\mu}}{\sigma^2}-1\right)}\, \exp\left(-\frac{2\,x}{\sigma^2}\right),$$

$$N = \left(\frac{\sigma^2}{2}\right)^{-\frac{2\bar{\mu}}{\sigma^2}}\left[\Gamma\left(\frac{2\bar{\mu}}{\sigma^2}\right)\right]^{-1}.$$

Moreover, the exact solution of the Stratonovich Logistic equation can be found using an appropriate change of variables. Dividing (1) by $X_t^2 dt$ and applying the chain rule (17), we have

$$-\frac{d}{dt}\left(\frac{1}{X_t}\right) = \frac{\bar{\mu}}{X_t} + \frac{\sigma}{X_t}\frac{dW_t}{dt} - 1.$$

Letting $Y_t = (X_t)^{-1}$, we obtain the differential equation

$$\frac{dY_t}{dt} + [\bar{\mu} + \sigma\xi(t)]\, Y_t = 1,$$

which can be solved to find

$$Y_t = Y_0\, e^{-\bar{\mu}t - \sigma W(t)} + e^{-\bar{\mu}t-\sigma W(t)}\int_0^t e^{\bar{\mu}s+\sigma W(s)}ds$$

$$\therefore \quad X_t = \frac{X_0\, e^{\bar{\mu}t+\sigma W(t)}}{1 + \displaystyle\int_0^t e^{\bar{\mu}s+\sigma W(s)}ds}.$$

Sample realisation of the SDE are shown in Figure 1, while Figure 2 illustrates the stationary transition density functions. For any noise amplitude σ, the process is driven by the exponential growth $e^{\bar{\mu}t}$, until saturation. The qualitative difference with Itō's interpretation of the same SDE (the solution of which can be found by substituting $\bar{\mu} \mapsto \bar{\mu} - \frac{1}{2}\sigma^2$, cf. Lecture 3) is remarkable: in the Itō's interpretation a stationary probability distribution ceases to exist when $\sigma^2 > 2\bar{\mu}$, i.e. when the exponential term $e^{\left(\bar{\mu}-\frac{1}{2}\sigma^2\right)t}$ decays in time. In terms of the population dynamics described by the SDE, this means that Itō's formulation predicts extinction (at least in the infinite-time limit) when $\sigma^2 > 2\bar{\mu}$, while the individuals are always alive for any σ according to the Stratonovich solution (see the case $\sigma = 2$ in Figure 1).

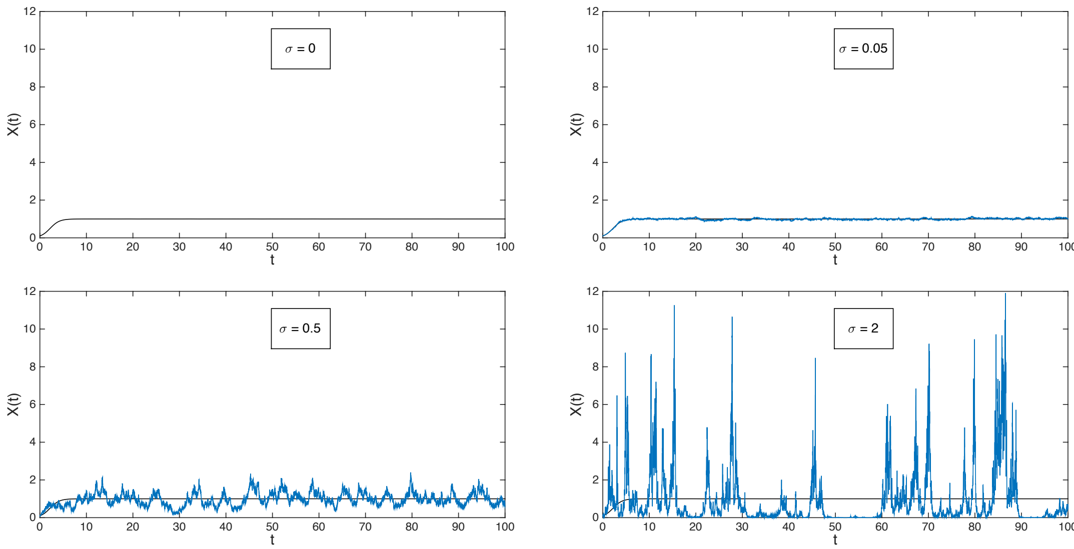

Figure 1: Sample realisation of the stochastic logistic equation, compared to the deterministic version (no noise) for $\bar{\mu} = 1$, $x_0 = 0.1$ and increasing noise amplitude σ. Clockwise (starting top-left): $\sigma = 0$, $\sigma = 0.05$, $\sigma = 0.5$, $\sigma = 2$.

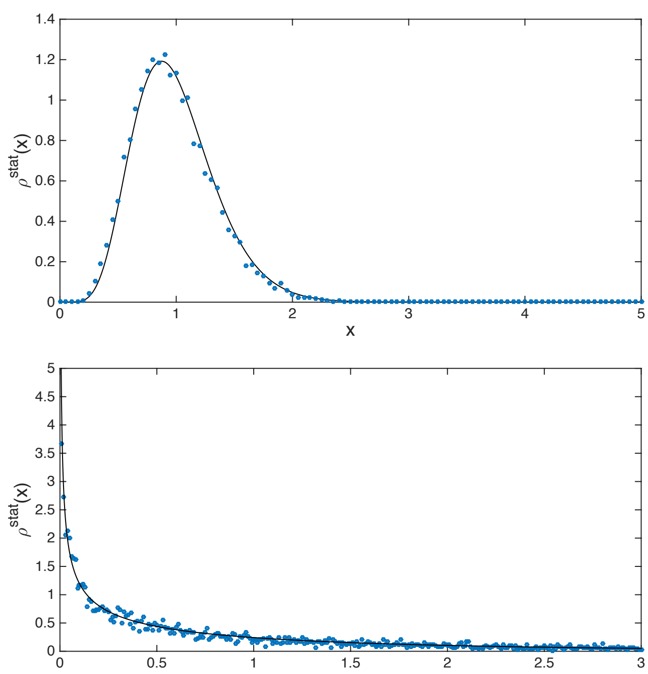

Figure 2: Comparison between the analytical transition density $\rho^{stat}(x)$ and the density function computed over 10^4 realisations of the SDE for $\sigma = 0.5$ (top) and $\sigma = 2$ (bottom).

3 Stratonovich SDEs for Vector-Valued Processes

The results presented in the previous section can be generalised to vector-valued processes. If $\mathbf{X}(t) = (X_1(t), X_2(t), ...)$ satisfies the set of SDEs

$$dX_i(t) = f_i(\mathbf{X}(t), t)dt + g_{ij}(\mathbf{X}(t), t) \circ dW_j(t), \quad i = 1, 2, ..., N, \quad j = 1, 2, ..., M, \tag{20}$$

then its transition density $\rho(\mathbf{x}, t|\mathbf{x}_0, t_0)$ can be computed with the Fokker-Planck equation

$$\frac{\partial}{\partial t}\rho = \frac{\partial}{\partial x_i}\left(-f_i(\mathbf{x}, t) + \frac{1}{2}g_{ik}(\mathbf{x}, t)\frac{\partial}{\partial x_j}g_{jk}(\mathbf{x}, t)\right)\rho. \tag{21}$$

Finally, we can translate this into the Itō's formulation (and vice versa) by modifying the drift in the same way as for the one-dimensional case.

4 White-noise Limit of a Dichotomous Markov Process

A symmetric dichotomous (a.k.a. two-step) Markov process $I(t)$ ($t \geq 0$) can take two values A and $-A$ (see Figure 3), the transition between one state and the other taking place at a constant rate α. The time intervals between state transitions are thus exponentially distributed and the probabilities $p_+(t) = \mathbb{P}[I(t) = A]$ and $p_-(t) = \mathbb{P}[I(t) = -A]$ evolve according to the master equation

$$\frac{d}{dt}\begin{pmatrix} p_+(t) \\ p_-(t) \end{pmatrix} = \begin{pmatrix} -\alpha & \alpha \\ \alpha & -\alpha \end{pmatrix}\begin{pmatrix} p_+(t) \\ p_-(t) \end{pmatrix}. \tag{22}$$

This equation can be solved to find

$$\begin{pmatrix} p_+(t) \\ p_-(t) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+e^{-2\alpha t} & 1-e^{-2\alpha t} \\ 1-e^{-2\alpha t} & 1+e^{-2\alpha t} \end{pmatrix}\begin{pmatrix} p_+(0) \\ p_-(0) \end{pmatrix}, \tag{23}$$

where $p_+(0)$ and $p_-(0)$ are the probabilities that the process starts at A or $-A$ respectively, with $p_+(0) + p_-(0) = 1$. The stationary distributions are immediately found as

$$\begin{pmatrix} p_+^{\mathrm{stat}} \\ p_-^{\mathrm{stat}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \tag{24}$$

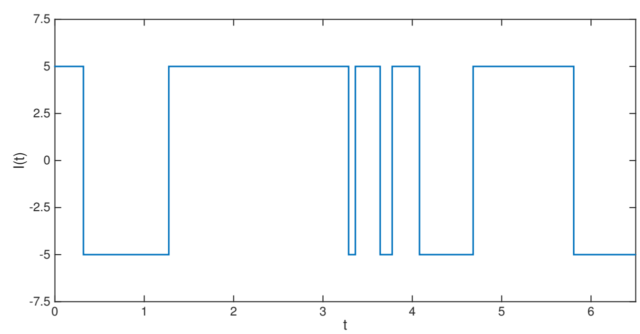

Figure 3: Sample realisation of $I(t)$ for $\alpha = 1$, $A = 5$.

and the conditional distributions $\mathbb{P}[I(t) = A|I(0) = \pm A]$, $\mathbb{P}[I(t) = -A|I(0) = \pm A]$ are obtained by setting $p_+(0)$ and $p_-(0)$ to 1 and 0 in turn. Specifically, one has

$$\begin{aligned} \mathbb{P}[I(t) = A|I(0) = \pm A] &= \frac{1}{2}\left(1 \pm e^{-2\alpha t}\right), \\ \mathbb{P}[I(t) = -A|I(0) = \pm A] &= \frac{1}{2}\left(1 \mp e^{-2\alpha t}\right). \end{aligned} \tag{25}$$

The conditional expectations can be used to compute the correlation

$$\begin{aligned} \mathbb{E}[I(t)I(0)] &= \sum_{n,m\in\{A,-A\}} n\, m\, \mathbb{P}[I(t) = n; I(0) = m] \\ &= \sum_{n,m\in\{A,-A\}} n\, m\, \mathbb{P}[I(t) = n|I(0) = m]\, \mathbb{P}[I(0) = m] \\ &= A^2\, e^{-2\alpha t}, \end{aligned} \tag{26}$$

for $t \geq 0$, which can be generalised to $\mathbb{E}[I(t)I(s)] = A^2\, e^{-2\alpha|t-s|}$ for any two time instants t and s. This means that the dichotomous process has the Lorenzian power spectrum

$$S(\omega) = \int_{-\infty}^{+\infty} A^2\, e^{-2\alpha|t|} e^{-i\omega t} dt = \frac{A^2}{\alpha} \frac{4\alpha^2}{4\alpha^2 + \omega^2} \quad . \tag{27}$$

Note that, as shown in Figure 4, $S(\omega)$ tends to a white spectrum in the limit $\alpha \to \infty$ if $A = \sqrt{\alpha}$.

Let us now consider a stochastic process X_t evolving according to

$$dX_t = f(X_t, t)dt + g(X_t, t)I(t). \tag{28}$$

The transition density of this process can be computed as $\rho(x, t|x_0, t_0) = \rho_+(x, t|x_0, t_0) + \rho_-(x, t|x_0, t_0)$, where the transition densities ρ_+ and ρ_- correspond to the mutually exclusive cases $I(t) = A$ and $I(t) = -A$ and satisfy the master equation

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho_+ \\ \rho_-\end{pmatrix} = \begin{pmatrix} -\alpha - \frac{\partial}{\partial x}(f + Ag) & \alpha \\ \alpha & -\alpha - \frac{\partial}{\partial x}(f - Ag) \end{pmatrix} \begin{pmatrix}\rho_+ \\ \rho_-\end{pmatrix}. \tag{29}$$

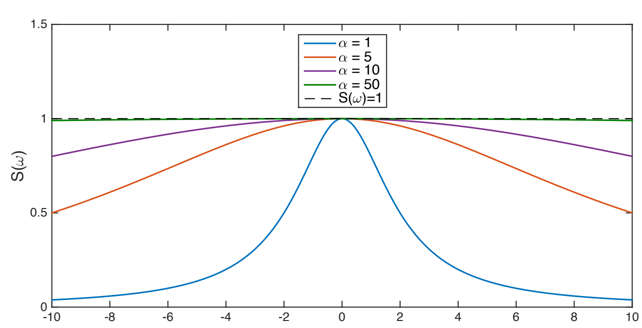

Figure 4: Correlation spectrum of $I(t)$ for $A = \sqrt{\alpha}$ and increasing values α.

Letting $\alpha = A^2$ in order to obtain the correct scaling of the noise spectrum, the equations for $\rho = \rho_+ + \rho_-$ and for $q = \rho_+ - \rho_-$ then read

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho \\ q\end{pmatrix} = \begin{pmatrix} -\frac{\partial}{\partial x}(f\rho + Agq) \\ -2A^2 q - \frac{\partial}{\partial x}(fq + Ag\rho)\end{pmatrix}. \tag{30}$$

When the noise amplitude A is very large, we argue that ρ and q can be expanded as

$$\begin{aligned} \rho(x,t) &= \rho_0(x,t) + \frac{1}{A}\rho_1(x,t) + \text{higher order terms} \\ q(x,t) &= q_0(x,t) + \frac{1}{A}q_1(x,t) + \text{higher order terms} \end{aligned} \tag{31}$$

so that the evolution of the random variable X_t in the white-noise limit is described by ρ_0. Substituting into (30) and collecting terms of the same order yields

$$O(A^2): \quad 2q_0 = 0, \tag{32a}$$

$$O(A): \quad \begin{cases} 2q_1 + \dfrac{\partial}{\partial x}(g\,\rho_0) = 0, \\ \dfrac{\partial}{\partial x}(g\,q_0) = 0, \end{cases} \tag{32b}$$

$$O(1): \quad \begin{cases} \dfrac{\partial q_0}{\partial t} + 2q_2 + \dfrac{\partial}{\partial x}(f q_0 + g\,\rho_1) = 0, \\ \dfrac{\partial q_0}{\partial t} + \dfrac{\partial}{\partial x}(f\rho_0 + g\,q_1) = 0, \end{cases} \tag{32c}$$

from which the following Fokker-Planck equation can be derived for ρ_0:

$$\frac{\partial \rho_0}{\partial t} = \frac{\partial}{\partial x}\left(-f + \frac{1}{2}g\frac{\partial}{\partial x}g\right)\rho_0. \tag{33}$$

This equation is the same as the Stratonovich Fokker-Planck equation. We conclude that the solution to the SDE

$$\frac{dX_t}{dt} = f(X_t,t) + g(X_t,t)\circ\xi(t), \tag{34}$$

where $\xi(t)$ is the white-noise limit of a dichotomous Markov process (a.k.a. dichotomous noise or DMN), is the process X_t whose transition density satisfies the Stratonovich Fokker-Planck equation.