

# Lecture 8: Horizontal Convection

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## 1 The Ocean: Constraints on a Thermal Circulation

We now turn our attention to the ocean. Our goal in this lecture and the next is to describe a theory for the deep circulation of the ocean, sometimes called the meridional overturning circulation (MOC) and occasionally the thermohaline circulation. We begin in this lecture by showing that there have to be winds or some other form of mechanical forcing in order to drive a substantial deep ocean circulation. The root effect goes back to Sandström, and although his rigour was suspect it seems his intuition was right.

## 2 Sandström's Effect

We first give an argument that is similar in spirit to the one that Sandström gave in his original papers (Sandström, 1908, 1916).

### 2.1 Maintaining a steady baroclinic circulation

The Boussinesq equations are

$$\frac{D\mathbf{v}}{Dt} = -\nabla\phi + b\mathbf{k} + \mathbf{F}, \quad \frac{Db}{Dt} = \dot{Q}, \quad \nabla \cdot \mathbf{v} = 0, \quad (1a,b,c)$$

where  $F$  represents frictional terms and  $\dot{Q} = J + \kappa\nabla^2 b$  (that is, the heating term here includes the effects of diffusion). The circulation,  $C$ , changes according to

$$\begin{aligned} \frac{DC}{Dt} &= \frac{D}{Dt} \oint \mathbf{v} \cdot d\mathbf{r} = \oint \left( \frac{D\mathbf{v}}{Dt} \cdot d\mathbf{r} + \mathbf{v} \cdot d\mathbf{v} \right) \\ &= \oint b\mathbf{k} \cdot d\mathbf{r} + \oint \mathbf{F} \cdot d\mathbf{r}, \end{aligned} \quad (2)$$

(Note that rotation does no work.) Furthermore, we can write rate of change of circulation itself as

$$\frac{DC}{Dt} = \oint \left( \frac{\partial\mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla\mathbf{v} \right) \cdot d\mathbf{r} = \oint \left( \frac{\partial\mathbf{v}}{\partial t} + \omega \times \mathbf{v} \right) \cdot d\mathbf{r}. \quad (3)$$

Let us assume the flow is steady, so that  $\partial\mathbf{v}/\partial t$  vanishes. Let us further choose the path of integration to be a streamline, which, since the flow is steady, is also a parcel trajectory.

The second term on the right-most expression of (3) then also vanishes and (2) becomes

$$\oint b \, dz = - \oint \mathbf{F} \cdot d\mathbf{r} = - \oint \frac{\mathbf{F}}{|\mathbf{v}|} \cdot \mathbf{v} \, dr, \quad (4)$$

where the last equality follows because the path is everywhere parallel to the velocity. Let us now assume that the friction retards the flow, and that  $\oint \mathbf{F} \cdot \mathbf{v} / |\mathbf{v}| \, dr < 0$ . (One form of friction that has this property is linear drag,  $\mathbf{F} = -C\mathbf{v}$  where  $C$  is a constant. The property is similar to, but not the same as, the property that the friction dissipates kinetic energy over the circuit.) Making this assumption, if we integrate the term on the left-hand side by parts we obtain

$$\oint z \, db < 0. \quad (5)$$

Now, because the integration circuit in (6) is a fluid trajectory, the change in buoyancy  $db$  is proportional to the heating of a fluid element as it travels the circuit  $db = \dot{Q} \, dt = dQ$ , where the heating includes diffusive effects.

$$\oint z \, dQ < 0. \quad (6)$$

Thus, the inequality implies that the net heating must be negatively correlated with height: that is, *the heating must occur, on average, at a lower level than the cooling in order that a steady circulation may be maintained against the retarding effects of friction.*

### A compressible fluid

A similar result can be obtained for a compressible fluid. We write the baroclinic circulation theorem as

$$\frac{DC}{Dt} = \oint p \, d\alpha + \oint \mathbf{F} \cdot d\mathbf{r} = \oint T \, d\eta + \oint \mathbf{F} \cdot d\mathbf{r}, \quad (7)$$

where  $\eta$  is the specific entropy. Then, by precisely the same arguments as led to (6), we are led to the requirements that

$$\oint T \, d\eta > 0 \quad \text{or equivalently} \quad \oint p \, d\alpha > 0. \quad (8a,b)$$

Equation (8a) means that parcels must gain entropy at high temperatures and lose entropy at low temperatures; similarly, from (6b), a parcel must expand ( $d\alpha > 0$ ) at high pressures and contract at low pressures.

For an ideal gas we can put these statements into a form analogous to (6) by noting that  $d\eta = c_p(d\theta/\theta)$ , where  $\theta$  is potential temperature, and using the definition of potential temperature for an ideal gas. With these we have

$$\oint T \, d\eta = \oint c_p \frac{T}{\theta} \, d\theta = \oint c_p \left( \frac{p}{p_R} \right)^\kappa \, d\theta, \quad (9)$$

and (8a) becomes

$$\oint c_p \left( \frac{p}{p_R} \right)^\kappa \, d\theta > 0. \quad (10)$$

Because the path of integration is a fluid trajectory,  $d\theta$  is proportional to the heating of a fluid element. Thus [and analogous to the Boussinesq result (6)], (10) implies that *the heating (the potential temperature increase) must occur at a higher pressure than the cooling in order that a steady circulation may be maintained against the retarding effects of friction.*

These results may be understood by noting that the heating must occur at a higher pressure than the cooling in order that work may be done, the work being necessary to convert potential energy into kinetic energy to maintain a circulation against friction. More informally, if the heating is below the cooling, then the heated fluid will expand and become buoyant and rise, and a steady circulation between heat source and heat sink can readily be imagined. On the other hand, if the heating is above the cooling there is no obvious pathway between source and sink.

## 2.2 A rigorous result

Following Paparella & Young (2002), we now show more rigorously that, if the diffusivity is small, the circulation is in a certain sense weak. Now including molecular viscosity and diffusivity, the equations of motion are

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{f} + 2\boldsymbol{\omega}) \times \mathbf{v} = -\nabla B + b\mathbf{k} + \nu \nabla^2 \mathbf{v}, \quad (11a)$$

$$\frac{Db}{Dt} = \frac{\partial b}{\partial t} + \dot{\div}(b\mathbf{v}) = \dot{Q} = J + \kappa \nabla^2 b, \quad (11b)$$

$$\dot{\div} \mathbf{v} = 0, \quad (11c)$$

Multiply the momentum equation by  $\mathbf{v}$  and integrate over a volume to give

$$\frac{d}{dt} \left\langle \frac{1}{2} \mathbf{v}^2 \right\rangle = \langle wb \rangle - \varepsilon, \quad (12)$$

where angle brackets denote a volume average and

$$\varepsilon = -\nu \langle \mathbf{v} \cdot \nabla^2 \mathbf{v} \rangle = \nu \langle \omega^2 \rangle, \quad (13)$$

after integrating by parts. The dissipation,  $\varepsilon$ , is a positive definite quantity.

Write the buoyancy equation as

$$\frac{Dbz}{Dt} = z \frac{Db}{Dt} + b \frac{Dz}{Dt} = z\dot{Q} + bw, \quad (14)$$

whence

$$\frac{d}{dt} \langle bz \rangle = \langle z\dot{Q} \rangle + \langle bw \rangle. \quad (15)$$

Subtracting (15) from (12) gives the energy equation

$$\frac{d}{dt} \left\langle \frac{1}{2} \mathbf{v}^2 - bz \right\rangle = -\langle z\dot{Q} \rangle - \varepsilon. \quad (16)$$

In a steady state:

$$\langle z\dot{Q} \rangle = -\varepsilon < 0. \quad (17)$$

This is an analogue of our earlier results. It says that if we want to have a dissipative, statistically steady flow there has to be a negative correlation between heating and  $z$ . Put simply, the heating has to be below the cooling. But note that the heating and cooling include the diffusive terms.

### With diffusion only

Take  $\dot{Q} = \kappa \nabla^2 b$  whence (17) becomes

$$\frac{\kappa}{V} \int z \nabla^2 b \, dV = -\varepsilon \quad (18)$$

The horizontal part of the integral vanishes so that

$$\frac{\kappa}{H} \int_{-H}^0 z \frac{\partial^2 \bar{b}}{\partial z^2} \, dz = -\varepsilon. \quad (19)$$

where an overbar is a horizontal average. Integrating the LHS by parts gives

$$\frac{\kappa}{H} [\bar{b}(0) - \bar{b}(-H)] = \varepsilon. \quad (20)$$

The LHS is bounded by the surface buoyancy gradient, so the KE dissipation goes to zero as  $\kappa \rightarrow 0$ .

The result is (at least from a physicist's point of view) quite rigorous. It can also be extended to a nonlinear equation of state (Nycander, 2010). It tells us that the dissipation of kinetic energy in a fluid diminishes with the diffusivity, and that if  $\kappa = 0$  then dissipation vanishes. It doesn't say there is no flow at all, but it is hard to envision a flow in a finite domain that does not dissipate kinetic energy. The result is often characterized as saying that the flow is *non turbulent*.

## 3 Buoyancy and Mixing Driven Scaling Theories

Now we talk about scaling, becoming a bit less rigorous. Interestingly the scaling, dating from Rossby (1965), predates the rigorous theories, and it also provides much stronger bounds. However, it is a scaling and not a rigorous result and therefore open to dispute.

### 3.1 Equations of motion

A non-rotating Boussinesq fluid heated and cooled from above obeys the equations.

$$\frac{D\mathbf{v}}{Dt} = -\nabla\phi + \nu\nabla^2\mathbf{v} + b\mathbf{k}, \quad (21)$$

$$\frac{Db}{Dt} = \kappa\nabla^2 b \quad (22)$$

$$\nabla \cdot \mathbf{v} = 0. \quad (23)$$

with boundary conditions

$$b(x, y, 0, t) = g(x, y), \quad (24)$$

For algebraic simplicity consider the two-dimensional version of these equations, in  $y$  and  $z$ . We can define a streamfunction

$$v = -\frac{\partial\psi}{\partial z}, \quad w = \frac{\partial\psi}{\partial y}, \quad \zeta = \nabla_x^2\psi = \left(\frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right), \quad (25)$$

Taking the curl of the momentum equation gives

$$\frac{\partial\nabla^2\psi}{\partial t} + J(\psi, \nabla^2\psi) = \frac{\partial b}{\partial y} + \nu\nabla^4\psi \quad (26a)$$

$$\frac{\partial b}{\partial t} + J(\psi, b) = \kappa\nabla^2b \quad (26b)$$

where  $J(a, b) \equiv (\partial_y a)(\partial_z b) - (\partial_z a)(\partial_y b)$ .

### Non-dimensionalization and scaling

We non-dimensionalize (26) by formally setting

$$b = \Delta b \hat{b}, \quad \psi = \Psi \hat{\psi}, \quad y = L \hat{y}, \quad z = H \hat{z}, \quad t = \frac{LH}{\Psi} \hat{t}, \quad (27)$$

where the hatted variables are non-dimensional,  $\Delta b$  is the temperature difference across the surface,  $L$  is the horizontal size of the domain, and  $\Psi$ , and ultimately the vertical scale  $H$ , are to be determined. Substituting (27) into (26) gives

$$\frac{\partial\widehat{\nabla}^2\hat{\psi}}{\partial\hat{t}} + \widehat{J}(\hat{\psi}, \nabla^2\hat{\psi}) = \frac{H^3\Delta b}{\Psi^2} \frac{\partial\hat{b}}{\partial\hat{y}} + \frac{\nu L}{\Psi H} \widehat{\nabla}^4\hat{\psi}, \quad (28a)$$

$$\frac{\partial\hat{b}}{\partial\hat{t}} + \widehat{J}(\hat{\psi}, \hat{b}) = \frac{\kappa L}{\Psi H} \widehat{\nabla}^2\hat{b}, \quad (28b)$$

where  $\widehat{\nabla}^2 = (H/L)^2\partial^2/\partial\hat{y}^2 + \partial^2/\partial\hat{z}^2$  and the Jacobian operator is similarly non-dimensional. If we now use (28b) to choose  $\Psi$  as

$$\Psi = \frac{\kappa L}{H}, \quad (29)$$

so that  $t = H^2\hat{t}/\kappa$ , then (28) becomes

$$\frac{\partial\widehat{\nabla}^2\hat{\psi}}{\partial\hat{t}} + \widehat{J}(\hat{\psi}, \widehat{\nabla}^2\hat{\psi}) = Ra \sigma \alpha^5 \frac{\partial\hat{b}}{\partial\hat{y}} + \sigma \widehat{\nabla}^4\hat{\psi}, \quad (30)$$

$$\frac{\partial\hat{b}}{\partial\hat{t}} + \widehat{J}(\hat{\psi}, \hat{b}) = \widehat{\nabla}^2\hat{b}, \quad (31)$$

and the non-dimensional parameters that govern the behaviour of the system are

$$Ra = \left(\frac{\Delta b L^3}{\nu \kappa}\right), \quad (\text{the Rayleigh number}), \quad (32a)$$

$$\sigma = \frac{\nu}{\kappa}, \quad (\text{the Prandtl number}), \quad (32b)$$

$$\alpha = \frac{H}{L}, \quad (\text{the aspect ratio}). \quad (32c)$$

## 3.2 Rossby's Scaling

For steady non-turbulent flows, and also perhaps for statistically steady flows, then we can demand that the buoyancy term in (30) is  $\mathcal{O}(1)$ . If it is smaller then the flow is not buoyancy driven, and if it is larger there is nothing to balance it. Our demand can be satisfied only if the vertical scale of the motion adjusts appropriately and, for  $\sigma = \mathcal{O}(1)$ , this suggests the scalings

$$H = L\sigma^{-1/5} Ra^{-1/5} = \left(\frac{\kappa^2 L^2}{\Delta b}\right)^{1/5}, \quad \Psi = Ra^{1/5} \sigma^{-4/5} \nu = (\kappa^3 L^3 \Delta b)^{1/5}. \quad (33a,b)$$

The vertical scale  $H$  arises as a consequence of the analysis, and the vertical size of the domain plays no direct role. [For  $\sigma \gg 1$  we might expect the nonlinear terms to be small and if the buoyancy term balances the viscous term in (30) the right-hand sides of (33) are multiplied by  $\sigma^{1/5}$  and  $\sigma^{-1/5}$ . For seawater,  $\sigma \approx 7$  using the molecular values of  $\kappa$  and  $\nu$ . If small scale turbulence exists, then the eddy viscosity will likely be similar to the eddy diffusivity and  $\sigma \approx 1$ .] Numerical experiments (Figs. 1 and 2, taken from Ilıcak & Vallis 2012) do provide some support for this scaling, and a few simple and robust points that have relevance to the real ocean emerge, as follows.

- Most of the box fills up with the densest available fluid, with a boundary layer in temperature near the surface required in order to satisfy the top boundary condition. The boundary gets thinner with decreasing diffusivity, consistent with (33). This is a diffusive prototype of the oceanic thermocline.
- The horizontal scale of the overturning circulation is large, being at or near the scale of the box.
- The downwelling regions (the regions of convection) are of smaller horizontal scale than the upwelling regions, especially as the Rayleigh number increases.

## 3.3 The importance of mechanical forcing

The above results do not, strictly speaking, prohibit there from being a thermal circulation, with fluid sinking at high latitudes and rising at low, even for zero diffusivity. However, in the absence of any mechanical forcing, this circulation must be laminar, even at high Rayleigh number, meaning that flow is not allowed to break in such a way that energy can be dissipated — a very severe constraint that most flows cannot satisfy. The scalings (33) further suggest that the magnitude of the circulation in fact scales (albeit nonlinearly) with the size molecular diffusivity, and if these scalings are correct the circulation will in fact diminish as  $\kappa \rightarrow 0$ . For small diffusivity, the solution most likely to be adopted by the fluid is for the flow to become confined to a very thin layer at the surface, with no abyssal motion at all, which is completely unrealistic vis-à-vis the observed ocean. Thus, the deep circulation of the ocean cannot be considered to be wholly forced by buoyancy gradients at the surface.

Suppose we add a mechanical forcing,  $\mathbf{F}$ , to the right-hand side of the momentum equation (11a); this might represent wind forcing at the surface, or tides. The kinetic

energy budget becomes

$$\varepsilon = \langle wb \rangle + \langle \mathbf{F} \cdot \mathbf{v} \rangle = H^{-1} \kappa [\bar{b}(0) - \bar{b}(-H)] + \langle \mathbf{F} \cdot \mathbf{v} \rangle. \quad (34)$$

In this case, even for  $\kappa = 0$ , there is a source of energy and therefore turbulence (i.e., a dissipative circulation) can be maintained. The turbulent motion at small scales then provides a mechanism of mixing and so can effectively generate an ‘eddy diffusivity’ of buoyancy. *Given* such an eddy diffusivity, wind forcing is no longer necessary for there to be an overturning circulation. Therefore, it is useful to think of mechanical forcing as having two distinct effects.

1. The wind provides a stress on the surface that may directly drive the large-scale circulation, including the overturning circulation.
2. Both tides and the wind provide a mechanical source of energy to the system that allows the flow to become turbulent and so provides a source for an eddy diffusivity and eddy viscosity.

In either case, we may conclude that the presence of mechanical forcing is necessary for there to be an overturning circulation in the world’s oceans of the kind observed.

## 4 The Relative Scale of Convective Plumes and Diffusive Upwelling

Why is the downwelling region narrower than the upwelling? The answer is that high Rayleigh number convection is much more efficient than diffusional upwelling, so that the convective buoyancy flux can match the diffusive flux only if the convective plumes cover a much smaller area than diffusion. (Tom Haine explained this to me.) Suppose that the basin is initially filled with water of an intermediate temperature, and that surface boundary conditions of a temperature decreasing linearly from low latitudes to high latitudes are imposed. The deep water will be convectively unstable, and convection at high latitudes (where the surface is coldest) will occur, quickly filling the abyss with dense water. After this initial adjustment, the deep, dense water at lower latitudes will be slowly warmed by diffusion, but at the same time surface forcing will maintain a cold high latitude surface, thus leading to high latitude convection. A steady state or statistically steady state is eventually reached with the deep water having a slightly higher potential density than the surface water at the highest latitudes, and so with continual convection, but convection that takes place only at the highest latitudes.

To see this more quantitatively consider the respective efficiencies of the convective heat flux and the diffusive heat flux. Consider an idealized re-arrangement of two parcels, initially with the heavier one on top as illustrated in Fig. 3. The potential energy released by the re-arrangement,  $\Delta P$  is given by

$$\Delta P = P_{\text{final}} - P_{\text{initial}} \quad (35)$$

$$= g [(\rho_1 z_2 + \rho_2 z_1) - (\rho_1 z_1 + \rho_2 z_2)] \quad (36)$$

$$= g(z_2 - z_1)(\rho_1 - \rho_2) = \rho_0 \Delta b \Delta z \quad (37)$$

where  $\Delta z = z_2 - z_1$  and  $\Delta b = g(\rho_1 - \rho_2)/\rho_0$ .

The kinetic energy gained by this re-arrangement,  $\Delta K$  is given by  $\Delta K = \rho_0 w^2$  and equating this to (35) gives

$$w^2 = -\Delta b \Delta z. \quad (38)$$

Note that if the heavier fluid is initially on top then  $\rho_2 > \rho_1$  and, as defined,  $\Delta b < 0$ . The vertical convective buoyancy flux per unit area,  $B_c$ , is given by  $B_c = w \Delta b$  and using (38) we find

$$B_c = (-\Delta b)^{3/2} (\Delta z)^{1/2}. \quad (39)$$

The upwards diffusive flux,  $B_d$ , per unit area is given by

$$B_d = \kappa \frac{\Delta b}{H} \quad (40)$$

where  $H$  is the thickness of the layer over which the flux occurs. In a steady state the total diffusive flux must equal the convective flux so that, from (39) and (40),

$$(-\Delta b)^{3/2} (\Delta z)^{1/2} \delta_A = \kappa \frac{\Delta b}{H}, \quad (41)$$

where  $\delta_A$  is the fractional area over which convection occurs. Thus If we set  $\Delta z = H$ , we get

$$\delta_A = \frac{\kappa}{(\Delta b)^{1/2} H^{3/2}} \quad (42)$$

This is a small number, although it is not quite right yet — we don't really know  $H$ . Let us use (33a), namely  $H = (\kappa^2 L^2 / \Delta b)^{1/5}$  then

$$\begin{aligned} \delta_A &= \frac{\kappa}{(\Delta b)^{1/2} (\kappa^2 L^2 / \Delta b)^{3/10}} \\ &= \left( \frac{\kappa^2}{\Delta b L^3} \right)^{1/5} = (Ra \sigma)^{-1/5}. \end{aligned} \quad (43)$$

For geophysically relevant situations this is a very small number, usually smaller than  $10^{-5}$ . Although the details of the above calculation may be questioned (for example, the use of the same buoyancy difference and vertical scale in the convection and the diffusion), the physical basis for the result is clear: for realistic choices of the diffusivity the convection is much more efficient than the diffusion and so will occur over a much smaller area. This result almost certainly transcends the limitations of its derivation.

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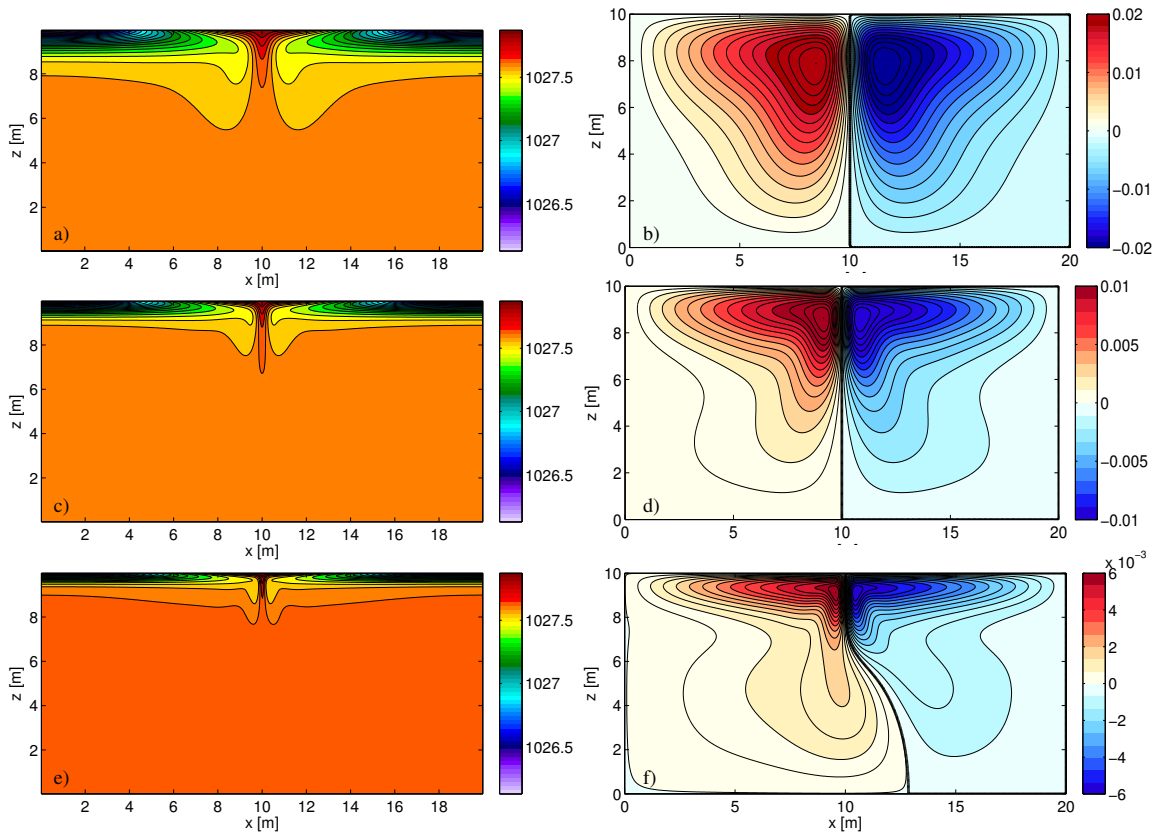


Figure 1: Temperature (left) and streamfunction (right) fields. From the top, the Rayleigh numbers are  $10^6$ ,  $10^7$ ,  $10^8$ .

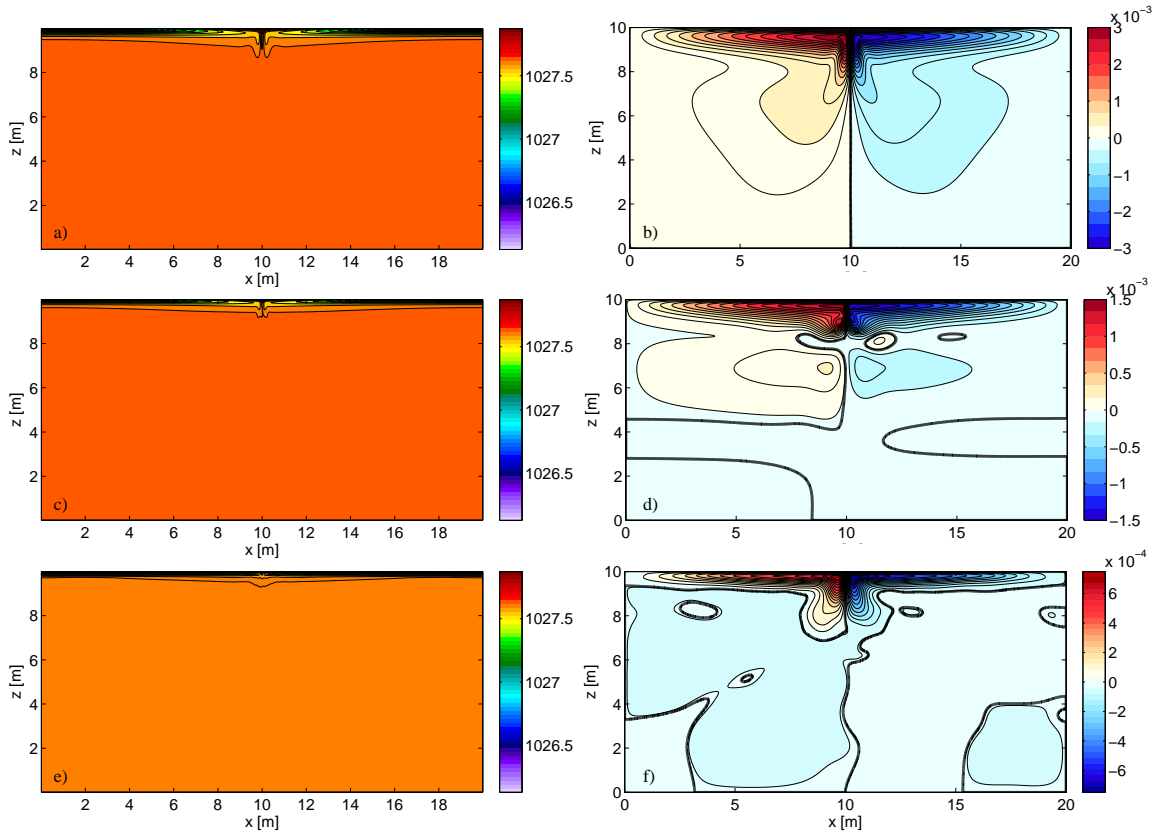


Figure 2: Temperature (left) and streamfunction (right) fields. From the top, the Rayleigh numbers are  $10^9$ ,  $10^{10}$ ,  $10^{11}$ .

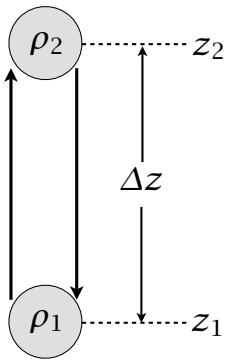


Figure 3: Two fluid parcels, of density  $\rho_1$  and  $\rho_2$  and initially at positions  $z_1$  and  $z_2$  respectively, are interchanged. If  $\rho_2 > \rho_1$  then the final potential energy is lower than the initial potential energy, with the difference being converted into kinetic energy.