Lecture 5: Convective Heat Transfer II

Kerry Emanuel; notes by Geoff J. Stanley and Jim Thomas

June 20

1 Introduction

In this lecture we will study classical Köhler theory and the effects of aerosols and their size distribution to explain raindrop formation. We will then study the stability of a moist atmosphere (building on the stability of a dry atmosphere in Lecture 3). We then conclude with a discussion of radiative moist convective equilibrium.

2 Raindrop Formation

How does rain actually form in Earth's atmosphere? The answer is quite complex, and there is a long history of scientists who have attempted to explain the phenomenon. In this lecture we will examine the process and along the way note its historical development.

We must begin with a phase diagram of water, which has been obtained through both theory and laboratory experiment. Figure 1 shows the phase of water as a function of pressure p and temperature T. When pressure and/or temperature change so as to cross a line on the diagram, water changes from one phase to another, as the theory goes. However, phase transitions in Earth's atmosphere, being catalysed by a distribution of aerosols, are more complex than in the laboratory.

First let us recall our definitions. Let ρ be the density of air and ρ_v the density of water vapour in the air. Let T be the air temperature. The gas constant for water vapor is $R_v = R^*/m_v = 462$ J kg^{-1} K^{-1}, with $R^* = 8.3144621$ J mol^{-1} K^{-1} the universal gas constant and $m_v = 18$ g mol^{-1} the molar mass of water. The latent heat of condensation of water is $L_v = 2.5 \times 10^6$ J / kg. The specific humidity q, which is the actual mass concentration of water vapor in the air, the vapor pressure e, and the saturation vapor pressure e^* are given by

$$q = \frac{\rho_v}{\rho},$$

$$e = \frac{\rho_v R^* T}{m_v},$$

$$e^* = e_0 \exp\left(\frac{L_v}{R_v}({T_0}^{-1} - T^{-1})\right). \tag{1}$$

For temperatures well below the critical temperature, the Clausius-Clapeyron relation takes the above exponential form and determines e^* as a function of T, with e_0 and T_0 empirical constants. This is the curve between the Triple Point and the Critical Point in the phase diagram (Figure 1). The above formulas treat L_v as constant, but it is, in fact, a weak function of temperature; correspondingly, there are more accurate semi-empirical formulas for e^*, but the above will suffice here. An air parcel holding water vapor becomes saturated when $e > e^*$, i.e. when it crosses this curve. This can occur by increasing its vapor pressure (by increasing q such as through sea spray, or by increasing the total pressure) or by decreasing T so as to decrease e^*. From the ideal gas law $p = \rho R^* T/\bar{m}$, with $\bar{m}$ the mean molecular weight of air (which changes very little between dry and saturated air, owing to the overwhelming concentration of N_2 and O_2) we may write

$$e = \rho_v \frac{R^* T}{m_v} = qp \frac{\bar{m}}{m_v}.$$

A subsaturated air parcel that ascends adiabatically (no heat content change) and reversibly (no change in q) will experience a lower p as well as T, such that both e and e^* decrease. While e decreases nearly perfectly exponentially with height (through $p \approx p_0 \exp(-z/H)$ with H the scale height and p_0 the surface pressure), e^* decreases as the exponential of a term $(-T^{-1})$ that itself decreases with height: $\partial_z \left(-T^{-1}\right) = \Gamma_D T^{-2}$ with Γ_d the dry lapse rate. Thus e^* decreases with height faster than e, so condensation routinely occurs by lifting a moist air parcel.

Cloud physicists realized long ago the difficulty of rain droplet formation. Forming a water droplet without a condensation nucleus, purely by condensation (homogeneous nucleation) is possible but requires supersaturation of the air by 10% or more, or equivalently (by Clausius-Clapeyron) supercooling of the air down to about -40°C. This supersaturation is required to overcome the tendency of small droplets to evaporate: water molecules on the edge of a small droplet, which has a small radius of curvature, are more exposed and separated from their neighbouring molecules; hence they experience weaker cohesive forces (surface tension), and will evaporate more easily. This is the Kelvin effect: the amount of supersaturation required for a droplet to be in condensation-evaporation equilibrium is larger for smaller droplets.

However, observations rarely show air in Earth's atmosphere to be supersaturated by more than about 1%. There had to be another process at work. Assistance comes from condensation nuclei (aerosols) which, by decreasing the number of liquid H_2O molecules at the interface (the edge of a droplet, in this case), reduce the tendency for evaporation. For a flat interface, this is known as the Raoult effect. In Earth's atmosphere, there are plenty of aerosols that catalyse this phase transition very well, so that it truly is a good approximation to assume that water vapor condenses when crossing this curve. However, simply condensing water vapor is not sufficient to form rain; droplets must grow in size until their terminal speed exceeds the updraft speed.

In a seminal paper, Köhler [1] transformed Raoult's law for a spherical interface (imitating a water droplet) and simultaneously added the Kelvin effect. Very small droplets—those for which the Kelvin effect inhibits condensational growth—that form on a condensation nuclei have a high particulate to water ratio. So by Raoult's effect the environment can be *sub*saturated and yet the droplet can maintain condensation-evaporation equilibrium. However, condensation will not grow such a droplet: if it grows the required vapor pressure for equilibrium is raised (Raoult's effect) and so, without adjusting the ambient vapor pres-

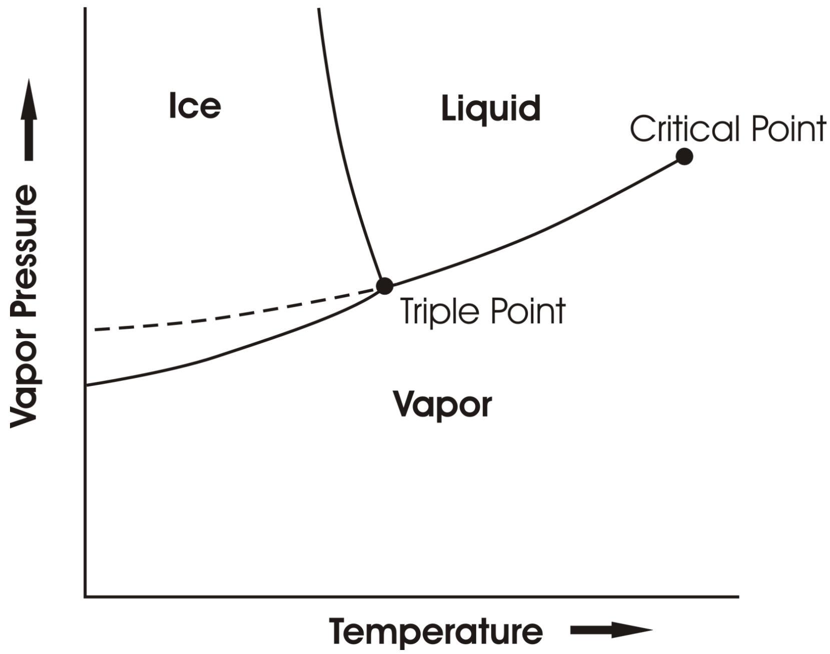

Figure 1: The phases of water dependent on temperature and pressure. At the triple point ($T = 0°$C and $p = 6.112$ hPa) all three phases can coexist in equilibrium. Beyond the critical point ($T > 647$ K and $p > 218$ atm, so not relevant for Earth's atmosphere), the liquid and gaseous phases become indistinguishable. Atmospheric aerosols that catalyse the condensation of water vapor to liquid droplets are plentiful in the atmosphere, so transitioning from vapor to liquid is just a matter of lowering the temperature or raising the vapor pressure to cross the phase transition line. However, these aerosols are not as well suited for catalysing deposition (vapor to ice). The dashed line indicates that in Earth's atmosphere, significant supersaturation or supercooling is required to form ice crystals from vapor.

sure, the droplet will evaporate back to its original size: a stable equilibrium (corresponding to positive slopes in Figure 2). When the ambient vapor pressure is above a critical supersaturation level, as Köhler's theory goes, Kelvin's effect begins to dominate: the edge of a growing water droplet becomes flatter and the vapor pressure required for equilibrium falls. If the ambient vapor pressure remains constant, an even larger vapor pressure differential drives more condensation onto the droplet: an unstable equilibrium (corresponding to negative slopes in Figure 2). The critical supersaturation, predicted by Köhler's theory for a given aerosol number density, is well below 1%; concerning observations, this theory of droplet growth was plausible.

Once the critical supersaturation has been passed and a water droplet has formed, one might expect that, since the saturation vapor pressure falls as the water droplet grows (from the Kelvin effect), that it will grow and grow without end—or until so large that it falls as rain. However, in the atmosphere CCN [cloud condenstation nuclei] are generally abundant, and are all competing for available water vapor, so none grow particularly large. Pure condensational growth does not produce water droplets large enough to fall as rain,

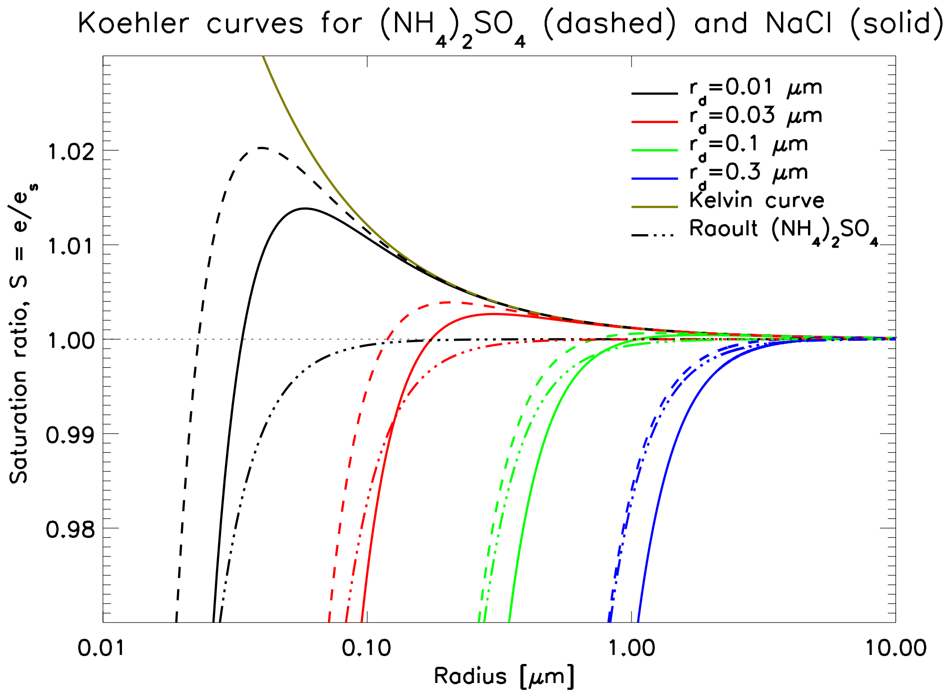

Figure 2: vapor pressure required for a droplets of different radii to be in condensation-evaporation equilibrium, as described by Köhler theory. The Kelvin effect (solid beige curve) increases the required vapor pressure with decreasing droplet size, due to an increasing radius of curvature. The Raoult effect (dash-dot curves) reduces the required vapor pressure with decreasing droplet size, due to higher aerosol concentration. Köhler theory (solid curves) combines these two effects, and accounts for both the type of aerosol (dashed vs. solid) and the size of the aerosol (colours). Figure from Ulrike Lohmann's lectures (IACETH); available at `http://www.iac.ethz.ch/edu/courses/bachelor/vertiefung/atmospheric_physics/Slides_2012/koehler.pdf`

and yet another process is required for rain formation.

Surely the collection of smaller droplets by larger droplets through collisions must be of some importance. However, not only does condensational growth produce droplets too small to rain out, those droplets are also roughly all the same size. That is, the distribution of droplet sizes narrows with time. Equal size droplets have the same buoyancy and drag forces and so will accelerate at the same rate (recall Stokes' flow past a sphere). All moving at similar velocities, then, they will not have much opportunity for collision. Turbulent air flows can force collisions of like-sized droplets; there is a significant body of literature on this topic but will not be pursued here. For the more laminar case, the efficiency of the collisional process for raindrop formation is a very strong function of not just the number density of droplets, but also of the droplet size distribution. A narrow distribution results in rare collisions. A variety of condensation nuclei with different sizes could initiate droplets with a distribution of sizes, but aerosols in continental regions are predominantly dust without much size variance. Again, another mechanism was needed to explain rain formation, and

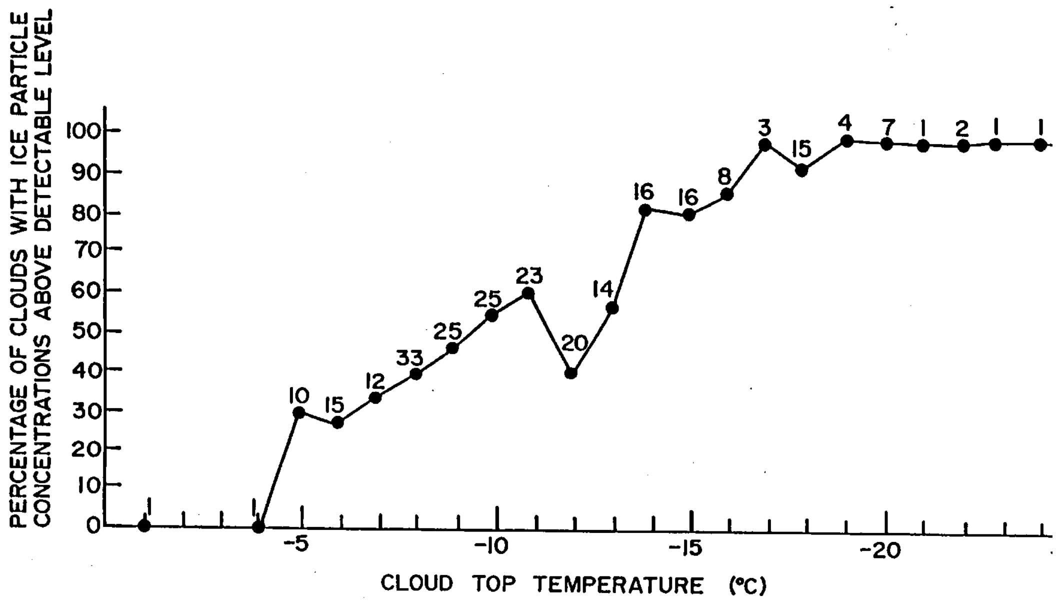

Figure 3: An observational study showing the fraction of clouds with ice in them as a function of cloud top temperature, being generally the coldest temperature in the cloud. The numbers indicate the sample number of clouds for each temperature. Above 0°C no ice exists, and below (for this study) −19°C all clouds contain ice.

it came from Bergeron in 1933.

There are two points to discuss before describing Bergeron's mechanism. First, we've said that atmospheric aerosols are well suited to catalyse water droplet formation; but those same aerosols are not so efficient for ice formation. Homogeneous nucleation of ice crystals from liquid water (i.e. in a perfectly clean atmosphere) also requires very low temperatures, around −40°C. Thus for ice formation, it is safe to assume water crystallizes below −40°C, and that none crystallizes above 0°C. But there is some complicated function (of temperature, and the aerosol population) that determines the fraction that crystallizes between these temperatures. Figure 3 shows this function from one study; the function will vary from study to study and is not meant to be taken as universal. The non-monotonicity found in this particular study indicates the potential difficulty in determining a more universal function. The relevant point for this discussion, however, is that this preference for liquid droplet formation over ice crystal formation leads to a large population of droplets but a small population of ice crystals in mixed phase clouds.

Second, deriving the Clausius-Clapeyron relation involved a thermodynamic study of the phase transition to water vapor from liquid water *or* from ice. Now, the latent heat of fusion for water is about one eighth of its latent heat of condensation[1], and so the latent heat

[1] The latent heat of fusion is often neglected in climate models, but is certainly non-negligible in detailed

of deposition (ice formation), being the sum of the latent heat of fusion and condensation, is larger than the latent heat for condensation alone. These different values for L go into (1), and this shows that the saturation vapor pressure over ice is always lower than that over liquid water. Thus for a given ambient vapor pressure between the two saturation vapor pressures, liquid water droplets evaporate and ice crystals grow by deposition.

Eleven years before presenting his theory Bergeron noted that a stratus cloud deck could extend down only to the canopy, not to the ground, when temperatures were below freezing. He proposed that a small ice crystal population could rapidly form large crystals—large enough to fall, and possibly melt back to rain—by scavenging water from a large population of water droplets. Given that temperatures below 0°C exist even in the tropics at high altitudes, this process had the potential to explain both mid-latitude and tropical rainfall.

Indeed, the Bergeron process does explain much rain formation, but not all. In the 1950s cloud physicists convened for a conference in Puerto Rico, where they observed intense rainfall from a cloud whose top was certainly warmer than 0°C. Indeed, in this case of a maritime climate, salt particles from the sea are plentiful in the atmosphere, giving a wide distribution of condensation nuclei sizes, and rain formation can occur via condensational growth and collision collection alone without any cold cloud microphysics.

We have seen how aerosols, and in particular their size distribution, play a critical role in rain drop formation, and thus in determining whether it will rain or not. We have also seen how cold cloud microphysics involving ice crystals can produce rain when the aerosol distribution does not particularly support rain drop formation. We have considered the condensational and collisional processes of rain droplet growth. It is important that we understand these fundamental processes of rain formation if we are to accurately predict the weather, as well as the climate.

3 Stability

In Lecture 3 we derived the entropy of a dry air parcel and considered adiabatic vertical displacements of this air parcel. Such a process is reversible so its entropy is conserved, and we found that the environment was stable (unstable) if the entropy increased (decreased) with height. Determining a stability criterion for moist air, however, turns out to be much more difficult. The difficulty arises because the density of moist air depends not just on its entropy and pressure, but also on the water content and the fraction of that water content that is vapor versus liquid.

We can define the entropy of moist air as a function of temperature, pressure, and specific humidity. This is well defined because, as discussed in the previous section, water vapor supersaturates very little. The moist air entropy s is that for dry air plus two additional terms (the last two, below):

$$s = c_p \ln \frac{T}{T_1} - R_d \ln \frac{p}{p_1} + \frac{L_v q}{T} - q R_v \ln \mathcal{H}. \tag{2}$$

Here $\mathcal{H} \equiv e/e^*$ is the relative humidity, c_p is the specific heat capacity for air at constant pressure, R_d (R_v) is the specific gas constant for dry air (water vapor), L_v is the latent heat

studies of cloud physics.

of condensation, and T_1 and p_1 are arbitrary constants (note that entropy is only relevant up to an additive constant). Let us also define the saturation entropy, which has $\mathcal{H} = 1$ so $\ln \mathcal{H} = 0$, giving

$$s^* = c_p \ln \frac{T}{T_1} - R_d \ln \frac{p}{p_1} + \frac{L_v q^*}{T}. \tag{3}$$

In addition to adding latter two terms, water content changes s by modifying the molar mass of the air mixture: hence the specific gas constant used should technically be a function of q. Similarly, the heat capacity of air is modified by water vapor, and even more strongly by condensed water: c_p should be a function of q. These modifications should be accounted for in numerical weather prediction models, but since these changes are more of a numerical nature than conceptual, we shall treat R_d and c_p as constants here. We have also chosen to neglect any ice processes, because of the difficulty in knowing at what temperature ice formation occurs (recall Figure 3). These, however, are the more minor difficulties.

The major difficulty in defining the stability of moist air comes from the additional dependence of density on the water content. The specific volume α is now a function of three variables: entropy and pressure, as well as a modification due to water. This makes it impossible to construct a 2D thermodynamic diagram (with axes of pressure and entropy) that classifies the stability of an atmospheric column. That is, there is no simple stability criterion based on $\partial s/\partial p$, as there was for dry air. An extra dimension, the water content, would be needed.

However, it is possible to define a modified entropy, s', that, to good approximation, solves this problem. Let us define $q_t = (\rho_v + \rho_l)/\rho$ to be the total mass fraction of water content in air, whether from water vapor (ρ_v) or liquid droplets (ρ_l). This q_t is conserved in a reversible vertical displacement, even under condensation or evaporation. Thus an arbitrary function of q_t is also conserved in a reversible process. It is possible to choose this function carefully such that, when added to s to define s', we get the specific volume as a function of just two variables: $\alpha \approx \alpha(p, s^{*\prime})$.

With this caveat, we may now consider the 2D thermodynamic diagram shown in Figure 4. Consider an adiabatic vertical displacement of an air parcel originating from the ground with (modified) entropy s'_0. If this is reversible, then s and s' are conserved. We need to know about the specific volume α of our parcel, and of the environment, to determine stability, and by the argument above we can get this from p and s'. Suppose that we know, from other thermodynamic information not shown in the figure, where the parcel will condense. This is the Lifting Condensation Level (LCL), above which the air parcel is saturated. Comparing its (modified) entropy (the rightmost vertical line) to the (modified) saturation entropy of the environment (dashed line) will tell us, through $\alpha(p, s^{*\prime})$, its buoyancy. Where $s'_0 > s^{*\prime}$ the air parcel will be positively buoyant: an unstable situation. Where $s'_0 < s^{*\prime}$ it is negatively buoyant and stable.

However, the moist atmosphere can exhibit meta-stability: it can be stable to small vertical displacements but unstable to large vertical displacements. An air parcel lifted from the ground to anywhere below the bottom intersection of the vertical s'_0 line and the $s^{*\prime}$ curve, called the Level of Free Convection (LFC), will be negatively buoyant and hence stable. But lift that same air parcel even slightly above the LFC and it will be positively buoyant, hence unstable. It will rise until it reaches the top intersection of the vertical s'_0 line and the $s^{*\prime}$ curve, called the Level of Neutral Buoyancy (LNB). There is no analogue

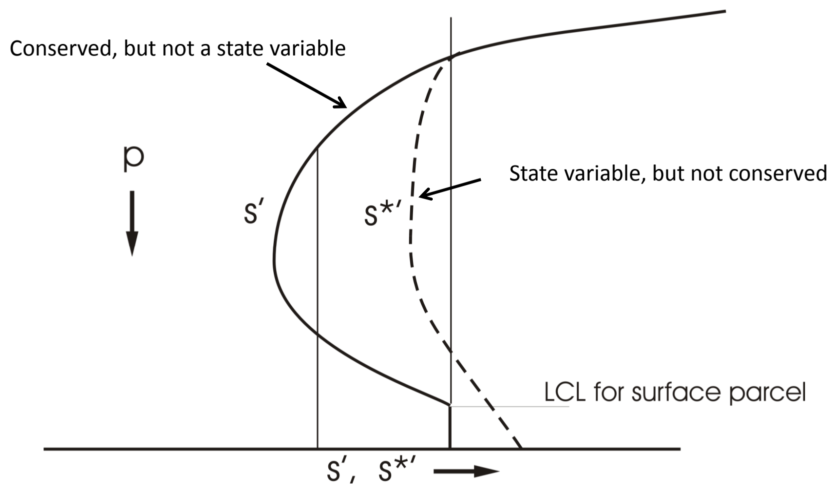

Figure 4: A schematic illustration of moist meta-stability. A typical environmental profile for s' is shown (thick solid line). In the lower and subsaturated air, turbulent mixing or convection has brought the environment to the dry adiabatic lapse rate, equivalent to a constant s'. Above the Lifting Condensation Level (LCL) water vapor condenses, q decreases in (2) and s' decreases. Higher up the decreasing pressure begins to dominate and s' begins to increase with decreasing p. Near the top of the diagram s' increases even more rapidly with decreasing p due to increasing T in the stratosphere. For full details, see the text.

of this in a homogeneous (e.g. dry) atmosphere.

Basic thermodynamics tells us that the heat added during a cyclic process is $\oint \mathrm{d}Q = \oint T\ \mathrm{d}s$. Thus if we transformed Figure 4 to have T as its vertical coordinate, the area between s'_0 and $s^{*\prime}$ curves is a measure of energy. Specifically, the area where $s'_0 > s^{*\prime}$ (above the LFC and below the LNB) is the the amount of potential energy available for convective release, termed the Convective Available Potential Energy (CAPE). Similarly, the area where $s'_0 < s^{*\prime}$ (below the LFC) is the amount of energy required for the air parcel to overcome the stable stratification and rise above the LFC. This is termed the Convective Inhibition (CIN).

Large-scale radiative cooling above the LFC lowers the entropy, thereby building up a reservoir of CAPE. Meanwhile CIN can be built up by large-scale subsidence and/or surface cooling (e.g. at night). CIN can be thought of as a potential barrier, and it is crucial to the meta-stability exhibited by moist air: without it, convection would immediately consume any CAPE, as it does in a dry atmosphere. During the day, surface heating raises the surface entropy which can reduce the CIN, or localized processes can add enough energy to overcome the CIN for a particular air parcel. When this occurs, air rises above the LFC and begins to tap the (potentially large) reservoir of CAPE. This can create extremely powerful thunderstorms, through a somewhat rare phenomenon called supercell convection, as well as tornadoes and hailstorms. The meteorological effects of CAPE, however, are beyond the scope of this lecture.

In the meteorological community there are two schools of thought. The first holds that CAPE is built up and released repeatedly during large events. The second holds that moist

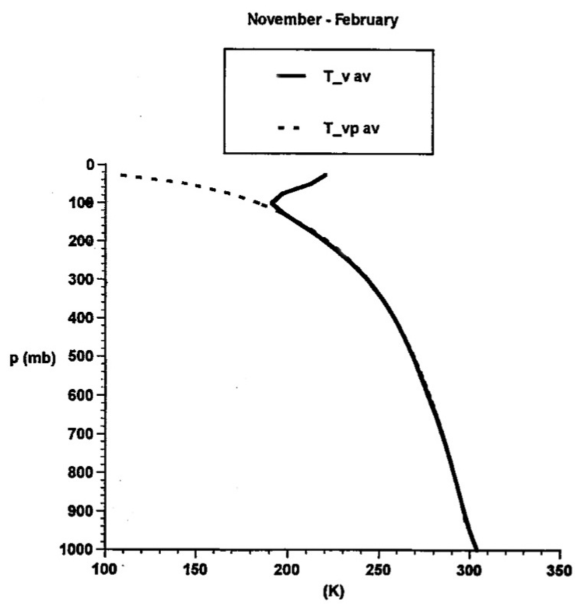

Figure 5: A typical tropical sounding. The solid line is a result of averaged measurements and the dashed line is a moist adiabat. The region where the slope changes sign is the tropopause.

convection is a process in quasi-statistical equilibrium (like dry convection), except in a few cases over continents.

Note that if we lift a different air parcel (i.e. from elsewhere in the column than the ground), then its LCL will be at a different pressure level, and its stability properties will also differ. Thus to fully characterize the stability of a moist water column, one must take air parcels from all pressure levels and subject them to all possible vertical displacements, large or small, which we shall explore in the next section.

4 Radiative moist convective equilibrium

The moist adiabatic lapse rate (s^*) is a function of pressure and temperature, though in the stratosphere where water vapor is scarce it is quite near to the (constant) dry adiabatic lapse rate. This is evident from a typical sounding data shown in Figure 5.

Figure 6 depicts the pressure variation experienced by fluid parcels that are raised from a range of starting pressure values. As seen in the figure, a parcel lifted from 900 hPa never becomes positively buoyant, because it has much less moisture to start with. For a parcel starting around 940 hPa it is striking that neutral buoyancy appears throughout the whole troposphere.

The process of forming a cumulus cloud is sketched in Figure 7. Air begins to rise and starts to form a cumulus cloud. Due to appreciable coalescence time, the precipitation formation occurs slowly as the air mass rises. A part of the precipitation falls outside the envelope during the mature stage. Re-evaporation can happen in this time, leading to

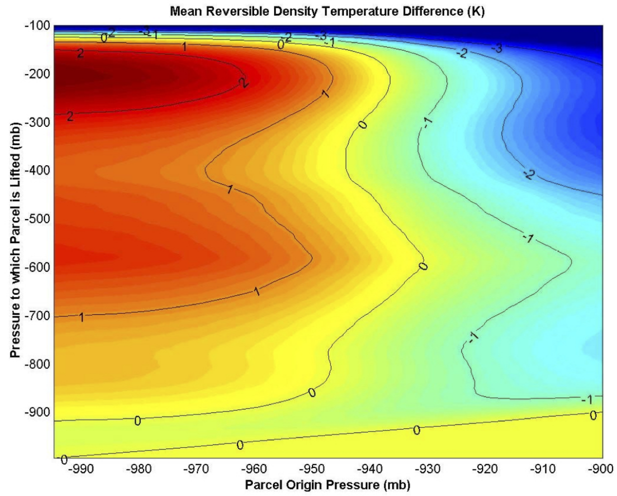

Figure 6: A plot showing the stability of air parcels being elevated. Red color indicates positive buoyancy and blue indicates negative. Note that a sample starting around 900 mb is positively buoyant, neutrally buoyant if it starts around 940 mb and negatively buoyant around 900 mb

chilling of surrounding air. In the final stage, a density current appears in the atmosphere, leading to 'raining' of low entropy air which in the process blocks the rise of high entropy air from the lower atmosphere. This leads to the destruction of the plume that formed, its life cycle being of the order of an hour.

Consider now a moist RCE situation where homogeneous conditions are set in the horizontal plane, i.e. constant ocean surface temperature, constant solar radiation along the horizontal and so on. On letting moist convection occur in such a system, interesting things happen. A cartoon representation of this is depicted in Figure 8, where a cross section is indicated. In spite of a spectrum of clouds appearing in the real world, only two are indicated here for simplicity. Air ascends through the deep clouds, precipitating in the process as rain. The air falling after reaching the top of the cloud is depleted of condensed water and is under saturated. The descent is therefore dry and hence the air is clear, as is seen in the figure between the clouds. The asymmetry in ascend and descending parts is interesting. There are strong updrafts in localized regions, which are widely spaced. Larger areas of clear air appear surrounding them, where the air descends at speeds roughly two orders of magnitude less than the ascending speeds. While the ascend can be on the order of a few m/s while the descends are usually of the order of a few cm/s.

Temperature variations with altitude are indicated in Figure 9 which were obtained from different models, holding relative humidity fixed. The calculations were carried out for pure radiative, dry radiative-convective and moist radiative-convective cases. Note that the height of the tropopause increases with these models. The RCE model fits well as a zeroth order approximation for a real data plotted at a typical station.

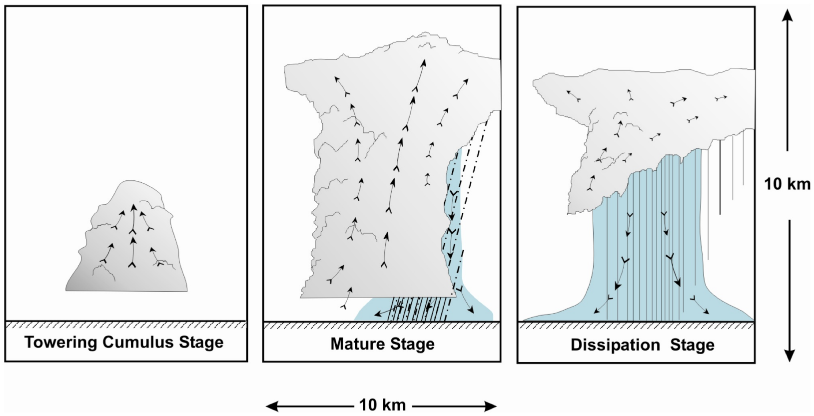

Figure 7: The formation of a cumulus cloud and its dissipation

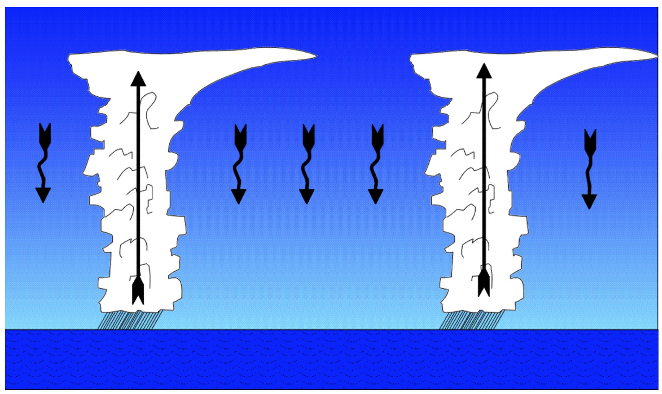

Figure 8: Precipitating widely spaced clouds

References

[1] Köhler, H., 1936. *The nucleus in and the growth of hygroscopic droplets.* Trans. Faraday Soc., 32, 11521161.

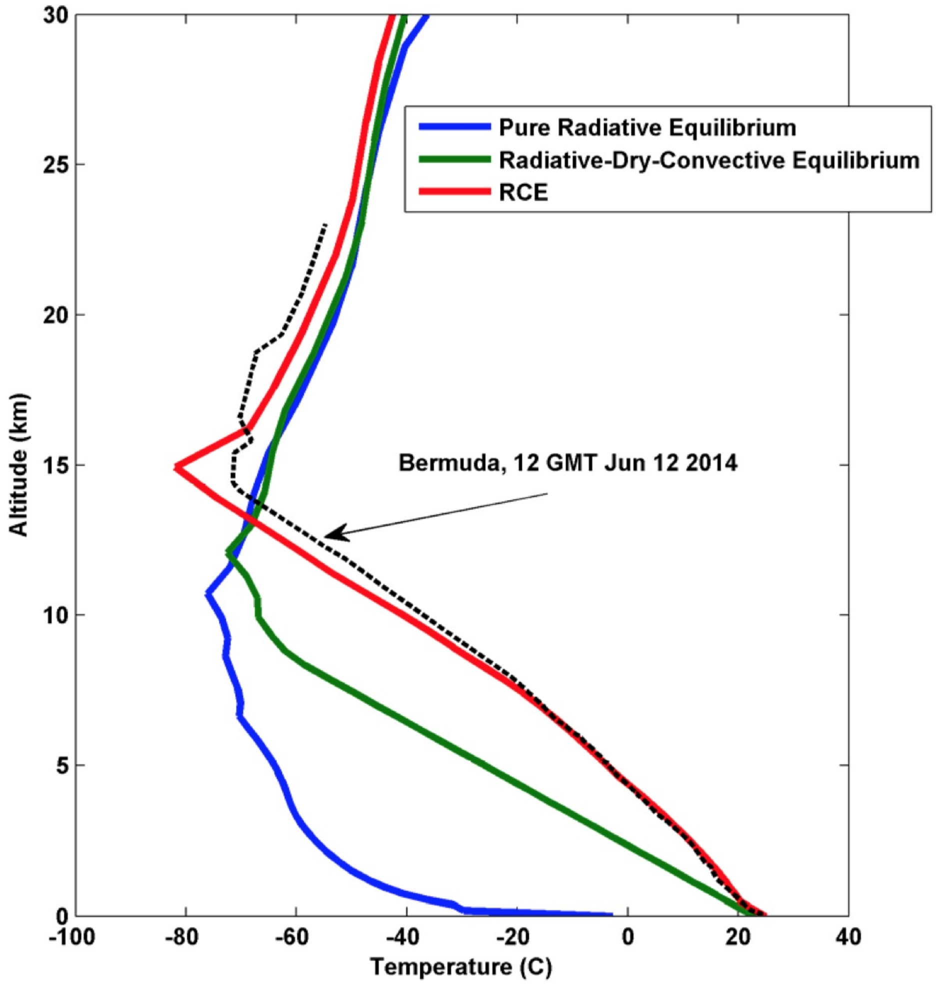

Figure 9: A comparison between different models predicting temperature as a function of altitude.