Lecture 9: The Deep Ocean Circulation

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In this lecture we try to understand the processes that give rise to a deep meridional overturning circulation. We'll present a zonally-averaged model of the meridional overturning circulation of the ocean, following Nikurashin & Vallis (2011, 2012). It is a quantitative model, and might even be called a theory, depending on what one's definition of theory is.

1 A Model of the Wind-driven Overturning Circulation

The model is motivated by the plot of the stratification shown in Fig. 1, and the schematic of water mass properties of the Atlantic shown in Fig. 3. The following features are apparent.

1. Two main masses of water, known as North Atlantic Deep Water (NADW) and Antarctic Bottom Water (AABW). Both are interhemispheric. NADW appears to outcrop in high northern latitudes and high southern latitudes, and AABW just at high Southern.

2. Isopycnals are flat over most of the ocean, and slope with a fairly uniform slope in the Southern Ocean.

3. The circulation is along isopycnals in much of the interior. There is some water mass transformation between AABW and NADW, but most of it occurs near the surface.

2 A Theory for the MOC in a Single Hemisphere

Let us first imagine there is a wall at the equator, and make a model of the circulation in the Southern Hemisphere (Figs. 4, 5, 6); that is, essentially of AABW. The model will have the following features, or bugs if you are being critical.

1. Zonally averaged.

2. Simple geometry. A zonally re-entrant channel at high latitudes, with an enclosed basin between it and the equator.

3. We solve the equations of motion separately in the two regions and match the solutions at the boundary.

4. Mesoscale eddies are parameterized with a very simple down-gradient scheme.

5. There are no wind-driven gyres.

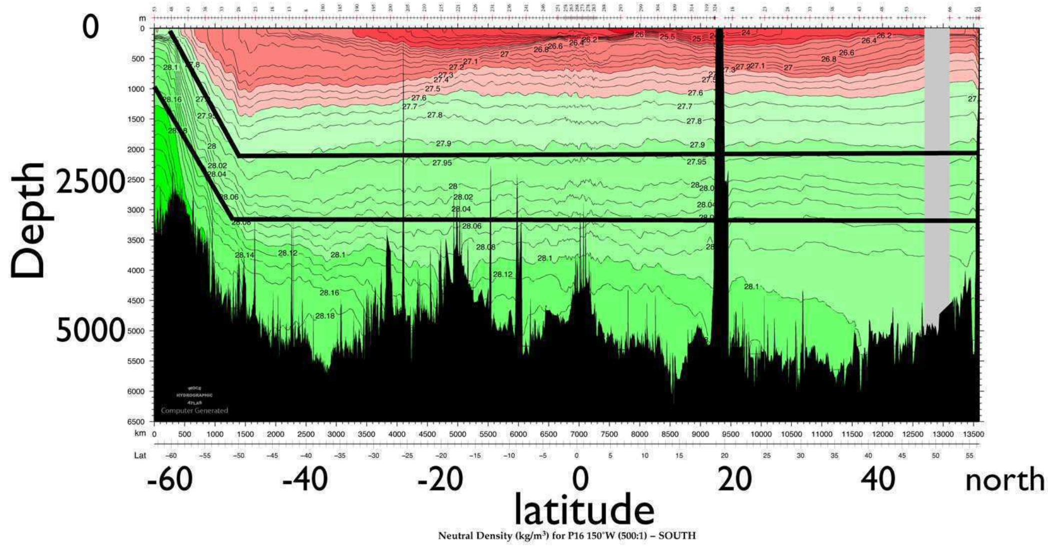

Figure 1: Stratification in the Pacific at 150° W

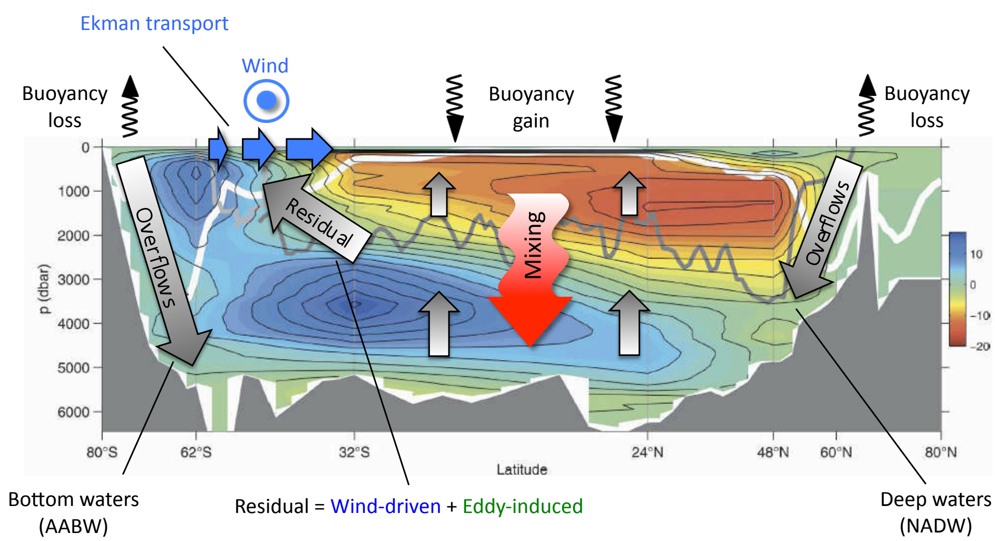

Figure 2: A schematic of deep ocean circulation.

2.1 Equations of motion

With quasi-geostrophic scaling the zonally-averaged zonal momentum and buoyancy equations are

$$\frac{\partial \overline{u}}{\partial t} - f_0\, \overline{v} = -\frac{\partial}{\partial y}\overline{u'v'} + \frac{\partial \tau}{\partial z}, \tag{1}$$

$$\frac{\partial \overline{b}}{\partial t} + N^2 \overline{w} = -\frac{\partial}{\partial y}\overline{v'b'} + \kappa_v \frac{\partial^2 \overline{b}}{\partial z^2}. \tag{2}$$

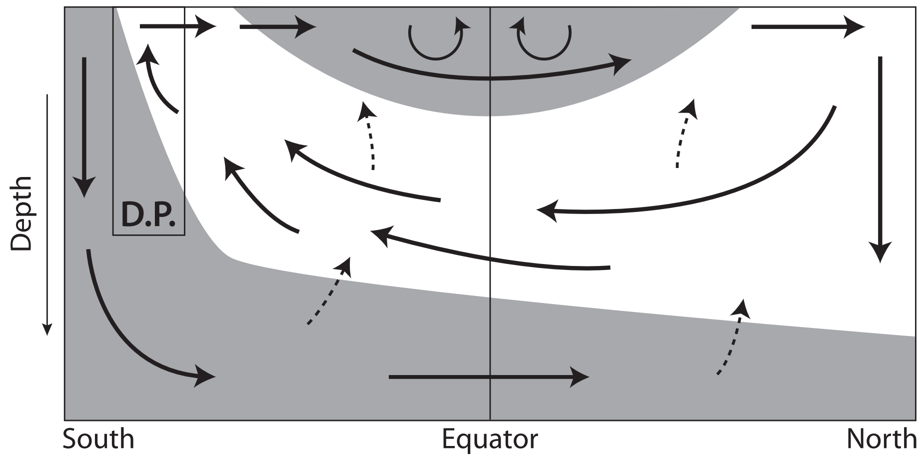

Figure 3: A simpler schematic of deep ocean circulation

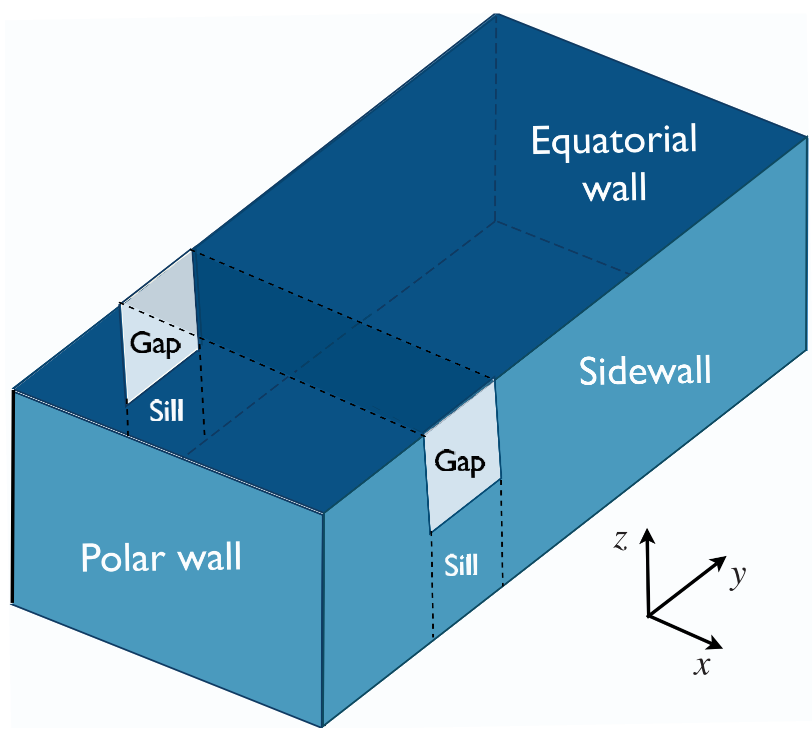

Figure 4: Idealized geometry of the Southern Ocean: a re-entrant channel, partially blocked by a sill, is embedded within a closed rectangular basin; thus, the channel has periodic boundary conditions. The channel is a crude model of the Antarctic Circumpolar Current, with the area over the sill analogous to the Drake Passage.

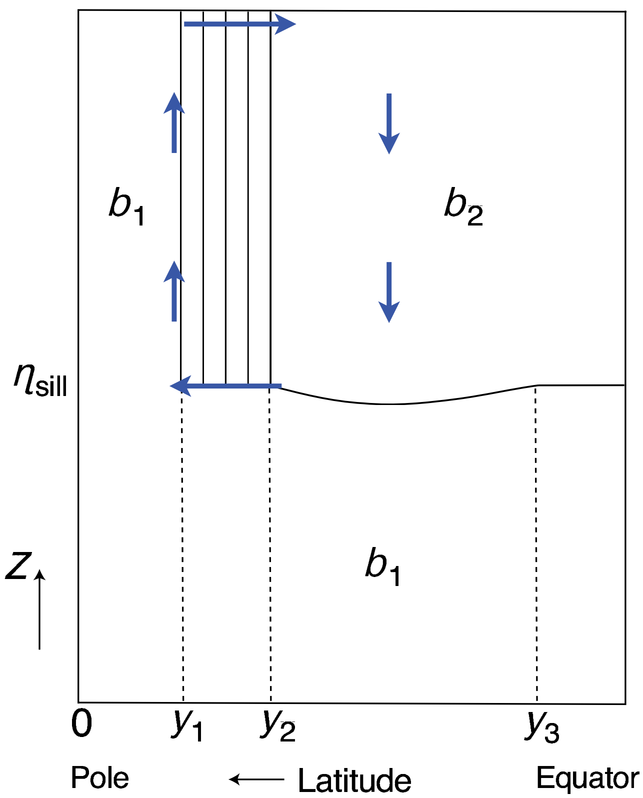

Figure 5: Cross-section of a structure of the single-hemisphere ocean model. There is a channel between y_1 and y_2. The arrows indicate the fluid flow driven by the equatorward Ekman transport in the channel, and the solid lines are isopycnals.

where b is buoyancy ('temperature') and $N^2 = \partial_z b_0$. To these we add the thermal wind relation and mass continuity:

$$f_0 \frac{\partial \overline{u}}{\partial z} = -\frac{\partial \overline{b}}{\partial y}, \qquad \frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial z} = 0.$$

Define a *residual flow* such that

$$\overline{v}^* = \overline{v} - \frac{\partial}{\partial z}\left(\frac{1}{N^2}\overline{v'b'}\right), \qquad \overline{w}^* = \overline{w} + \frac{\partial}{\partial y}\left(\frac{1}{N^2}\overline{v'b'}\right).$$

whence

$$\frac{\partial \overline{u}}{\partial t} - f_0\, \overline{v}^* = \overline{v'q'} + \frac{\partial \tau}{\partial z} \tag{3a}$$

$$\frac{\partial \overline{b}}{\partial t} + N^2 \overline{w}^* = \kappa_v \frac{\partial^2 \overline{b}}{\partial z^2}. \tag{3b}$$

These are the so-called transformed Eulerian mean (TEM) equations. The theory of them is extensive and suffice it to say here that $\overline{v}^*$ and $\overline{w}^*$ more nearly represent the trajectories

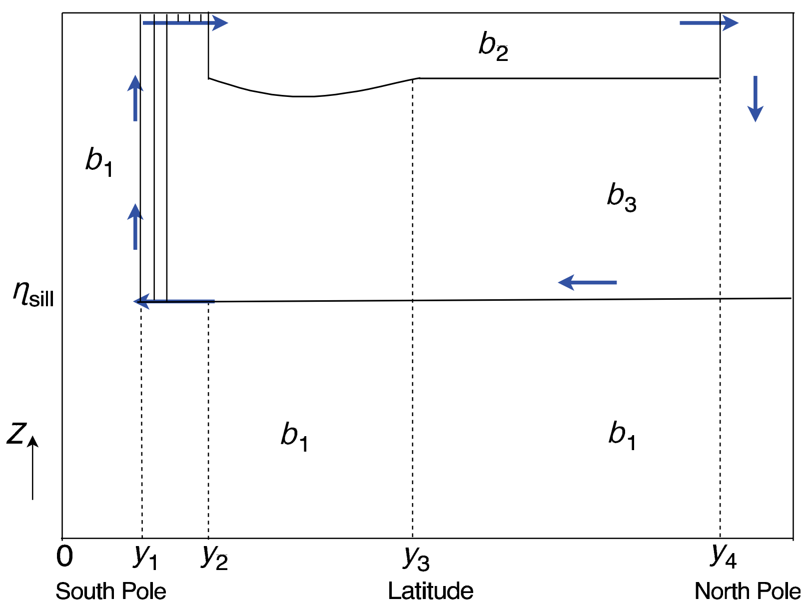

Figure 6: As for Fig. 5, but now for a two-hemisphere ocean with a source of dense water, b_3, at high northern latitudes.

of fluid parcels. Note that there are no fluxes in the buoyancy equation and that only the PV flux, $\overline{v'q'}$, need be parameterized. If we now put back some of the terms we omitted, our complete equations are

$$\frac{\partial \overline{u}}{\partial t} - f\overline{v}^* = \overline{v'q'} + \frac{\partial \tau}{\partial z} \tag{4a}$$

$$\frac{\partial \overline{b}}{\partial t} + \overline{v}^* \frac{\partial \overline{b}}{\partial y} + \overline{w}^* \frac{\partial \overline{b}}{\partial z} = \kappa_v \frac{\partial^2 \overline{b}}{\partial z^2}. \tag{4b}$$

where $(\overline{v}^*, \overline{w}^*) = (-\partial \psi/\partial z, \partial \psi/\partial y)$ and $f\partial \overline{u}/\partial z = -\partial \overline{b}/\partial y$. The stress τ in only non-zero near the top (wind-stress) and bottom (Ekman drag), and τ integrates to zero. We'll look for steady state solutions and drop the * notation so that all variables are residuals and zonal averages.

Equations in the channel

We parameterize

$$\overline{v'q'} = -K_e \frac{\partial \overline{q}}{\partial y}. \tag{5}$$

where, approximately, for the large-scale ocean

$$\overline{q} \approx f \frac{\partial}{\partial z}\left(\frac{\overline{b}}{\overline{b}_z}\right), \quad \text{so that} \quad \frac{\partial \overline{q}}{\partial y} \approx f \frac{\partial}{\partial z}\left(\frac{\overline{b}_y}{\overline{b}_z}\right) = -f\frac{\partial S}{\partial z} \tag{6}$$

where $S = -\overline{b}_y/\overline{b}_z$ is the slope of the isopycnals. Thus

$$\overline{v'q'} = fK_e \frac{\partial S}{\partial z}. \tag{7}$$

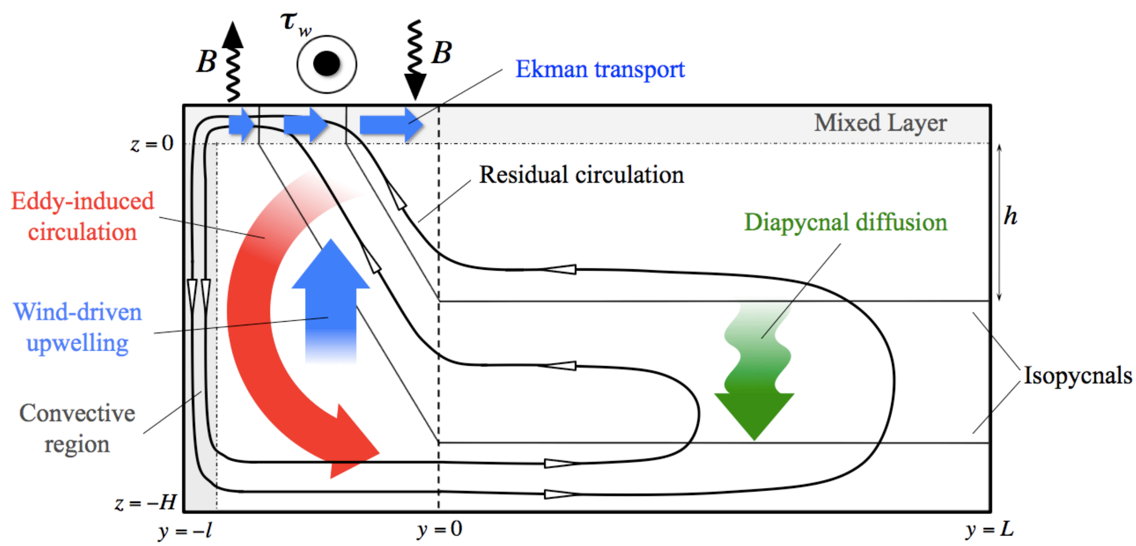

Figure 7: Schematic of the single hemisphere meridional overturning circulation crudely representing AABW. Thin black lines are the isopycnals, thick black line is a overturning streamfunction, dashed vertical line is the northern edge of the channel, shaded gray areas are the convective region and the surface mixed layer.

and the momentum equation becomes

$$-f\overline{v} = fK_e\frac{\partial S_b}{\partial z} + \frac{\partial \tau}{\partial z}. \tag{8}$$

Since $\overline{v}^* = -\partial\psi/\partial z$ we integrate this from the top to a level z and obtain

$$\psi = -\frac{\tau_w}{f} + K_e S. \tag{9}$$

We have assumed $\psi = 0$ and $S = 0$ at the top, and $\tau = \tau_w$ at the top (base of mixed layer) and $\tau = 0$ in the interior.

The buoyancy equation in terms of streamfunction is

$$\overline{\mathbf{v}} \cdot \nabla\overline{b} = \kappa_v\frac{\partial^2\overline{b}}{\partial z^2} \qquad \Longrightarrow \qquad \frac{\partial\psi}{\partial y}\frac{\partial\overline{b}}{\partial z} - \frac{\partial\psi}{\partial z}\frac{\partial\overline{b}}{\partial y} = \kappa_v\frac{\partial^2\overline{b}}{\partial z^2} \tag{10}$$

or

$$\frac{\partial\psi}{\partial y} + S\frac{\partial\psi}{\partial z} = \frac{\partial_z^2\overline{b}}{\partial_z\overline{b}} . \tag{11}$$

The boundary condition on ψ for this will be supplied by the basin! The other boundary condition we will need is the buoyancy distribution at the top, and so we specify

$$\overline{b}(y, z = 0) = b_0(y). \tag{12}$$

Equations in the basin

In the basin the slope of the isopycnals is assumed zero and (11) becomes the conventional upwelling diffusive balance,

$$w\frac{\partial \overline{b}}{\partial z} = \kappa \frac{\partial^2 \overline{b}}{\partial z^2} \qquad \text{or} \qquad \frac{\partial \psi}{\partial y}\frac{\partial \overline{b}}{\partial z} = \kappa_v \frac{\partial^2 \overline{b}}{\partial z^2}. \tag{13}$$

Integrate this from the edge of the channel, $y = 0$, to the northern edge, $y = L$, and obtain

$$\psi|_{y=0} = -\kappa_v L \frac{\overline{b}_{zz}}{\overline{b}_z} \ . \tag{14}$$

2.2 Scaling

Let hats denote non-dimensional values and let

$$z = h\widehat{z}, \qquad y = l\widehat{y}, \qquad \tau_w = \tau_0 \widehat{\tau}_w, \tag{15}$$

$$f = f_0 \widehat{f}, \qquad \psi = \frac{\tau_0}{f_0}\widehat{\psi}, \qquad S = \frac{h}{l}\widehat{S}, \tag{16}$$

where h is a characteristic vertical scale such that $S \sim h/l$. It will emerge as part of the solution. If we have scaled properly then variables with hats on are of order one. The nondimensional equations of motion are then

$$\text{Buoyancy evolution:} \qquad \partial_{\widehat{y}}\widehat{\psi} + \widehat{S}\partial_{\widehat{z}}\widehat{\psi} = \epsilon \left(\frac{l}{L}\right) \frac{\partial_{\widehat{z}\widehat{z}}\widehat{b}}{\partial_{\widehat{z}}\widehat{b}}, \tag{17a}$$

$$\text{Momentum balance:} \qquad \widehat{\psi} = -\frac{\widehat{\tau}}{\widehat{f}} + \Lambda \widehat{S}, \tag{17b}$$

$$\text{Boundary condition:} \qquad \widehat{\psi}|_{\widehat{y}=0} = -\epsilon \frac{\partial_{\widehat{z}\widehat{z}}\widehat{b}}{\partial_{\widehat{z}}\widehat{b}}, \tag{17c}$$

where

$$\Lambda = \frac{\text{Eddies}}{\text{Wind}} = \frac{K_e}{\tau_0/f_0}\frac{h}{l} \sim 1. \qquad \text{and} \qquad \epsilon = \frac{\text{Mixing}}{\text{Wind}} = \frac{\kappa_v}{\tau_0/f_0}\frac{L}{h} \sim 0.1 - 1. \tag{18a,b}$$

These are the two important nondimensional numbers in the problem and we can obtain estimates of their values by using some observed values for the other parameters. Thus, with $\kappa_v = 10^{-5}$ m^2 s^{-1}, $K_e = 10^3$ m^2 s^{-1}, $\tau_0 = 0.1$ N m^{-2}, $f_0 = 10^{-4}$ s^{-1}, $\rho_0 = 10^3$ kg m^{-3}, $L = 10,000$ km, $l = 1,000$ km, and $h = 1$ km we find

$$\Lambda = 1, \qquad \epsilon = 0.1, \qquad \text{and} \qquad \frac{l}{L} = 0.1. \tag{19}$$

Note that Λ and ϵ are not independent of each other for they both depend on the vertical scale of stratification h which is a part of the solution, and for that we must look at some limiting cases.

The weak diffusiveness limit

Suppose that mixing is small and that $\epsilon \ll 1$. We can then *require* that $\Lambda = 1$ in order that the eddy-induced circulation nearly balance the wind-driven circulation (because the diffusive term is small), whence the vertical scale h is given by

$$\frac{h}{l} = \frac{\tau_0/f_0}{K_e}. \tag{20}$$

If K_e does become small then h becomes large, meaning that the isopycnals are near vertical. Using the above value for h we find that

$$\epsilon = \frac{\kappa_v K_e}{(\tau_0/f_0)^2}\frac{L}{l} \tag{21}$$

This is an appropriate measure of the strength of the diapycnal diffusion in the ocean. Using (17c) we see that $\widehat{\psi} \sim \epsilon$ so that the dimensional strength of the circulation goes as

$$\Psi = \epsilon\frac{\tau_0}{f_0} = \kappa_v \frac{K_e}{\tau_0/f_0}\frac{L}{l}. \tag{22}$$

Another way to obtain this is to note that for weak diffusion the balance in the dimensional momentum equation is between wind forcing and eddy effects (because they must nearly cancel) so that

$$\frac{\tau_w}{f} \sim K_e S, \tag{23}$$

which may be written as

$$\frac{h}{l} \sim \frac{\tau_w}{K_e f}. \tag{24}$$

Advective-diffusive balance in the basin gives

$$\frac{\partial\psi}{\partial y}\frac{\partial\overline{b}}{\partial z} = \kappa_v\frac{\partial^2\overline{b}}{\partial z^2} \qquad\qquad \Longrightarrow \qquad\qquad \Psi = \frac{\kappa_v L}{h} \tag{25}$$

and (24) and (25) together give (22).

The strong diffusiveness limit

This limit may be appropriate for the abyssal ocean and in any case it is worth doing, so let us take $\epsilon \gg 1$ and the circulation in the basin will in some sense be strong. As before the nondimensional strength of the circulation is given by

$$\widehat{\psi} = \mathcal{O}(\epsilon) \gg 1. \tag{26}$$

The fact that $\widehat{\psi} \neq \mathcal{O}(1)$ means we haven't scaled things in an ideal fashion, but let's proceed anyway. Dimensionally

$$\Psi = \epsilon\frac{\tau_0}{f_0} \qquad \text{or} \qquad \Psi = \frac{\kappa_v L}{h} \tag{27a,b}$$

but h and ϵ are both different than before.

Now, if $\widehat{\psi} \sim \epsilon \gg 1$ the diffusion driven circulation in the basin cannot be matched by a purely wind-driven circulation in the channel, since the latter is $\mathcal{O}(1)$. We can only match the circulation with an eddy-driven circulation and therefore we require

$$\Lambda = \mathcal{O}(\epsilon). \tag{28}$$

In particular, if we set $\Lambda = \epsilon$ then

$$\epsilon = \Lambda = \sqrt{\left(\frac{K_e \kappa_v L}{(\tau_0/f)^2 l}\right)}. \tag{29}$$

This is the square root of the expression for ϵ in the weak diffusiveness limit. Using (29) and (27a) we find

$$h = \sqrt{\frac{\kappa_v}{K_e} L l} \quad \text{and} \quad \Psi = \sqrt{\frac{K_e \kappa_v L}{l}}. \tag{30a,b}$$

Discussion of limits

If diffusion is weak the stratification itself is set by the eddies. Thus, upwelling-diffusion gives $\psi \sim \kappa_v L/h$, with h being set by the wind as in (24), thus giving a circulation strength that is linearly proportional to diffusivity, as in (22). In the strong diffusion case the diffusion itself affects the stratification, and so we get a weaker dependence of the circulation strength on κ_v. In this limit diapycnal mixing deepens the isopycnals in the basin away from the channel, so that the isopycnals are steeper in the channel. This steepening is balanced by the slumping effects of baroclinic instability, and wind only has a secondary effect. From an asymptotic perspective in the small ϵ limit the residual circulation is zero to lowest order. At next order it follows the isopycnals except in the mixed layer.

Instead of varying diffusivity we can think of the wind changing. In the weak wind limit the circulation is diffusively driven and independent of the wind strength. In the strong wind limit the circulation actually *decreases* as the wind increases. This is because the wind steepens the isopycnals so the diffusive term ($\sim \kappa_v \overline{b}_{zz}$) gets smaller and hence the circulation weaker.

3 An Interhemispheric Circulation

We now introduce another 'water mass' into the mix — *North Atlantic Deep Water*, or NADW. We will construct a model of similar type to what we did in the previous lecture, but now we will divide the ocean into three regions, namely

1. a southern channel, say from 50° S to 70° S

2. a basin region, say from from 50° S to 60° N

3. a northern convective region

(Fig. 8). The idea will be to write down the dynamics in these three regions and match them at the boundaries. The main difference, and it is an important one, between this

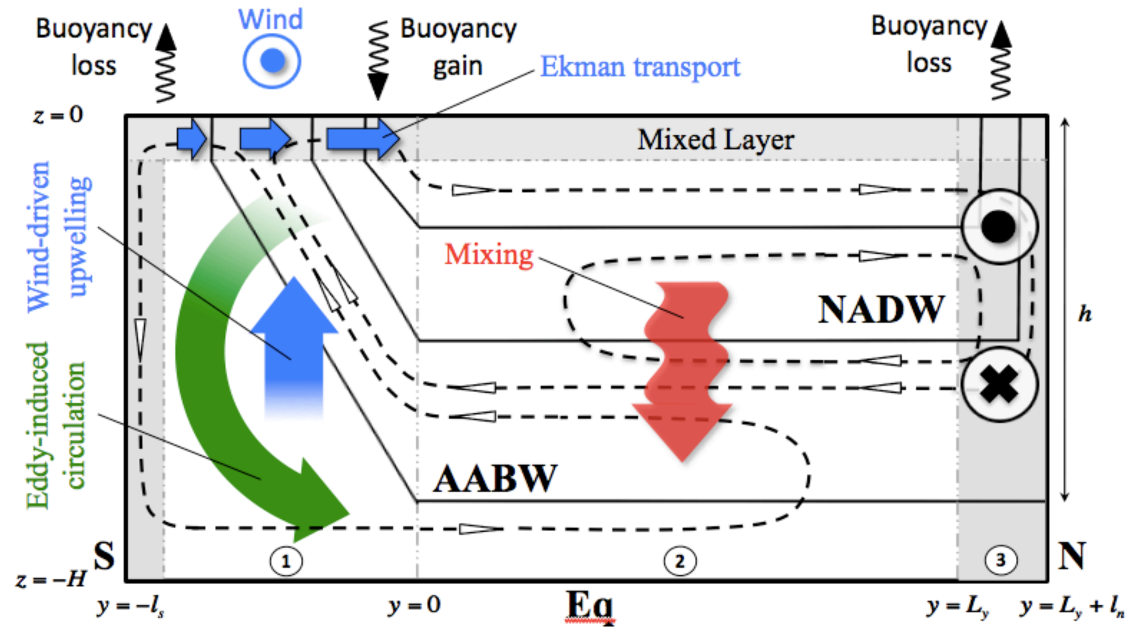

Figure 8: Schematic of the interhemispheric MOC. Thin solid black lines are the isopycnals, dashed lines with arrows are the streamlines, dashed vertical lines are the boundaries between adjacent regions, shaded gray areas are the convective regions at high latitudes and the surface mixed layer,and the red arrow represents downward diffusive heat flux. Labels 1, 2, and 3 (in circles) denote the circumpolar channel, ocean basin, and isopycnal outcrop regions.

model and the previous one is the presence of an interhemispheric cell that is primarily wind driven, and sits on top of the lower cell. It is convenient to write down the equations of motion for each region separately, with the first two regions being similar to those of the previous section.

3.1 Equations of motion

In the equations below use restoring conditions at the top, but a specified buoyancy would work too.

Region 1, the southern channel

In the channel the buoyancy equation takes its full advective-diffusive form (although we later find that in some circumstances diffusion is unimportant). The momentum equation has a wind-driven component and and eddy-driven component, as before. In dimensional form the equations are

$$\text{Buoyancy advection:} \qquad J(\psi_1, b_1) = \kappa_v \frac{\partial^2 b_1}{\partial z^2} \tag{31a}$$

$$\text{Momentum:} \qquad \psi_1 = -\frac{\tau(y)}{f} - K_e S \tag{31b}$$

$$\text{Surface boundary, at } z = 0: \qquad -\kappa \frac{\partial b_1}{\partial z} = \lambda(b^*(y) - b_1) \tag{31c}$$

$$\text{Buoyancy match to interior:} \qquad b_1(z)|_{y=0} = b_2(z) \tag{31d}$$

$$\text{Streamfunction match to interior} \qquad \psi|_{y=0} = -\kappa_v L \frac{\overline{b}_{zz}}{\overline{b}_z} \tag{31e}$$

These are more-or-less the same as those for the channel in the previous section, although we have explicitly added a buoyancy condition and we use a restoring condition on temperature.

Region 2, the basin

In the basin the isopycnals are flat and the buoyancy equation is an upwelling-diffusive balance. We won't need the momentum equation, and so we have

$$\text{Upwelling diffusive:} \qquad \frac{(\psi_3 - \psi_1)}{L} \frac{\partial b_2}{\partial z} = \kappa_v \frac{\partial^2 b_2}{\partial z^2} \tag{32a}$$

$$\text{Surface boundary:} \qquad -\kappa \frac{\partial b_2}{\partial z}\bigg|_{z=0} = \lambda(b^* - b_2) \tag{32b}$$

The upwelling diffusive balance is just $w\partial_z b = \partial_z^2 b$ with flat isopycnals, with ψ_1 and ψ_3 being the streamfunctions at the southern and northern ends of the basin, respectively. If $\psi_1 \neq \psi_2$ there is a net convergence and hence an upwelling. If $\kappa_v = 0$ then either there is no upwelling or no vertical buoyancy gradient. However, there can be an interhemispheric flow; the properties of the water mass do not change, and we expect the meridional flow to occur in a western boundary current.

Region 3, the northern convective region

In this region the values of buoyancy at the surface (i.e., $b_3(y, z = 0)$) are mapped on to the flat isopycnals in the interior (i.e, $b_2(z)$). We assume this matching occurs by convection. That is, the surface waters convect downward to the level of neutral buoyancy and then move meridionally. By thermal wind the outcropping isopycnals give rise to a zonal flow, with the total zonal transport being determined by the meridional temperature gradient and the depth to which flow convects, which is a function of such things as the winds, eddy strength and diapycnal mixing in the Southern channel. The zonal flow is thus

$$u_3(y, z) = -\frac{1}{f} \int_{-h}^{z} \frac{\partial b_3}{\partial y} \, \mathrm{d}z' + C \tag{33}$$

where C is determined by the requirement that $\int_{-h}^{0} u_3 \, \mathrm{d}z = 0$. When the relatively shallow eastward moving zonal flow collides with the eastern wall it subducts and returns. When the deep westward flow collides with the western wall it may move equatorward in a frictional deep western boundary current. It is the upper, northward moving branch of the deep western boundary current that feeds the eastward moving flow. The total volume transport in these zonal flows thus translates to a meridional streamfunction that has the value

$$\psi_3(z) = \int_{-h}^{z} \mathrm{d}z' \int_{L_y}^{L_n} u_3 \, \mathrm{d}y \tag{34}$$

where L_y is the latitude of the southern edge of the convecting region and $L_n = L_y + l_n$ is the northern edge of the domain. Summing up, the equations in the convective region are

$$\text{Convective matching:} \qquad b_2(z) \Rightarrow b_3(y, z = 0) \tag{35}$$

$$\text{Thermal wind:} \qquad f u_3(y, z) = -\int_{-h}^{z} \frac{\partial b_3}{\partial y} \, \mathrm{d}z' + C \tag{36}$$

$$\text{Mass Continuity:} \qquad \psi_3(z) = \iint u_3 \, \mathrm{d}y \, \mathrm{d}z \tag{37}$$

We now discuss how all this fits together.

3.2 Scaling and Dynamics

Our main focus is on the upper cell, since the lower cell has essentially the same dynamics as in the single hemisphere case. We proceed by writing down some parametric expressions for the streamfunctions in the three domains.

$$\Psi_1 = \left(\frac{\tau_0}{f_1} - K_e \frac{h}{l_s} \right) L_x, \tag{38a}$$

$$\Psi_2 = \Psi_3 - \Psi_1 = \frac{\kappa_v}{h} L_x L_y, \tag{38b}$$

$$\Psi_3 = \frac{\Delta b h^2}{f_3}. \tag{38c}$$

We don't like these equations because when doing scaling we don't like having additive expressions but for now we damn the torpedoes. The four unknowns are Ψ_1, Ψ_2, Ψ_3 and h and there are four equations (note that (38b) is two equations). If we combine them we obtain

$$\frac{\Delta b h^2}{f_3} - \left(\frac{\tau_0}{f_1} - K_e \frac{h}{l_s} \right) L_x = \frac{\kappa_v}{h} L_x L_y \tag{39}$$

This expression is very similar to one obtained by Gnanadesikan (1999).

With no northern source

Suppose that $\Delta b = 0$ and that there is no deep water formation in the North Atlantic. If also κ_v is small then we obtain $h/l_s = (\tau_0/f_1)/K_e$, which is essentially the same as (20), obtained previously. If κ_v is large then we find $h^2 = \kappa_v L l_s / K_e$; that is, we recover (30a). Pretty much everything is the same as it was section 2.

With no southern channel

If there is no southern channel then $\Psi_1 = 0$ and we have

$$\frac{\Delta b h^2}{f_3} = \frac{\kappa_v}{h} L_x L_y \tag{40}$$

and

$$h^3 = \kappa_v \left(\frac{f_3 L_x L_y}{\Delta b} \right) \quad \text{and} \quad \Psi_3 = \Psi_2 = (\kappa_v L_x L_y)^{2/3} \left(\frac{\Delta b}{f_3} \right)^{1/3}. \tag{41}$$

These are classical expressions for the thickness of a diffusive thermocline and the strength of a diffusively-driven overturning circulation, going back to Robinson and Stommel. This is also the same as the strong diffusivity limit.

With all three regions

This is the new bit. The weak diffusivity limit is the interesting case, as the strong diffusivity limit is really just the case with no southern channel.

In this case the upwelling is weak and $|\Psi_3| \approx \Psi_1|$ and

$$\frac{\Delta b h^2}{f_3} - \left(\frac{\tau_0}{f_1} - K_e \frac{h}{l_s} \right) L_x = 0. \tag{42}$$

In this case the basin is just a 'pass-through' region: water formed in the North Atlantic just passes through the basin without change, and upwells in the Southern Ocean. For the moment let us also assume that K_e is small and then

$$\frac{\Delta b h^2}{f_3} = \frac{\tau_0}{\rho_0 f_1} L_x, \tag{43}$$

which results in a depth scale h for the stratification,

$$h = \left(\frac{\tau_0 f_3 L_x}{f_1 \Delta b} \right)^{1/2} \tag{44}$$

Putting in the numbers, we find $h \approx 320\,\mathrm{m}$. Furthermore, the strength of the circulation is just determined by the wind stress,

$$\Psi_1 = \Psi_3 = \left(\frac{\tau_0 L_x}{f_1} \right) \tag{45}$$

which is about $10\,\mathrm{Sv}$.

In the more general case we solve (42) to give

$$h = \left(\frac{\tau_0 f_3 L_x}{f_1 \Delta b} \right)^{1/2} \left(-\alpha + \sqrt{1 + \alpha^2} \right) \tag{46}$$

where α is the nondimensional number given by the ratio of the wind to eddy effects

$$\alpha = \frac{1}{2} \frac{K_e}{l_s} \left(\frac{L_x f_1 f_3}{\tau_0 \Delta b} \right)^{1/2} = \frac{1}{2} \frac{\Psi^*}{\overline{\Psi}}. \tag{47}$$

where

$$\overline{\Psi} = \frac{\tau_0}{f_1} L_x, \qquad \Psi^* = -K_e h \frac{L_x}{l_s} = -k_e \frac{L_x}{l_s} \left(\frac{\tau_0 f_3 L_x}{f_1 \Delta b} \right)^{1/2}. \tag{48}$$

If we put in numbers then $\alpha \approx 0.08$, $\Psi^* \approx 1.6\,\mathrm{Sv}$ and $\overline{\Psi} \approx 10\,\mathrm{Sv}$. That is, the wind-induced circulation is the dominant factor in the meridional overturning circulation, and if we take α to small then we have

$$h \approx \left(\frac{\tau_0 f_3 L_x}{f_1 \Delta b} \right)^{1/2}, \qquad \Psi \approx \frac{\tau_0}{f_1} L_x \tag{49}$$

Discussion

Although there can be no certainties when eddy diffusivities are present, the use of representative parameters suggests that the eddy-induced circulation is indeed smaller than the wind-driven circulation in the Southern Ocean. That is, putting in numbers, we find $\alpha \approx 0.08$ with $\Psi^* \approx 1.6\,\mathrm{Sv}$ and $\overline{\Psi} \approx 10\,\mathrm{Sv}$. This suggests that, for typical oceanic parameters, the strength of the eddy-induced circulation on isopycnals corresponding to the middepth overturning cell is only about 10-20% of the wind-driven circulation. Thus, rather than the residual circulation vanishing as is sometimes assumed, the middepth residual circulation is comparable to the wind-driven circulation and acts to pull $O(10)$ Sv of deep water formed at high northern latitudes in the North Atlantic back up to the surface. As a result, the depth scale of stratification h is not linearly proportional to the wind stress τ, as one would obtain from the vanishing residual circulation argument with a simple eddy parameterization, but rather it scales with τ as $\tau^{1/2}$ and is dependent on Δb which is the buoyancy range for isopycnals which are shared between the circumpolar channel and the isopycnal outcrop region in the Northern Hemisphere.

In summary, in the limit of weak diapycnal mixing, relevant to the present middepth ocean, the strength of the middepth overturning circulation is primarily determined by the Ekman transport in the Southern Ocean. The rest of the ocean is essentially forced to adjust and produce the amount of deep water demanded by the Ekman transport and the associated wind-driven upwelling in the Southern Ocean. For instance, during the transient adjustment, the Ekman transport in the circumpolar channel, in conjunction with the surface buoyancy flux, pulls dense waters up from the deep ocean, converts them into light waters at the surface, and pumps these waters into, or just below, the main thermocline in the ocean basin. The rate at which these light waters are then imported into the deep water formation region in the North Atlantic, converted back into dense waters, and exported to the ocean basin at middepth, is controlled by the meridional pressure gradient set up by the outcropping isopycnals in the north. Hence, light waters pumped into the ocean basin by the Ekman transport in the south accumulate in, or just below, the main thermocline, therefore deepening the middepth isopycnals and increasing the transport of light water into the deep water formation region in the north until the transports in the north and south match. The established interhemispheric balance sets the depth of the isopycnals in the ocean basin and thus stratification throughout the entire ocean.

In the case when deep waters are not produced in the north, as observed in the Pacific Ocean, light waters pumped into the ocean basin by the Southern Ocean wind will deepen the mid-depth isopycnal in the ocean basin, and thus steepen their slopes in the Southern Ocean, until the eddy-induced circulation in the Southern Ocean cancels the wind-driven circulation resulting in a zero residual circulation and water mass transformation.

4 Theory of the Main Thermocline (And why it is not really like the tropopause)

Our goal in this lecture is to give a sense of the structure of the thermocline, and to draw out similarities and differences (mainly differences) with the tropopause.

A simple kinematic model

The fact that cold water with polar origins upwells into a region of warmer water suggests that we consider the simple one-dimensional advective–diffusive balance,

$$w\frac{\partial T}{\partial z} = \kappa\frac{\partial^2 T}{\partial z^2}, \tag{50}$$

where w is the vertical velocity, κ is a diffusivity and T is temperature. In mid-latitudes, where this might hold, w is positive and the equation represents a balance between the upwelling of cold water and the downward diffusion of heat. If w and κ are given constants, and if T is specified at the top ($T = T_T$ at $z = 0$) and if $\partial T/\partial z = 0$ at great depth ($z = -\infty$) then the temperature falls exponentially away from the surface according to

$$T = (T_T - T_B)\mathrm{e}^{wz/\kappa} + T_B, \tag{51}$$

here T_B is a constant. This expression cannot be used to estimate how the thermocline depth scales with either w or κ, because the magnitude of the overturning circulation depends on κ. However, it is reasonable to see if the observed ocean is broadly consistent with this expression. The diffusivity κ can be measured; it is an eddy diffusivity, maintained by small-scale turbulence, and measurements produce values that range between $10^{-5}\,\mathrm{m}^{2}\,\mathrm{s}^{-1}$ in the main thermocline and $10^{-4}\,\mathrm{m}^{2}\,\mathrm{s}^{-1}$ in abyssal regions over rough topography and in and near continental margins, with still higher values locally.

The vertical velocity is too small to be measured directly, but various estimates based on deep water production suggest a value of about $10^{-7}\,\mathrm{m\ s}^{-1}$. Using this and the smaller value of κ in (51) gives an e-folding vertical scale, κ/w, of just 100 m , beneath which the stratification is predicted to be very small (i.e., nearly uniform potential density). Using the larger value of κ increases the vertical scale to 1000 m , which is probably closer to the observed value for the total thickness of the thermocline (look at Fig. 9), but using such a large value of κ in the main thermocline is not supported by the observations. Similarly, the deep stratification of the ocean is rather larger than that given by (50), except with values of diffusivity on the large side of those observed.

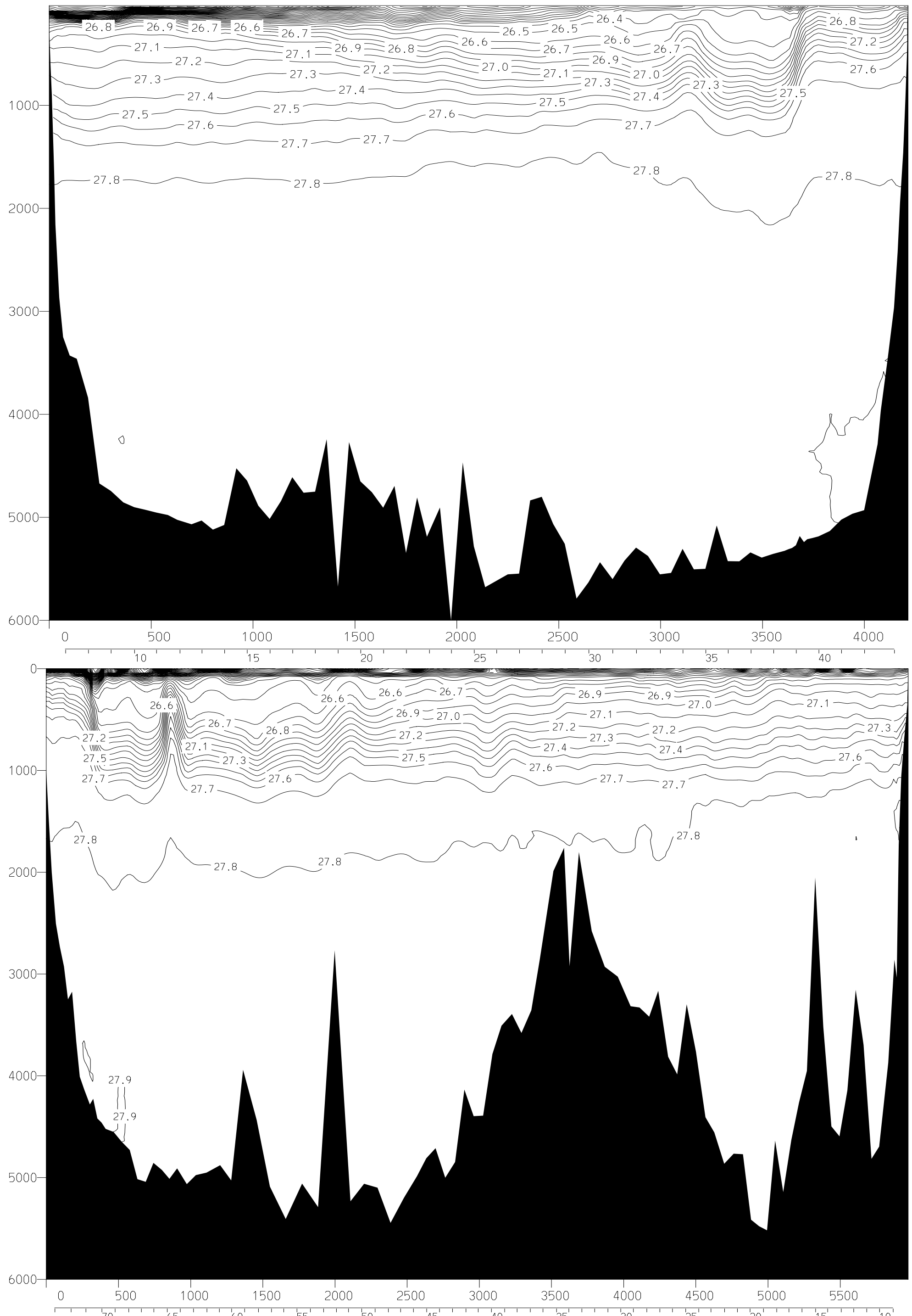

Figure 9: Sections of potential density (σ_θ) in the North Atlantic. Upper panel: meridional section at 53° W , from 5° N to 45° N , across the subtropical gyre. Lower panel: zonal section at 36° N , from about 75° W to 10° W .

5 Scaling and Simple Dynamics of the Main Thermocline

The Rossby number of the large-scale circulation is small and the scale of the motion large, and the flow obeys the planetary-geostrophic equations:

$$\mathbf{f}\times\mathbf{u} = -\nabla\phi, \qquad \frac{\partial\phi}{\partial z} = b, \tag{52a,b}$$

$$\div\mathbf{v} = 0, \qquad \frac{\mathrm{D}b}{\mathrm{D}t} = \kappa\frac{\partial^2 b}{\partial z^2}. \tag{53a,b}$$

5.1 An advective scale

If there is upwelling ($w > 0$) from the abyss, and Ekman downwelling ($w < 0$) at the surface, there is some depth D_a at which $w = 0$. By cross-differentiating (52a) we obtain $\beta v = -f \div [z]\mathbf{u}$, or

$$\beta v = f\frac{\partial w}{\partial z} \qquad \to \qquad \beta V = f\frac{W}{D_a} = f\frac{W_E}{D_a}. \tag{54}$$

Thermal wind

$$\mathbf{f}\times\frac{\partial\mathbf{u}}{\partial z} = -\nabla b \qquad \to \qquad \frac{U}{D_a} = \frac{1}{f}\frac{\Delta b}{L}, \tag{55}$$

where Δb is the scaling value of variations of buoyancy in the horizontal. Assuming the vertical scales are the same in (54) and (55) and that $V \sim U$ then

$$D_a = W_E^{1/2}\left(\frac{f^2 L}{\beta\,\Delta b}\right)^{1/2}. \tag{56}$$

5.2 A diffusive scale

The estimate (56) cares nothing about the thermodynamic equation, so let's now include some and construct a scaling from from advective–diffusive balance in the thermodynamic equation, the linear geostrophic vorticity equation, and thermal wind balance:

$$w\frac{\partial b}{\partial z} = \kappa\frac{\partial^2 b}{\partial z^2}, \qquad \beta v = f\frac{\partial w}{\partial z}, \qquad f\frac{\partial\mathbf{u}}{\partial z} = \mathbf{k}\times\nabla b, \tag{57a,b,c}$$

with corresponding scales

$$\frac{W}{\delta} = \frac{\kappa}{\delta^2}, \qquad \beta U = \frac{fW}{\delta}, \qquad \frac{U}{\delta} = \frac{\Delta b}{fL}, \tag{58a,b,c}$$

where δ is the vertical scale. Because there is now one more equation than in the advective scaling theory we cannot take the vertical velocity as a given, otherwise the equations would be overdetermined. We therefore take it to be the abyssal upwelling velocity, which then becomes part of the *solution,* rather than being imposed. From (58) we obtain the diffusive vertical scale,

$$\delta = \left(\frac{\kappa f^2 L}{\beta\,\Delta b}\right)^{1/3}. \tag{59}$$

With $\kappa = 10^{-5}\,\mathrm{m}^{\ 2}\,\mathrm{s}^{\ -2}$ and with the other parameters taking the values given following (56), (59) gives $\delta \approx 150\,\mathrm{m}$ and, using (58a), $W \approx 10^{-7}\,\mathrm{m\ \ s^{-1}}$, which is an order of magnitude smaller than the Ekman pumping velocity W_E.

6 The Internal thermocline

6.1 The M equation

The planetary-geostrophic equations can be written as a single partial differential equation in a single variable, although the resulting equation is of quite high order and is nonlinear. We write the equations of motion as

$$-fv = -\frac{\partial \phi}{\partial x}, \qquad fu = -\frac{\partial \phi}{\partial y}, \qquad b = \frac{\partial \phi}{\partial z}, \tag{60a,b,c}$$

$$\div \mathbf{v} \ = 0, \qquad \frac{\partial b}{\partial t} + \mathbf{v} \ \cdot \nabla b = \kappa \nabla^2 b, \tag{61a,b}$$

where we take $f = \beta y$. Cross-differentiating the horizontal momentum equations and using (61a) gives the linear geostrophic vorticity relation $\beta v = f \partial w / \partial z$ which, using (60a) again, may be written as

$$\frac{\partial \phi}{\partial x} + \frac{\partial}{\partial z}\left(-\frac{f^2}{\beta} w\right) = 0. \tag{62}$$

This equation is the divergence in (x, z) of $(\phi, -f^2 w/\beta)$ and is automatically satisfied if

$$\phi = M_z \qquad \text{and} \qquad \frac{f^2 w}{\beta} = M_x. \tag{63a,b}$$

where the subscripts on M denote derivatives. Then straightforwardly

$$u = -\frac{\partial_y \phi}{f} = -\frac{M_{zy}}{f}, \qquad v = \frac{\partial_x \phi}{f} = \frac{M_{zx}}{f}, \qquad b = \partial_z \phi = M_{zz}. \tag{64a,b,c}$$

The thermodynamic equation, (61b) becomes

$$\frac{\partial M_{zz}}{\partial t} + \left(\frac{-M_{zy}}{f} M_{zzx} + \frac{M_{zx}}{f} M_{zzy}\right) + \frac{\beta}{f^2} M_x M_{zzz} = \kappa M_{zzzz} \tag{65}$$

or

$$\frac{\partial M_{zz}}{\partial t} + \frac{1}{f} J(M_z, M_{zz}) + \frac{\beta}{f^2} M_x M_{zzz} = \kappa M_{zzzz}. \tag{66}$$

where J is the usual horizontal Jacobian. This is the M *equation,*[1] somewhat analogous to the potential vorticity equation in quasi-geostrophic theory in that it expresses the entire dynamics of the system in a single, nonlinear, advective–diffusive partial differential equation, although note that M_{zz} is materially conserved (in the absence of diabatic effects) by the three-dimensional flow. Because of the high differential order and nonlinearity of the system analytic solutions of (66) are very hard to find, and from a numerical perspective it is easier to integrate the equations in the form (60) and (61) than in the form (66). Nevertheless, it is possible to move forward by approximating the equation to one or two dimensions, or by a priori assuming a boundary-layer structure.

[1] Welander (1971).

A one-dimensional model

Let us consider an illustrative one-dimensional model (in z) of the thermocline. Merely setting all horizontal derivatives in (66) to zero is not very useful, for then all the advective terms on the left-hand side vanish. Rather, we look for steady solutions of the form $M = M(x,z)$, and the M equation then becomes

$$\frac{\beta}{f^2} M_x M_{zzz} = \kappa M_{zzzz}, \tag{67}$$

which represents the advective–diffusive balance

$$w\frac{\partial b}{\partial z} = \kappa \frac{\partial^2 b}{\partial z^2}. \tag{68}$$

(We must also suppose that the value of κ varies meridionally in the same manner as does β/f^2; without this technicality M would be a function of y , violating our premise.) If the ocean surface is warm and the abyss is cold, then (67) represents a balance between the upward advection of cold water and the downward diffusion of warm water. The horizontal advection terms vanish because the zonal velocity, u, and the meridional buoyancy gradient, b_y, are each zero. Let us further consider the special case

$$M = (x - x_e) W(z), \tag{69}$$

where the domain extends from $0 \le x \le x_e$, so satisfying $M = 0$ on the eastern boundary. Equation (67) becomes the ordinary differential equation

$$\frac{\beta}{f^2} W W_{zzz} = \kappa W_{zzzz}, \tag{70}$$

where W has the dimensions of velocity squared. We non-dimensionalize this by setting

$$z = H\widehat{z}, \qquad \kappa = \widehat{\kappa}(HW_S), \qquad W = \left(\frac{f^2 W_S}{\beta}\right)\widehat{W}, \tag{71a,b,c}$$

where the hatted variables are non-dimensional and W_S is a scaling value of the dimensional vertical velocity, w (e.g., the magnitude of the Ekman pumping velocity W_E). Equation (70) becomes

$$\widehat{W}\widehat{W}_{\widehat{z}\widehat{z}\widehat{z}} = \widehat{\kappa}\widehat{W}_{\widehat{z}\widehat{z}\widehat{z}\widehat{z}}, \tag{72}$$

The parameter $\widehat{\kappa}$ is a non-dimensional measure of the strength of diffusion in the interior, and the interesting case occurs when $\widehat{\kappa} \ll 1$; in the ocean, typical values are $H = 1\,\mathrm{km}$, $\kappa = 10^{-5}\,\mathrm{m\ s}^{-2}$ and $W_S = W_E = 10^{-6}\,\mathrm{m\ s}^{-1}$ so that $\widehat{\kappa} \approx 10^{-2}$, which is indeed small. (It might appear that we could completely scale away the value of κ in (70) by scaling W appropriately, and if so there would be no meaningful way that one could say that κ was small. However, this is a chimera, because the value of κ would still appear in the boundary conditions.)

The time-dependent form of (72), namely

$$\widehat{W}_{\widehat{z}\widehat{z}t} + \widehat{W}\widehat{W}_{\widehat{z}\widehat{z}\widehat{z}} = \widehat{\kappa}\widehat{W}_{\widehat{z}\widehat{z}\widehat{z}\widehat{z}} \tag{73}$$

is similar to Burger's equation

$$V_t + VV_z = \nu V_{zz} \tag{74}$$

which is known to develop fronts. (In the inviscid Burger's equation, $\mathrm{D}V/\mathrm{D}t = 0$, where the advective derivative is one-dimensional, and therefore the velocity of a given fluid parcel is preserved on the line. Suppose that the velocity of the fluid is positive but diminishes in the positive z-direction, so that a fluid parcel will catch-up with the fluid parcel in front of it. But since the velocity of a fluid parcel is fixed, there are two values of velocity at the same point, so a singularity must form. In the presence of viscosity, the singularity is tamed to a front.) Thus, we might similarly expect (72) to produce a front, but because of the extra derivatives the argument is not as straightforward and it is simplest to obtain solutions numerically.

Equation (72) is fourth order, so four boundary conditions are needed, two at each boundary. Appropriate ones are a prescribed buoyancy and a prescribed vertical velocity at each boundary, for example

$$\begin{aligned} &\widehat{W} = \widehat{W}_E, \quad -\widehat{W}_{\widehat{z}\widehat{z}} = B_0, \qquad && \text{at top} \\ &\widehat{W} = 0, \qquad -\widehat{W}_{\widehat{z}\widehat{z}} = 0, \qquad && \text{at bottom,} \end{aligned} \tag{75}$$

where $\widehat{W}_E$ is the (non-dimensional) vertical velocity at the base of the top Ekman layer, which is negative for Ekman pumping in the subtropical gyre, and B_0 is a constant, proportional to the buoyancy difference across the domain. We obtain solutions numerically by Newton's method. The solutions here are obtained using about 1000 uniformly spaced grid points to span the domain, taking just a few seconds of computer time. Because of the boundary layer structure of the solutions employing a non-uniform grid would be even more efficient for this problem, but there is little point in designing a streamlined hat to reduce the effort of walking. These are shown in Figs. 10 and 11. The solutions do indeed display fronts, or boundary layers, for small diffusivity. If the wind forcing is zero (Fig. 11), the boundary layer is at the top of the fluid. If the wind forcing is non-zero, an internal boundary layer — a front — forms in the fluid interior with an adiabatic layer above and below. In the real ocean, where wind forcing is of course non-zero, the frontal region is known as the *internal thermocline.*

6.2 Boundary-layer analysis

The reasoning and the numerical solutions of the above sections suggest that the internal thermocline has a boundary-layer structure whose thickness decreases with κ. If the Ekman pumping at the top of the ocean is non-zero, the boundary layer is internal to the fluid. To learn more, let us perform a boundary layer analysis.

One-dimensional model

Let us now *assume* a steady two-layer structure of the form illustrated in Fig. 12, and that the dynamics are governed by (72) in a domain that extends from 0 to -1. The buoyancy thus varies rapidly only in an internal boundary layer of non-dimensional thickness $\widehat{\delta}$ located at $\widehat{z} = -h$; above and below this the buoyancy is assumed to be only very slowly

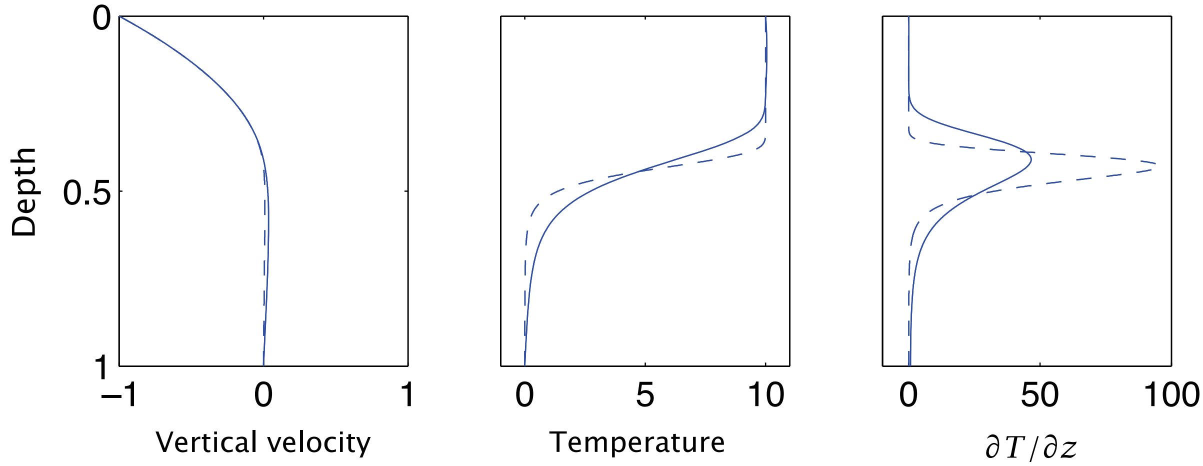

Figure 10: Solution of the one-dimensional thermocline equation, (72), with boundary conditions (75), for two different values of the diffusivity: $\widehat{\kappa} = 3.2 \times 10^{-3}$ (solid line) and $\widehat{\kappa} = 0.4 \times 10^{-3}$ (dashed line), in the domain $0 \leq \widehat{z} \leq -1$.

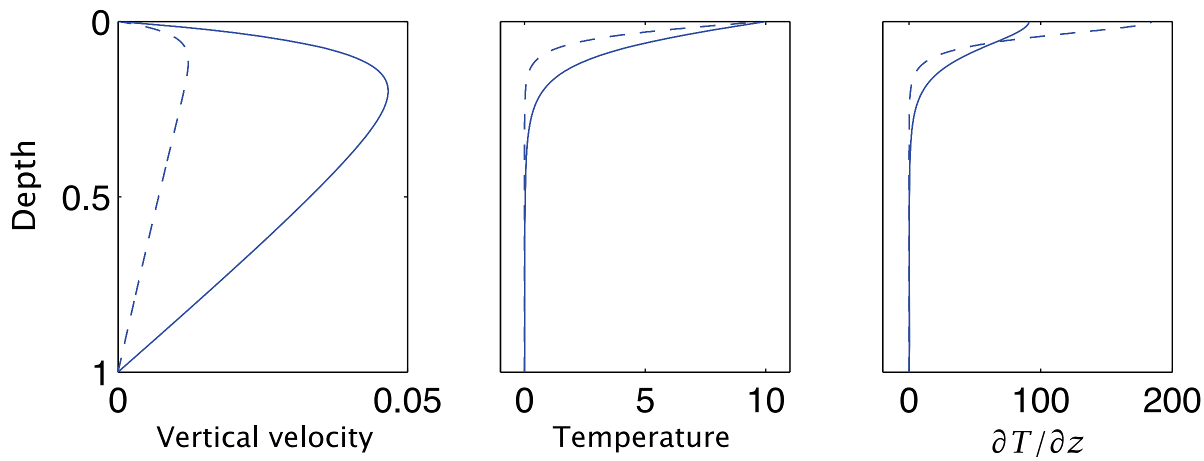

Figure 11: As for Fig. 10, but with no imposed Ekman pumping velocity at the upper boundary ($\widehat{W}_E = 0$), again for two different values of the diffusivity.

varying. Following standard boundary layer procedure we introduce a stretched boundary layer coordinate ζ where

$$\widehat{\delta}\zeta = \widehat{z} + h. \tag{76}$$

That is, ζ is the distance from $\widehat{z} = -h$, scaled by the boundary layer thickness $\widehat{\delta}$, and within the boundary layer ζ is an order-one quantity. We also let

$$\widehat{W}(\widehat{z}) = \widehat{W}_I(\widehat{z}) + \widetilde{W}(\zeta), \tag{77}$$

where $\widehat{W}_I$ is the solution away from the boundary layer and $\widetilde{W}$ is the boundary layer correction. Because the boundary layer is presumptively thin, $\widehat{W}_I$ is effectively constant through

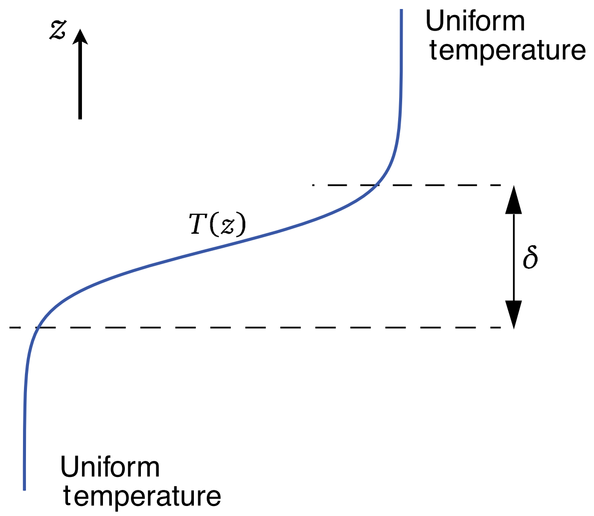

Figure 12: The simplified boundary-layer structure of the internal thermocline. In the limit of small diffusivity the internal thermocline forms a boundary layer, of thickness δ in the figure, in which the temperature and buoyancy change rapidly.

it and, furthermore, for $\widehat{z} < -h$, $\widehat{W}$ vanishes in the limit as $\kappa = 0$. We thus take $\widehat{W}_I = 0$ throughout the boundary layer. (The small diffusively-driven upwelling below the boundary layer is part of the boundary layer solution, not the interior solution.) Now, buoyancy varies rapidly in the boundary layer but it remains an order-one quantity throughout. To satisfy this we explicitly scale $\widetilde{W}$ in the boundary layer by writing

$$\widetilde{W}(\zeta) = \widehat{\delta}^2 B_0 A(\zeta), \tag{78}$$

where B_0 is defined by (75) and A is an order-one field. The derivatives of W are

$$\frac{\partial \widehat{W}}{\partial \widehat{z}} = \frac{1}{\widehat{\delta}}\frac{\partial \widetilde{W}}{\partial \zeta} = \widehat{\delta} B_0 \frac{\partial A}{\partial \zeta}, \qquad \frac{\partial^2 \widehat{W}}{\partial \widehat{z}^2} = B_0 \frac{\partial^2 A}{\partial \zeta^2}, \tag{79}$$

so that $\widehat{W}_{\widehat{z}\widehat{z}}$ is an order-one quantity. Far from the boundary layer the solution must be able to match the external conditions on temperature and velocity, (75); the buoyancy condition on $W_{\widehat{z}\widehat{z}}$ is satisfied if

$$A_{\zeta\zeta} \to \begin{cases} 1 & \text{as } \zeta \to +\infty \\ 0 & \text{as } \zeta \to -\infty. \end{cases} \tag{80}$$

On vertical velocity we require that $W \to (\widehat{z}/h + 1)W_E$ as $\zeta \to +\infty$, and $W \to$ constant as $\zeta \to -\infty$. The first matches the Ekman pumping velocity above the boundary layer, and the second condition produces the abyssal upwelling velocity, which as noted vanishes for $\kappa \to 0$.

Substituting (77) and (78) into (72) we obtain

$$B_0 A A_{\zeta\zeta\zeta} = \frac{\widehat{\kappa}}{\widehat{\delta}^3} A_{\zeta\zeta\zeta\zeta}. \tag{81}$$

Because all quantities are presumptively $\mathcal{O}(1)$, (81) implies that $\widehat{\delta} \sim (\widehat{\kappa}/B_0)^{1/3}$. We restore the dimensions of δ by using $\kappa = \widehat{\kappa}(HW_S)$ and $\Delta b = B_0 L f^2 W_S/(\beta H^2)$, where Δb is the dimensional buoyancy difference across the boundary layer [note that $b = M_{zz} = (x - 1)W_{zz} \sim LW_{zz} \sim LB_0 f^2 W_S/(\beta H^2)$ using (71)]. The dimensional boundary layer thickness, δ, is then given by

$$\delta \sim \left(\frac{\kappa f^2 L}{\Delta b \, \beta} \right)^{1/3}, \tag{82}$$

which is the same as the heuristic estimate (59). The dimensional vertical velocity scales as

$$W \sim \frac{\kappa}{\delta} \sim \kappa^{2/3} \left(\frac{\Delta b \, \beta}{f^2 L} \right)^{1/3}, \tag{83}$$

this being an estimate of strength of the upwelling velocity at the base of the thermocline and, more generally, the strength of the diffusively-driven component of meridional overturning circulation of the ocean.

The qualitative features of these models transcend their detailed construction, and in particular:

- the thickness of the internal thermocline increases with increasing diffusivity, and decreases with increasing buoyancy difference across it, and as the diffusivity tends to zero the thickness of the internal thermocline tends to zero.
- the strength of the upwelling velocity, and hence the strength of the meridional overturning circulation, increases with increasing diffusivity and increasing buoyancy difference.

The three-dimensional equations

We now apply boundary layer techniques to the three-dimensional M equation.[2] The main difference is that the depth of the boundary layer is now a function of x and y , so that the stretched coordinate ζ is given by

$$\widehat{\delta}\zeta = z + h(x, y). \tag{84}$$

[The coordinates (x, y, z) in this subsection are non-dimensional, but we omit their hats to avoid too cluttered a notation.] Just as in the one-dimensional case we rescale M in the boundary layer and write

$$M = B_0 \widehat{\delta}^2 \widehat{A}(x, y, \zeta), \tag{85}$$

where the scaling factor $\widehat{\delta}^2$ again ensures that the temperature remains an order-one quantity. In the boundary layer the derivatives of M become

$$\frac{\partial M}{\partial z} = \frac{1}{\widehat{\delta}} \frac{\partial A}{\partial \zeta}, \tag{86}$$

and

$$\frac{\partial M}{\partial x} = \widehat{\delta}^2 B_0 \left(\frac{\partial A}{\partial \zeta} \frac{\partial \zeta}{\partial x} + \frac{\partial A}{\partial x} \right) = \widehat{\delta}^2 B_0 \left(\frac{\partial A}{\partial \zeta} \frac{1}{\widehat{\delta}} \frac{\partial h}{\partial x} + \frac{\partial A}{\partial x} \right). \tag{87}$$

[2] Following Samelson (1999).

Substituting these into (65) we obtain, omitting the time-derivative,

$$
\begin{aligned}
\widehat{\delta}\left[\frac{1}{f}\left(A_{\zeta x}A_{\zeta\zeta y}-A_{\zeta y}A_{\zeta\zeta x}\right)+\frac{\beta}{f^2}A_xA_{\zeta\zeta\zeta}\right]+\frac{\beta}{f^2}h_xA_\zeta A_{\zeta\zeta\zeta}& \\
+\frac{1}{f}\left[h_x\left(A_{\zeta\zeta}A_{\zeta\zeta y}-A_{\zeta y}A_{\zeta\zeta\zeta}\right)+h_y\left(A_{\zeta x}A_{\zeta\zeta\zeta}-A_{\zeta\zeta}A_{\zeta\zeta x}\right)\right]& \\
=\frac{\kappa}{B_0\widehat{\delta}^2}A_{\zeta\zeta\zeta\zeta},&
\end{aligned}
\tag{88}
$$

where the subscripts on A and h denote derivatives. If $h_x = h_y = 0$, that is if the base of the thermocline is flat, then (88) becomes

$$
\frac{1}{f}\left[A_{\zeta x}A_{\zeta\zeta y}-A_{\zeta y}A_{\zeta\zeta x}\right]+\frac{\beta}{f^2}A_xA_{\zeta\zeta\zeta}=\frac{\kappa}{B_0\widehat{\delta}^3}A_{\zeta\zeta\zeta\zeta}. \tag{89}
$$

Since all the terms in this equation are, by construction, order one, we immediately see that the non-dimensional boundary layer thickness $\widehat{\delta}$ scales as

$$
\widehat{\delta}\sim\left(\frac{\kappa}{B_0}\right)^{1/3}, \tag{90}
$$

just as in the one-dimensional model. On the other hand, if h_x and h_y are order-one quantities then the dominant balance in (88) is

$$
\frac{1}{f}\left[h_x(A_{\zeta\zeta}A_{\zeta\zeta y}-A_{\zeta y}A_{\zeta\zeta\zeta})+h_y(A_{\zeta x}A_{\zeta\zeta\zeta}-A_{\zeta\zeta}A_{\zeta\zeta x})\right]=\frac{\kappa}{B_0\widehat{\delta}^2}A_{\zeta\zeta\zeta\zeta} \tag{91}
$$

and

$$
\widehat{\delta}\sim\left(\frac{\kappa}{B_0}\right)^{1/2}, \tag{92}
$$

confirming the heuristic scaling arguments. Thus, if the isotherm slopes are fixed independently of κ (for example, by the wind stress), then as $\kappa \to 0$ an internal boundary layer will form whose thickness is proportional to $\kappa^{1/2}$. We expect this to occur at the base of the main thermocline, with purely advective dynamics being dominant in the upper part of the thermocline, and determining the slope of the isotherms (i.e., the form of h_x and h_y). Interestingly, the balance in the three-dimensional boundary layer equation does not in general correspond locally to $wT_z \approx \kappa T_{zz}$. Both at $\mathcal{O}(1)$ and $\mathcal{O}(\delta)$ the horizontal advective terms in (88) are of the same asymptotic size as the vertical advection terms. In the boundary layer the thermodynamic balance is thus $\mathbf{u}\cdot\nabla_z T + wT_z \approx \kappa T_{zz}$, whether the isotherms are sloping or flat. We might have anticipated this, because the vertical velocity passes through zero within the boundary layer.

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