Upper Bounds for Convection in an Internally Heated Fluid Layer

Lu Lu
University of Michigan

1 Introduction

Finding the bounds of certain global quantities is an important approach in the theoretical study of turbulence. Particularly, maximum convective heat transport in Rayleigh-Bénard convection is a much-studied subject. The idea dates back to Malkus [1]. Following this idea, a rigorous upper bound of convective heat transport in Rayleigh-Bénard convection without continuity constraint was first derived by Howard [2], using a variational approach. In the same paper [2], a single-wavenumber boundary layer approximation is used to study the asymptotic solutions of the Euler-Lagrange equations. Busse [3] extended this asymptotic technique and introduced multi-α-solutions, which has been proved fruitful in studying other fluid dynamics problems [4], [5], [6], and this approach is reviewed in [7]. A new approach to derive the rigorous bounds on turbulent flow quantities, the background method (Deoring-Contantin approach), appeared in 1992 [8], and subsequently applied to Rayleigh-Bénard convection [9], [10]. In this project report, these two above-mentioned techniques are applied to bound the minimum average temperature in a fluid layer with internal heating.

2 Governing Equations

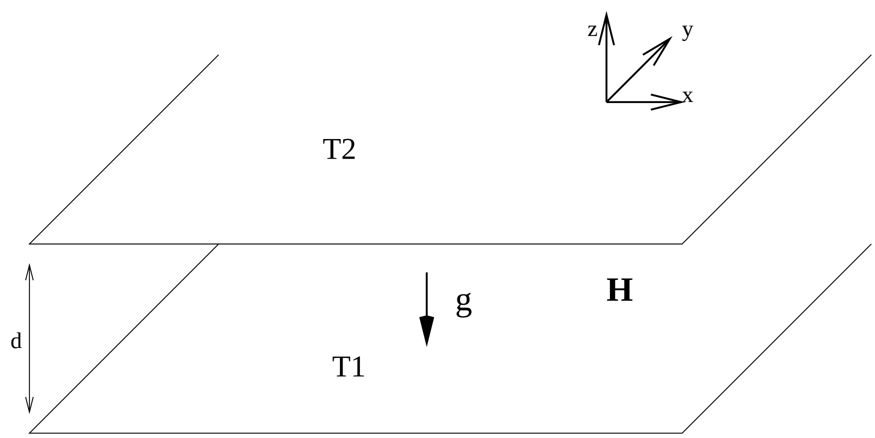

Figure 1: Geometry of convection with uniform internal heating.

The geometry is shown in Fig. (1). The setup is very similar to Rayleigh-Bénard convection. A fluid layer is confined between two parallel plates with a distance d. However,a uniform volumetric heat flux H(with unit $\frac{J}{s \cdot m^3}$) is pumped into the fluid layer. The upper and lower plates are held at fixed temperatures T_0 and T_1 respectively, and there is no restriction on the temperature difference $\Delta T = T_0 - T_1$. With internal heating, the governing equations are identical to Rayleigh-Bénard convection except for the additional term γ in the heat equation:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \nu\nabla^2\mathbf{u} + \hat{\mathbf{k}}g\alpha T, \tag{1a}$$

$$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = \kappa\nabla^2 T + \gamma, \tag{1b}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{1c}$$

with the boundary conditions

$$\mathbf{u}|_{z=0,1} = 0, \quad T|_{z=0} = T_0, \quad T|_{z=1} = T_1, \tag{1d}$$

where

$$\gamma = \frac{H}{\rho c}. \tag{1e}$$

By introducing $\frac{d^2}{\kappa}$ as the unit of time, d of length, $\frac{\kappa}{d}$ of velocity and $\frac{\kappa\nu}{g\alpha d^3}$ as the unit of temperature, the governing equations are put into the non-dimensional form

$$Pr^{-1}\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) + \nabla p = \nabla^2\mathbf{u} + T\hat{\mathbf{k}} \tag{2a}$$

$$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = \nabla^2 T + R \tag{2b}$$

where

$$Pr = \frac{\nu}{\kappa}, \quad R = \frac{g\alpha d^5\gamma}{\kappa^2\nu} \tag{2c}$$

R is the *heat Rayleigh number*, which is proportional to the internal heating rate. The boundary conditions in non-dimensional form are

$$\mathbf{u}|_{z=0,1} = 0, \quad T|_{z=0} = -T_0, \quad T|_{z=1} = 0, \tag{2d}$$

where $T_0 = \frac{g\alpha d^3}{\kappa\nu}|\Delta T|$, which shows that the non-dimensional T_0 is equivalent to the role of the Rayleigh number Ra. A negative non-dimensional temperature at the bottom plate means the upper plate is hotter. If $T|_{z=0}$ is 0, then two plates have the same temperature. The case when the bottom plate is hotter corresponds to a positive non-dimensional temperature at the bottom plate.

3 Linear and Energy Stability

A static solution can be found easily:

$$T = -\frac{1}{2}Rz^2 + (\frac{1}{2}R + T_0)z - T_0 \tag{3}$$

where $T|_{z=0} = -T_0$. Fig (2) shows several possibilities of the of the static solutions. When two plates have the same temperature, the profile is symmetric about $z = \frac{1}{2}$. The maximum

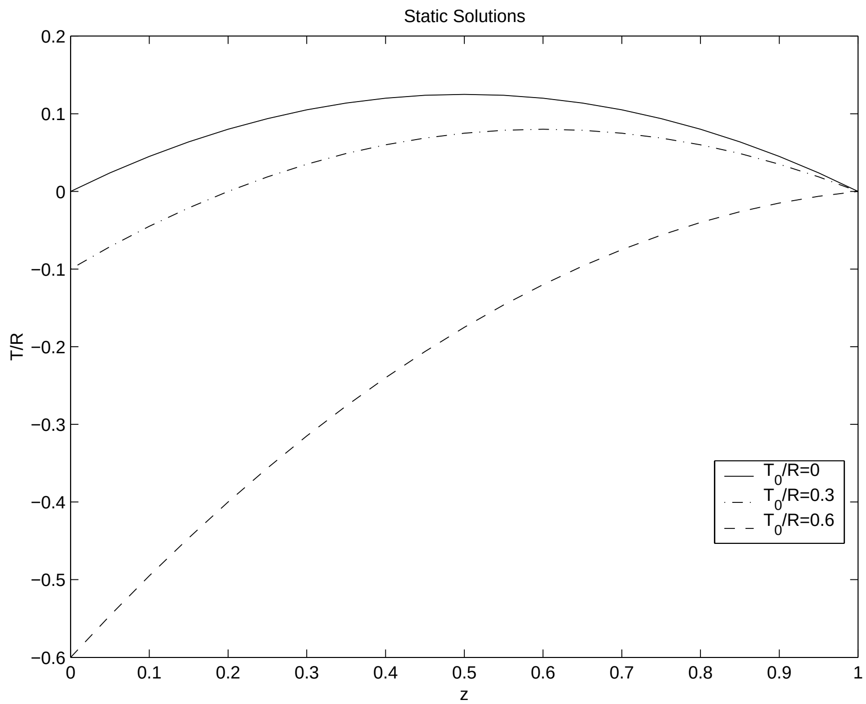

Figure 2: Static solutions of an internally heated fluid layer.

temperature occurs at $z = \frac{1}{2}$. Thus temperature gradient for the lower half of the fluid layer is positive, corresponding to a stable stratification. While in the upper half, the fluid layer is unstably stratified. This profile provides the possibility, when the internal heating rate R is big enough, that convection starts in the upper half of the fluid layer. When T_0 increases, but not exceeding $\frac{1}{2}R$,the unstable stratification persists with the position of maximum temperature shifting toward the upper plate. Eventually, when $T_0 \geq \frac{1}{2}R$, the temperature gradient is positive everywhere which suggests that the fluid layer is linearly stable, a fact that will be established in the analysis of linear stability bellow.

Let $T = -\frac{1}{2}Rz^2 + (\frac{1}{2}R + T_0)z - T_0 + \theta$, where θ is the temperature disturbance. We can write down the linearized equations governing the growth of the disturbances θ and $\mathbf{u}$

$$Pr^{-1}\left(\frac{\partial \mathbf{u}}{\partial t}\right) + \nabla p = \nabla^2 \mathbf{u} + \theta \hat{\mathbf{k}} \tag{4a}$$

$$\frac{\partial \theta}{\partial t} = \left(z - \frac{1}{2} - T_0\right) w + \nabla^2 \theta \tag{4b}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{4c}$$

with the boundary conditions

$$\mathbf{u}|_{z=0,1} = 0, \qquad \theta|_{z=0,1} = 0. \tag{4d}$$

Taking the z component of the **curl curl** of the $\mathbf{u}$ equation leads to

$$Pr^{-1}\frac{\partial}{\partial t}\nabla^2 w = \nabla^4 w + \Delta_2 \theta, \tag{5}$$

where Δ_2 is the horizontal Laplacian, defined as

$$\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{6}$$

Then we make the ansatz

$$w = e^{st} f(x,y) W(z), \qquad \theta = e^{st} f(x,y)\Theta(z), \tag{7a}$$

and

$$\Delta_2 f(x,y) = -a^2 f(x,y). \tag{7b}$$

The exchange of stability holds in this problem ([11]). Hence to determine the critical R, we can set $s = 0$:

$$\left(D^2 - a^2\right)^2 W = a^2 R\Theta \tag{8a}$$

$$\left(D^2 - a^2\right)\Theta = -\left(z - \frac{1}{2} - T_0\right) W \tag{8b}$$

with the boundary conditions

$$W = DW = \theta = 0 \quad \text{at} \quad z = 0, 1. \tag{8c}$$

In Rayleigh-Bénard convection, the equations for linear stability turn out to be identical to those of energy stability. Thus it is of interest to investigate the energy stability of the internally heated fluid layer. The equations for the disturbances $\mathbf{u}$ and θ are

$$Pr^{-1}\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p\right) = \nabla^2\mathbf{u} + \theta\hat{\mathbf{k}}, \tag{9a}$$

$$\frac{\partial\theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = \left(Rz - \frac{1}{2}R - T_0\right) w + \nabla^2\theta, \tag{9b}$$

$$\nabla\cdot\mathbf{u} = 0. \tag{9c}$$

Using $\mathbf{u}\cdot$ equation (9a), $\theta\times$ equation (9b), we integrate over the whole layer to get

$$\frac{1}{2}\frac{d}{dt}\frac{\left\langle \mathbf{u}^2\right\rangle}{Pr} = -\left\langle |\nabla\mathbf{u}|^2\right\rangle + \langle w\theta\rangle, \tag{10a}$$

$$\frac{1}{2}\frac{d}{dt}\frac{\left\langle\theta^2\right\rangle}{R} = \left\langle\left(z - \frac{1}{2} - \frac{T_0}{R}\right) w\theta\right\rangle - \frac{\left\langle|\nabla\theta|^2\right\rangle}{R}. \tag{10b}$$

Introducing a balance parameter s and adding the above equations yield

$$\frac{d}{dt}\frac{1}{2}\left\{s\cdot\frac{\left\langle\mathbf{u}^2\right\rangle}{Pr} + \frac{\left\langle\theta^2\right\rangle}{R}\right\} = -\left\{s\cdot\left\langle|\nabla\mathbf{u}|^2\right\rangle - \left\langle\left(z - \frac{1}{2} - \frac{T_0}{R} + s\right) w\theta\right\rangle + \frac{\left\langle|\nabla\theta|^2\right\rangle}{R}\right\} \tag{11}$$

Let

$$\lambda = s\cdot\frac{\left\langle|\nabla\mathbf{u}|^2\right\rangle - \left\langle\left(z - \frac{1}{2} - \frac{T_0}{R} + s\right) w\theta\right\rangle + \frac{\left\langle|\nabla\theta|^2\right\rangle}{R}}{\frac{1}{2}\left\{s\cdot\frac{\left\langle\mathbf{u}^2\right\rangle}{Pr} + \frac{\left\langle\theta^2\right\rangle}{R}\right\}}, \tag{12}$$

then if $\min \lambda \leq 0$ among the fields with

$$\nabla \cdot \mathbf{u} = 0, \quad \mathbf{u}|_{z=0,1} = 0, \quad \theta|_{z=0,1} = 0,$$

the disturbances decay exponentially in time, which implies energy stability. To determine the critical value of R, we need the Euler-Lagrange equations:

$$s \cdot \nabla^2 \mathbf{u} + \frac{1}{2}\left(z - \frac{1}{2} - \frac{T_0}{R} + s\right)\theta\hat{\mathbf{k}} + \nabla p + \lambda\frac{\mathbf{u}}{R} = 0 \tag{13a}$$

$$\nabla^2\theta + \frac{1}{2}\left(z + \frac{1}{2} - \frac{T_0}{R}\right)w + \lambda\theta = 0 \tag{13b}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{13c}$$

Setting $\lambda = 0$ and applying the ansatz (7) yields

$$s \cdot (D^2 - a^2)^2 W - \frac{1}{2}\left(z - \frac{1}{2} - \frac{T_0}{R} + s\right)a^2\Theta = 0 \tag{14a}$$

$$(D^2 - a^2)\Theta + \frac{1}{2}\left(z - \frac{1}{2} - \frac{T_0}{R} + s\right)RW = 0 \tag{14b}$$

with the boundary conditions

$$W = DW = \Theta = 0 \qquad \text{at} \qquad z = 0, 1 \tag{14c}$$

For each s, the critical $R_E(s)$ can be found by minimizing the eigenvalue R with respect to the horizontal wave-number a. Then s can be chosen such that it gives a maximal

$$R_E(T_0) = \max_s \min_a R(a, s)$$

.

The linear stability and energy stability equations are solved numerically, using a Chebyshev spectral method. The results are shown in Fig. (3) and Fig. (4). The first figure (with the bottom plate colder) shows the convergence of the linear stability to the straight line $T_0 = \frac{1}{2}R$, which corresponds the temperature difference beyond which the entire fluid layer is stably stratified. However, the energy stability increases with T_0 but does not converge to the line $T_0 = \frac{1}{2}R$. In the case of a hotter bottom plate, both the linear stability and energy stability lines converge to the critical Rayleigh number 1708 for Rayleigh-Bénard convection. This is expected since the fluid layer is linearly unstable even without any internal heating in that case. In both figures, the critical R for energy stability is lower than that of linear stability. This suggests the possibility of subcritical bifurcations.

As shown above, even though the bottom plate is colder and the lower half of the fluid is stably stratified, the system can still be linearly unstable. Once the convection starts, it tries to lower the average temperature of the fluid layer. Thus it is of interest to investigate the scaling of the minimum average temperature. In the following two sections, this scaling will be studied with the background method and the multi-α-solution approach.

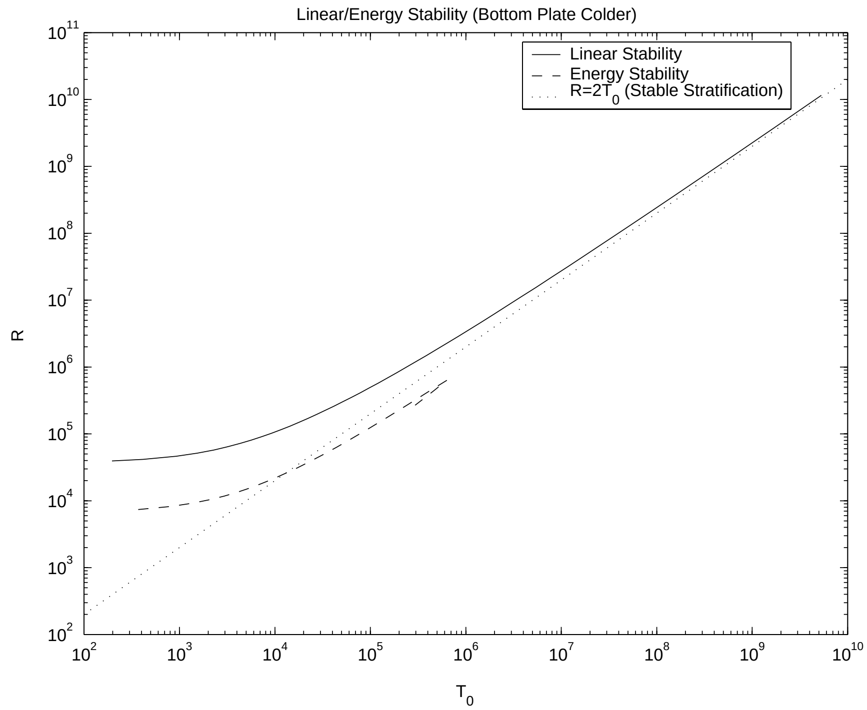

Figure 3: Linear and energy stability.

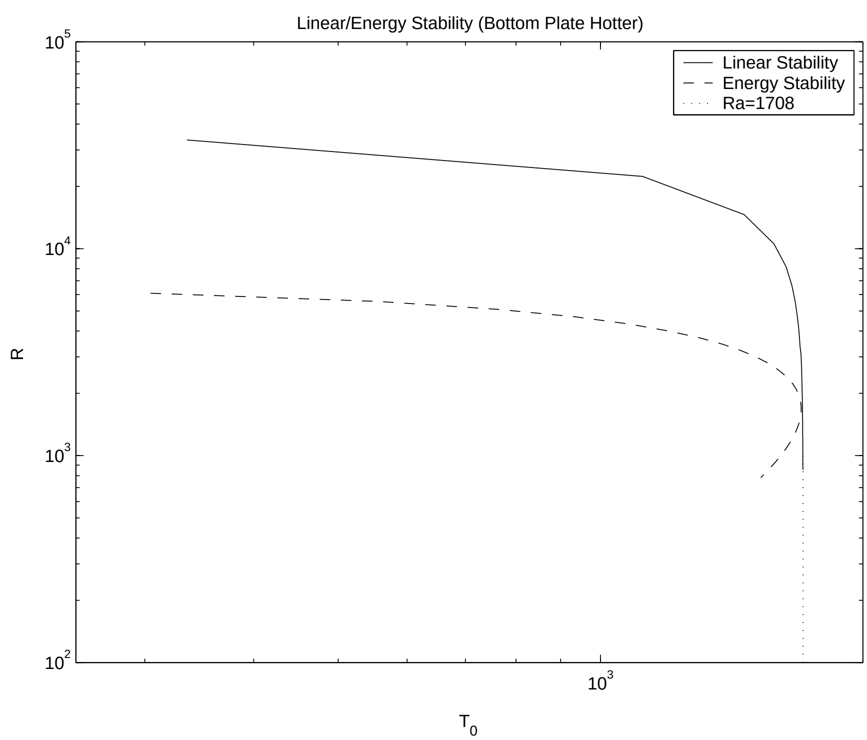

Figure 4: Linear and energy stability (Hotter lower plate).

4 Background method

To apply the background method, first we decompose the temperature field T into a background profile $\tau(z)$ and a fluctuating part $\theta(x,y,z,t)$:

$$T = \tau(z) + \theta(x,y,z,t). \tag{15}$$

The boundary conditions of T are contained in $\tau(z)$:

$$\tau(0) = \tau(1) = 0. \tag{16}$$

The velocity field $\mathbf{u}$ still satisfies divergence-free and no-slip boundary conditions:

$$\nabla \cdot \mathbf{u} = 0, \qquad \mathbf{u}|_{z=0,1} = 0. \tag{17}$$

Then the governing equations (2) become

$$Pr^{-1}\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) + \nabla p = \nabla^2\mathbf{u} + \tau\hat{\mathbf{k}} + T\hat{\mathbf{k}}, \tag{18a}$$

$$\frac{\partial \theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = \nabla^2\theta + \tau'' + R - w\tau'. \tag{18b}$$

Define $< \cdot >= \lim_{t=1\to\infty} \frac{1}{t}\int_0^t dt' \frac{1}{L_x L_y}\int dxdydz\cdot$ to be the average over both space and time. Then $\langle \mathbf{u}\cdot(18a)\rangle$ yields

$$\left\langle |\nabla\mathbf{u}|^2 \right\rangle = \langle w\theta\rangle, \tag{19}$$

and $\langle\theta\cdot(18b)\rangle$, $\langle\tau\cdot(18b)\rangle$ yield, respectively,

$$\left\langle w\theta\tau' \right\rangle = -\left\langle |\nabla\theta|^2 \right\rangle - \left\langle \theta'\tau' \right\rangle + R\langle\theta\rangle, \tag{20}$$

$$-\left\langle w\theta\tau' \right\rangle = -\left\langle \theta'\tau' \right\rangle + R\langle\tau\rangle - \left\langle \tau'^2 \right\rangle. \tag{21}$$

The difference of the above two identities is

$$R\left(\langle\theta\rangle - \langle\tau\rangle\right) = \left\langle |\nabla\theta|^2 \right\rangle + 2\left\langle w\theta\tau' \right\rangle - \left\langle \tau'^2 \right\rangle. \tag{22}$$

Since $\langle T\rangle = \langle\tau\rangle + \langle\theta\rangle$, we have

$$R\langle T\rangle = \left\langle |\nabla\theta|^2 \right\rangle + 2\left\langle w\theta\tau' \right\rangle + 2R\langle\tau\rangle - \left\langle \tau'^2 \right\rangle. \tag{23}$$

The identity (23) can also be written as

$$0 = a\left\langle |\nabla\mathbf{u}|^2 \right\rangle - a\langle w\theta\rangle, \tag{24}$$

where a is a positive number used as an optimization parameter to be adjusted to yield the best prefactor. Adding equation (24) to equation (23) enables us to express the average temperature as follows:

$$R\langle T\rangle = 2R\langle\tau\rangle - \left\langle \tau'^2 \right\rangle + H, \tag{25}$$

where

$$H = \left\{ \left\langle |\nabla\theta|^2 \right\rangle + \left\langle (2\tau' - a)w\theta \right\rangle + a \left\langle |\nabla \mathbf{u}|^2 \right\rangle \right\}. \tag{26}$$

We require the functional H to be semi positive definite among the fields $\mathbf{u}$ and θ satisfying

$$\nabla \cdot \mathbf{u} = 0, \quad , \mathbf{u}|z = 0, 1 = 0, \quad \theta|_{z=0,1} = 0.$$

A bound can readily be obtained by applying the following background profile

$$\tau(z) = \begin{cases} \frac{a}{2} z, & 0 \le z \le 1 - \delta \\ \frac{a(1-\delta)}{2\delta}(1 - z), & 1- \quad \delta < z \le 1 \end{cases} \tag{27}$$

With this background profile, $2\tau' - a$ vanishes in the interval $1 \le z \le 1 - \delta$. Thus we only need to estimate $|\langle (2\tau' - a)w\theta \rangle| = \frac{a}{\delta} \langle w\theta \rangle$ in the region $1 - \delta < z \le 1$, and adjust δ and a to make H semi positive definite. Following ([9]), an estimate is given by

$$|\left\langle (2\tau' - a)w\theta \right\rangle| \le \frac{a}{\delta} \frac{\delta^2}{4} \left[\frac{c}{4} \left\langle |\nabla \mathbf{u}|^2 \right\rangle + \frac{\delta^2}{4c} \left\langle |\nabla\theta|^2 \right\rangle \right]. \tag{28}$$

Then

$$\left\{ \left\langle |\nabla\theta|^2 \right\rangle + \left\langle (2\tau' - a)w\theta \right\rangle + a \left\langle |\nabla \mathbf{u}|^2 \right\rangle \right\} \ge \left(a - \frac{a}{\delta} \frac{\delta^2 c}{16} \right) \left\langle |\nabla \mathbf{u}|^2 \right\rangle + \left(1 - \frac{a}{\delta} \frac{\delta^2}{4c} \right) \left\langle |\nabla\theta|^2 \right\rangle. \tag{29}$$

The choice

$$\frac{a\delta^2}{64} = 1, \qquad c = \frac{a\delta}{4} \tag{30}$$

makes the right hand side equal to zero, which ensures $H \ge 0$. Then

$$\begin{aligned} R\langle T \rangle &\ge 2R\langle \tau \rangle - \left\langle \tau'^2 \right\rangle \\ &= \frac{aR}{2}(1 - \delta) - \frac{a^2}{4} \frac{1 - \delta}{\delta}. \end{aligned} \tag{31}$$

Minimizing the right hand side with respect to a gives the optimal choice of the parameter a,

$$a = \delta R. \tag{32}$$

And δ can be solved from equation (30),

$$\delta = \frac{4}{\sqrt[3]{R}}. \tag{33}$$

Finally,

$$\begin{aligned} \langle T \rangle &\ge 2\langle \tau \rangle - \frac{\left\langle \tau'^2 \right\rangle}{R} \\ &= \frac{1}{4} R\delta(1 - \delta) \\ &= R^{2/3} - 4R^{1/3}. \end{aligned} \tag{34}$$

This shows that as $R \to \infty$, $\langle T \rangle \sim R^{2/3}$, which will be verified in the next section by using the multi-α solution approach.

5 Howard-Busse Approach

5.1 Formulation of the Functional

The governing equations are repeated here again

$$Pr^{-1}\left(\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}\right)+\nabla p=\nabla^2\mathbf{u}+T\hat{\mathbf{k}}, \tag{35a}$$

$$\frac{\partial T}{\partial t}+\mathbf{u}\cdot\nabla T=\nabla^2 T+R, \tag{35b}$$

with

$$\mathbf{u}|_{z=0,1}=0,\quad T(0)=-T_0,\quad T(1)=0. \tag{35c}$$

Assume the turbulence is statistically stationary and the velocity field $\mathbf{v}$ and temperature T can be decomposed into a time independent horizontal average and a meanless fluctuating part:

$$T=\overline{T}+\theta,\qquad \overline{\theta}=0\quad\text{with}\quad \overline{\mathbf{u}}=0,\quad \overline{\theta}=0.. \tag{36}$$

After taking horizontal average of (35b), we have

$$\frac{d\overline{w\theta}}{dz}=\frac{d^2\overline{T}}{d^2z}+R. \tag{37}$$

Integrate once,

$$\overline{w\theta}=\frac{d\overline{T}}{dz}+Rz+c. \tag{38}$$

The integration constant c is determined by integrating above equation over $[0,1]$. This yields

$$\frac{d\overline{T}}{dz}=\overline{w\theta}-\langle w\theta\rangle-R\left(z-\frac{1}{2}\right)+T_0 \tag{39}$$

With the decomposition (36), equation (35b) can be written as

$$\frac{\partial\theta}{\partial t}+w\frac{d\overline{T}}{dz}+\mathbf{u}\cdot\nabla\theta=\nabla^2\theta+\frac{d^2\overline{T}}{d^2z}+R=\nabla^2\theta+\frac{d\overline{w\theta}}{dz} \tag{40}$$

where equation (37) has been used. Multiply both sides with θ and integrate over the bulk, we have

$$\left\langle w\theta\frac{d\overline{T}}{dz}\right\rangle=-\left\langle|\nabla\theta|^2\right\rangle. \tag{41}$$

Together with equation (39), the above expression yields

$$R\left\langle\left(z-\frac{1}{2}\right)w\theta\right\rangle=\left\langle|\nabla\theta|^2\right\rangle+\left\langle(\overline{w\theta}-<w\theta>)^2\right\rangle+T_0\langle w\theta\rangle \tag{42}$$

Another power integral is derived by multiplying equation (35a) by $\mathbf{u}$ and integrating over the bulk:

$$\left\langle|\nabla\mathbf{u}|^2\right\rangle = \langle w\theta\rangle \tag{43}$$

We can find an expression of the average temperature by multiplying equation (39) by z and integrate over $[0,1]$:

$$\langle T\rangle = -\left\langle\left(z-\frac{1}{2}\right)w\theta\right\rangle + \frac{1}{12}R - \frac{1}{2}T_0 \tag{44}$$

In summary, we have the following expressions:

$$\left\langle|\nabla\mathbf{u}|^2\right\rangle = \langle w\theta\rangle \tag{45}$$

$$R\left\langle\left(z-\frac{1}{2}\right)w\theta\right\rangle = \left\langle|\nabla\theta|^2\right\rangle + \left\langle(\overline{w\theta}-\langle w\theta\rangle)^2\right\rangle + T_0\langle w\theta\rangle \tag{46}$$

$$\langle T\rangle = -\left\langle\left(z-\frac{1}{2}\right)w\theta\right\rangle + \frac{1}{12}R - \frac{1}{2}T_0 \tag{47}$$

From equation (46), R can be expressed as

$$R = \frac{\left\langle|\nabla\theta|^2\right\rangle + <(\overline{w\theta}-\langle w\theta\rangle)^2> + T_0\langle w\theta\rangle}{\left\langle\left(z-\frac{1}{2}\right)w\theta\right\rangle}. \tag{48}$$

Substitute the above expression into equation (47),

$$\begin{aligned}\langle T\rangle &= -\left\langle\left(z-\frac{1}{2}\right)w\theta\right\rangle + \frac{1}{12}\frac{\left\langle|\nabla\theta|^2\right\rangle + \left\langle(\overline{w\theta}-\langle w\theta\rangle)^2\right\rangle + T_0\langle w\theta\rangle}{\left\langle\left(z-\frac{1}{2}\right)w\theta\right\rangle} - \frac{1}{2}T_0 \\ &= \frac{\left\langle|\nabla\theta|^2\right\rangle + \left\langle(\overline{w\theta}-<w\theta\rangle)^2> -12\left\langle\left(z-\frac{1}{2}\right)w\theta\right\rangle}{12\left\langle\left(z-\frac{1}{2}\right)w\theta\right\rangle^2} + T_0\left(\frac{\langle w\theta\rangle}{12\left\langle\left(z-\frac{1}{2}\right)w\theta\right\rangle} - \frac{1}{2}\right).\end{aligned} \tag{49}$$

Let

$$h(z) = \sqrt{12}\left(z-\frac{1}{2}\right). \tag{50}$$

Notice that

$$\langle h\rangle = 0, \quad \left\langle h^2\right\rangle = 1. \tag{51}$$

This property of $h(z)$ yields the following identity:

$$\left\langle\left(\overline{w\theta}-h\langle hw\theta\rangle-\langle w\theta\rangle\right)^2\right\rangle = \left\langle\overline{w\theta}^2\right\rangle - \langle w\theta\rangle^2 - \langle hw\theta\rangle^2. \tag{52}$$

Together with equation (45), the average temperature can be expressed as follows,

$$\begin{aligned}<\sqrt{12}T> &= \frac{\left\langle|\nabla\theta|^2\right\rangle + <(\overline{w\theta}-<w\theta>)^2> - \langle hw\theta\rangle^2}{\langle hw\theta\rangle} + T_0\left(\frac{<w\theta>}{\langle hw\theta\rangle} - \sqrt{3}\right) \\ &= \frac{\left\langle|\nabla\theta|^2\right\rangle\left\langle|\nabla\mathbf{u}|^2\right\rangle}{<w\theta><hw\theta>} + \frac{<(\overline{w\theta}-h<hw\theta>-<w\theta>)^2>}{<hw\theta>} + T_0\left(\frac{<w\theta>}{\langle hw\theta\rangle} - \sqrt{3}\right).\end{aligned} \tag{53}$$

Thus the variational problem ($T_0 = 0$ case) can be formulated as
Given $\mu =< hw\theta >$, find the minimum of the functional

$$\mathcal{F} = \frac{\langle|\nabla\theta|^2\rangle \langle|\nabla\mathbf{u}|^2\rangle}{\langle hw\theta\rangle \langle w\theta\rangle} + \mu\frac{\langle(\overline{w\theta} - h\,\langle hw\theta\rangle - \langle w\theta\rangle)^2\rangle}{\langle hw\theta\rangle^2} \tag{54}$$

among the $\mathbf{u}, \theta$ fields with

$$\nabla\cdot\mathbf{u} = 0, \qquad \mathbf{u}|_{z=0,1} = 0, \qquad \theta|_{z=0,1} = 0, \tag{55}$$

where

$$w = \mathbf{u}\cdot\hat{\mathbf{k}}, \qquad h(z) = \sqrt{12}\left(z - \frac{1}{2}\right).$$

Since the functional $\mathcal{F}$ is homogeneous in both w and θ, we can impose two normalization conditions

$$\langle hw\theta\rangle = 1, \qquad \langle w^2\rangle = \langle\theta^2\rangle. \tag{56}$$

5.2 Multi-α Solution

We are seeking the minimum of the functional $\mathcal{F}$ as $\mu\to\infty$. This implies that $\overline{w\theta} = h + \langle w\theta\rangle$ (here and in the following discussion the normalization condtions (55) have been assumed.) in most of the interval $0 < z < 1$, which makes the second term in the functional vanish in this interval. Only near the boundary $z = 0, 1$ the boundary conditions prevent a close appoach of $\overline{w\theta}$ to $h + \langle w\theta\rangle$. And the contribution to the functional is thus from possible boundary layers at $z = 0, 1$. Since $h(1) + \langle w\theta\rangle = \sqrt{3} + \langle|\nabla\mathbf{u}|^2\rangle > 0$ (equation (45 and definition (50)), there must be a boundary layer at $z = 1$. At $z = 0$, $h(0) + \langle w\theta\rangle = -\sqrt{3} + \langle w\theta\rangle$ is indefinite. Thus the existence of a boundary layer at $z = 0$ depends on whether $h(0) + \langle w\theta\rangle$ is zero. Without loss of generality, we assume there are two boundary layers at $z = 0, 1$ respectively, and make the ansatz

$$w = \sum w_n\phi_n + w_n^*\phi_n^*, \qquad \theta = \sum\theta_n\phi_n + \theta_n^*\phi_n^*, \tag{57}$$

where ϕ_n and ϕ_n^* satify

$$\Delta_2\phi_n = -\alpha_n^2\phi_n, \qquad \Delta_2\phi_n^* = -\alpha_n^{*2}\phi_n^*. \tag{58}$$

We introduce the following boundary layer variables:

$$w = \begin{cases}\mu^{-p_n}\hat{w}(\zeta_n) & \text{for } 1 - z = O(\mu^{-r_n}),\\ \mu^{-s_n}\tilde{w}(\zeta_{n-1}) & \text{for } 1 - z = O(\mu^{-r_{n-1}})\end{cases} \tag{59}$$

$$\theta = \begin{cases}\mu^{p_n}\hat{\theta}(\zeta_n) & \text{for } 1 - z = O(\mu^{-r_n}),\\ \mu^{s_n}\tilde{\theta}(\zeta_{n-1}) & \text{for } 1 - z = O(\mu^{-r_{n-1}})\end{cases} \tag{60}$$

$$w^* = \begin{cases}\mu^{-p_n}\hat{w}^*(\zeta_n^*) & \text{for } z = O(\mu^{-r_n}),\\ \mu^{-s_n}\tilde{w}^*(\zeta_{n-1}^*) & \text{for } z = O(\mu^{-r_{n-1}})\end{cases} \tag{61}$$

$$\theta^* = \begin{cases}\mu^{p_n}\hat{\theta}^*(\zeta_n^*) & \text{for } z = O(\mu^{-r_n}),\\ \mu^{s_n}\tilde{\theta}^*(\zeta_{n-1}^*) & \text{for } z = O(\mu^{-r_{n-1}})\end{cases} \tag{62}$$

(63)

where

$$\zeta_n = (1-z)\mu^{r_n}, \qquad \zeta_n^* = z\mu^{r_n} \tag{64}$$

The boudnary layer structure is such that in the interior

$$\tilde{w}_1\tilde{\theta}_1 + \tilde{w}_1^*\tilde{\theta}_1^* \approx h + \langle w\theta \rangle\,, \tag{65}$$

and in the boundary layers

$$\tilde{w}_n\tilde{\theta}_n + \hat{w}_{n-1}\hat{\theta}_{n-1} \approx h_1 + \langle w\theta \rangle\,, \qquad \tilde{w}_n^*\tilde{\theta}_n^* + \hat{w}_{n-1}^*\hat{\theta}_{n-1}^* \approx h_0 + \langle w\theta \rangle\,, \qquad \text{for} \quad n = 1, \ldots, N-1, \tag{66}$$

where

$$h_0 = h(0) = -\sqrt{3}, \qquad h_1 = h(1) = \sqrt{3}.$$

With boundary layer approximations, the functional becomes:

$$\begin{aligned}
\hat{\mathcal{F}}_N = \frac{1}{\langle w\theta\rangle} &\left\{ \sum_1^N \mu^{2p_n+r_n} \left(\int_0^\infty \hat{\theta}_n'^{\,2} d\zeta_n + \int_0^\infty \hat{\theta}_n^{*\prime 2} d\zeta_n^* \right) \right. \\
&\left. + \sum_2^N \mu^{q_n-r_n+2s_n} \left(b_n^2 \int_0^\infty \tilde{\theta}_n^2 d\zeta_{n-1} + b_n^{*2} \int_0^\infty \tilde{\theta}_n^{*2} d\zeta_{n-1}^* \right) + \mu^{q_1} \left(b_1^2 \left\langle \tilde{\theta}_1^2 \right\rangle + b_1^{*2} \left\langle \tilde{\theta}_1^{*2} \right\rangle \right) \right\} \\
&\cdot \left\{ \sum_1^N \mu^{3r_n-2p_n-q_n} \left(\frac{1}{b_n^2} \int_0^\infty \hat{w}_n''^2 d\zeta_n + \frac{1}{b_n^{*2}} \int_0^\infty \hat{w}_n^{*\prime\prime 2} d\zeta_n^* \right) \right. \\
&\left. + \sum_2^N \mu^{q_n-r_{n-1}-2s_n} \left(b_n^2 \int_0^\infty \tilde{w}_n^2 d\zeta_{n-1} + b_n^{*2} \int_0^\infty \tilde{w}_n^{*2} d\zeta_{n-1}^* \right) \mu^{q_1} \left(b_1^2 < \tilde{w}_1^2 > + b_1^{*2} \left\langle \tilde{w}_1^{*2} \right\rangle \right) \right\} \\
&+ \left\{ \mu^{1-r_N} \left(\int_0^\infty (\hat{w}_N\hat{\theta}_N - h_1 - < w\theta >)^2 d\zeta_N + \int_0^\infty (\hat{w}_N^*\hat{\theta}_N^* - h_0 - < w\theta >)^2 d\zeta_N \right) \right\}
\end{aligned} \tag{67}$$

Balancing the exponents in the above exrepssion yields

$$r_n = \frac{1-4^{-n}}{3-4^{-n}}, \quad q_n = \frac{2-4^{-n}}{3-4^{-n}}, \quad s_n = 0, \quad 2p_n = \frac{4^{-n}}{3-4^{-n}}. \tag{68}$$

Then we have

$$\hat{\mathcal{F}}_N = \mu^{\frac{2}{3-4^{-N}}} F_N \tag{69}$$

where

$$F_N = \frac{1}{\langle w\theta\rangle} \left\{ \sum_1^N \left(\int_0^\infty \hat{\theta}_n'^{\,2} d\zeta_n + \int_0^\infty \hat{\theta}_n^{*\prime 2} d\zeta_n^* \right) \right.$$

$$\left. \sum_2^N \left(b_n^2 \int_0^\infty \tilde{\theta}_n^2 d\zeta_{n-1} + b_n^{*2} \int_0^\infty \tilde{\theta}_n^{*2} d\zeta_{n-1}^* \right) + \left(b_1^2 < \tilde{\theta}_1^2 > + b_1^{*2} < \tilde{\theta}_1^{*2} > \right) \right\}$$

$$\cdot \left\{ \sum_1^N \left(\frac{1}{b_n^2} \int_0^\infty \hat{w}_n''^2 d\zeta_n + \frac{1}{b_n^{*2}} \int_0^\infty \hat{w}_n^{*\prime\prime 2} d\zeta_n^* \right) \right.$$

$$\left. \sum_2^N \left(b_n^2 \int_0^\infty \tilde{w}_n^2 d\zeta_{n-1} + b_n^{*2} \int_0^\infty \tilde{w}_n^{*2} d\zeta_{n-1}^* \right) + \left(b_1^2 < \tilde{w}_1^2 > + b_1^{*2} < \tilde{w}_1^{*2} > \right) \right\}$$

$$+ \left\{ \int_0^\infty (\hat{w}_N \hat{\theta}_N - h_1 - < w\theta >)^2 d\zeta_N + \int_0^\infty (\hat{w}_N^* \hat{\theta}_N^* - h_0 - < w\theta >)^2 d\zeta_N^* \right\}. \quad (70)$$

Now the Euler-Lagrange equatins for the functional F_N can be written down:

$$\frac{1}{\langle w\theta\rangle} \frac{D_\theta}{b_n^2} \hat{w}_n^{(4)} - \mu^{r_N - r_n} \hat{\theta}_n \left(h_1 + \langle w\theta\rangle - \hat{w}_n \hat{\theta}_n - \tilde{w}_{n+1} \tilde{\theta}_{n+1} \right) = 0, \quad (71)$$

$$\frac{1}{\langle w\theta\rangle} D_w \hat{\theta}_n'' + \mu^{r_N - r_n} \hat{w}_n \left(h_1 + < w\theta > - \hat{w}_n \hat{\theta}_n - \tilde{w}_{n+1} \tilde{\theta}_{n+1} \right) = 0, \qquad n = 1, \ldots, N \quad (72)$$

$$\frac{b_{n+1}^2}{\langle w\theta\rangle} D_\theta \tilde{w}_{n+1} - \mu^{r_N - r_n} \tilde{\theta}_{n+1} \left(h_1 + < w\theta > - \hat{w}_n \hat{\theta}_n - \tilde{w}_{n+1} \tilde{\theta}_{n+1} \right) = 0, \quad (73)$$

$$\frac{b_{n+1}^2}{\langle w\theta\rangle} D_w \tilde{\theta}_{n+1} - \mu^{r_N - r_n} \tilde{w}_{n+1} \left(h_1 + \langle w\theta\rangle - \hat{w}_n \hat{\theta}_n - \tilde{w}_{n+1} \tilde{\theta}_{n+1} \right) = 0, \qquad n = 1, \ldots, N-1 \quad (74)$$

And for $\tilde{w}_1$, $\tilde{\theta}_1$,

$$\frac{D_\theta}{\langle w\theta\rangle} b_1^2 \tilde{w}_1 - \tilde{\theta}_1 \left\{ \frac{D_\theta D_w}{2 \langle w\theta\rangle^2} (h \langle w\theta\rangle + 1) + \mu^{r_N} (h + \langle w\theta\rangle - \tilde{w}_1 \tilde{\theta}_1 - \tilde{w}_1^* \tilde{\theta}_1^*) \right.$$

$$\left. + h \left(\int_0^\infty (\hat{w}_N \hat{\theta}_N - h_1 - \langle w\theta\rangle)^2 d\zeta_N + \int_0^\infty (\hat{w}_N^* \hat{\theta}_N^* - h_0 - \langle w\theta\rangle)^2 d\zeta_N^* \right) \right\} = 0, \quad (75)$$

$$\frac{D_w}{\langle w\theta\rangle} b_1^2 \tilde{\theta}_1 - \tilde{w}_1 \left\{ \frac{D_\theta D_w}{2 \langle w\theta\rangle^2} (h \langle w\theta\rangle + 1) + \mu^{r_N} (h + \langle w\theta\rangle - \tilde{w}_1 \tilde{\theta}_1 - \tilde{w}_1^* \tilde{\theta}_1^*) \right.$$

$$\left. + h \left(\int_0^\infty (\hat{w}_N \hat{\theta}_N - h_1 - < w\theta >)^2 d\zeta_N + \int_0^\infty (\hat{w}_N^* \hat{\theta}_N^* - h_0 - \langle w\theta\rangle)^2 d\zeta_N^* \right) \right\} = 0, \quad (76)$$

The same set of equations are also satisfied by the starred quantities $\tilde{w}_n^*, \tilde{\theta}_n^*, \hat{w}_n^*, \hat{\theta}_n^*$.

From equation (75) and (76), we have

$$D_\theta \tilde{w}_1^2 = D_w \tilde{\theta}_1^2, \quad (77)$$

$$D_\theta \tilde{w}_1^{*2} = D_w \tilde{\theta}_1^{*2}. \quad (78)$$

Adding these two identities yields

$$D_\theta \left\langle \tilde{w}_1^2 + \tilde{w}_1^{*2} \right\rangle = D_w \left\langle \tilde{\theta}_1^2 + \tilde{\theta}_1^{*2} \right\rangle. \tag{79}$$

Hence the normalization condition $< w^2 >=< \theta^2 >$ implies

$$D_\theta = D_w = D. \tag{80}$$

This identity together with equation (75) and (76) yields

$$\tilde{w}_1^2 = \tilde{\theta}_1^2, \qquad b_1 = b_1^*. \tag{81}$$

Equation (73) together with equation (74) gives

$$D_\theta \tilde{w}_{n+1}^2 = D_w \tilde{\theta}_{n+1}^2.$$

Same identity holds for $\tilde{w}_{n+1}^*$ and $\tilde{\theta}_{n+1}^*$. Therefore

$$\tilde{w}_{n+1}^2 = \tilde{\theta}_{n+1}^2, \tilde{w}_{n+1}^{*2} = \tilde{\theta}_{n+1}^{*2} \qquad \text{for } n = 1, \ldots, N-1. \tag{82}$$

Substitute the above identity back into equation (73), we have

$$h_1 + < w\theta > - \hat{w}_n \hat{\theta}_n - \tilde{w}_{n+1} \tilde{\theta}_{n+1} = \mu^{r_n - r_N} b_{n+1}^2 \frac{D}{< w\theta >}, \qquad n = 1, \ldots, N-1, \tag{83}$$

$$h_0 + < w\theta > - \hat{w}_n^* \hat{\theta}_n^* - \tilde{w}_{n+1}^* \tilde{\theta}_{n+1}^* = \mu^{r_n - r_N} b_{n+1}^{*2} \frac{D}{< w\theta >}, \qquad n = 1, \ldots, N-1. \tag{84}$$

Then equation (71) and equation (72) become

$$\frac{1}{b_n^2} \hat{w}_n^{(4)} - b_{n+1}^2 \hat{\theta}_n = 0, \tag{85}$$

$$\hat{\theta}_n'' + b_{n+1}^2 \hat{w}_n = 0, \qquad n = 1, \ldots, N-1. \tag{86}$$

The above equations hold in the region where $\hat{w}_n \hat{\theta}_n \neq h_1 + \langle w\theta \rangle$. When the equality holds, then from equation (71) and equation (72) we can derive:

$$\frac{\hat{w}_n^{(4)}}{b_n^2} = -\frac{\hat{\theta}_n \hat{\theta}_n''}{w_n} = (h_1 + \langle w\theta \rangle)^2 \, \frac{\hat{w}_n'' \hat{w}_n - 2\hat{w}_n'^{\,2}}{\hat{w}_n^5} \tag{87}$$

With the following change of variables,

$$\begin{cases} \zeta = b_n^{1/3} b_{n+1}^{2/3} \zeta_n \\ \hat{\Omega} = b_n^{-1/3} b_{n+1}^{1/3} \left(h_1 + \langle hw\theta \rangle \right)^{-1/2} \hat{w}_n \\ \hat{\Theta} = b_n^{1/3} b_{n+1}^{-1/3} \left(h_1 + \langle hw\theta \rangle \right)^{-1/2} \hat{\theta}_n \end{cases} \tag{88}$$

equations (85), (86) and (87) become:

$$\begin{cases} \hat{\Omega}^{(4)} - \hat{\Theta} = 0 \\ \hat{\Theta}'' + \hat{\Omega} = 0 \\ \hat{\Omega}^{(4)} = \frac{\hat{\Omega}'' \hat{\Omega} - 2\hat{\Omega}'^2}{\hat{\Omega}^5} \end{cases} \tag{89}$$

Starred quantities satisfy the same equations with h_1 replaced by h_0. This set of differential equations have been studied in [3], which gives the constant β:

$$3\beta = \int_0^\infty \hat{\Omega}''^2 d\zeta + \int_0^\infty (1 - \hat{\Omega}\hat{\Theta}) d\zeta. \tag{90}$$

And the integral

$$\int_0^\infty \frac{w_n''^2}{b_n^2} d\zeta_n + \int_0^\infty b_{n+1}^2 \tilde{w}_{n+1}^2 d\zeta_n = 3\beta \left(h_1 + \langle w\theta\rangle\right) b_n^{-1/3} b_{n+1}^{4/3}, \qquad n = 1, \ldots, N-1. \tag{91}$$

When $n = N$, the differential equations for $\hat{w}_N$ and $\hat{\theta}_N$ are

$$\frac{D}{< w\theta > b_N^2} \hat{w}_N^{(4)} - (h_1 + \langle w\theta\rangle - \hat{w}_N\hat{\theta}_N)\hat{\theta}_N = 0, \tag{92}$$

$$\frac{D}{< w\theta > b_N^2} \hat{\theta}_N'' + (h_1 + \langle w\theta\rangle - \hat{w}_N\hat{\theta}_N)\hat{w}_N = 0, \tag{93}$$

With the following change of variables,

$$\begin{cases} \zeta = b_n^{1/3} \left(h_1 + \langle w\theta\rangle\right)^{1/3} \left(\frac{D}{\langle w\theta\rangle}\right)^{-1/3} \zeta_N \\ \Omega = b_n^{-1/3} \left(h_1 + \langle hw\theta\rangle\right)^{-1/3} \left(\frac{D}{\langle w\theta\rangle}\right)^{-1/6} \hat{w}_N \\ \Theta = b_n^{1/3} \left(h_1 + \langle hw\theta\rangle\right)^{-2/3} \left(\frac{D}{\langle w\theta\rangle}\right)^{1/6} \hat{\theta}_N \end{cases}, \tag{94}$$

equation (92) and (93) become

$$\Omega^{(4)} - (1 - \Omega\Theta)\Theta = 0, \tag{95}$$

$$\Theta'' + (1 - \Omega\Theta)\Omega = 0. \tag{96}$$

In ([2]) the following result is given:

$$\sigma = \int_0^\infty \Omega''^2 d\zeta = \int_0^\infty \Theta'^2 d\zeta = \frac{1}{4}\int_0^\infty (1 - \Omega\Theta)^2 d\zeta. \tag{97}$$

Thus the following integrals can be expressed in σ:

$$\int_0^\infty \frac{\left(\hat{w}^{(4)}\right)^2}{b_N^2} d\zeta_N = \sigma \left(h_1 + \langle w\theta\rangle\right)^{5/3} \left(\frac{D}{\langle w\theta\rangle}\right)^{-2/3} b_N^{-1/3}, \tag{98}$$

$$\int_0^\infty \hat{\theta}_N'^2 d\zeta_N = \sigma \left(h_1 + \langle w\theta\rangle\right)^{5/3} \left(\frac{D}{\langle w\theta\rangle}\right)^{-2/3} b_N^{-1/3}, \tag{99}$$

$$\int_0^\infty \left(h_1 + \langle hw\theta\rangle - \hat{w}_N\hat{\theta}_N\right)^2 d\zeta_N = 4\sigma \left(h_1 + \langle w\theta\rangle\right)^{5/3} \left(\frac{D}{\langle w\theta\rangle}\right)^{1/3} b_N^{-1/3}. \tag{100}$$

Putting the above integrals together, the functional F_N can be expressed as

$$F_N = \frac{D^2}{\langle w\theta\rangle} + 4\sigma \left(h_1 + \langle w\theta\rangle\right)^{5/3} \left(\frac{D}{\langle w\theta\rangle}\right)^{1/3} b_N^{-1/3} + 4\sigma \left(h_0 + \langle w\theta\rangle\right)^{5/3} \left(\frac{D}{\langle w\theta\rangle}\right)^{1/3} b_N^{*-1/3}, \tag{101}$$

and

$$D = \sum_{n=1}^{N-1} 3\beta \left\{ \left[\frac{b_{n+1}^4}{b_n}\right]^{1/3} (h_1 + \langle w\theta\rangle) + \left[\frac{b_{n+1}^{*4}}{b_n^*}\right]^{1/3} (h_0 + \langle w\theta\rangle) \right\}$$
$$+ \sigma \left(\frac{D}{\langle w\theta\rangle}\right)^{-2/3} \left\{ (h_1 + \langle w\theta\rangle)^{5/3} b_N^{-1/3} + (h_0 + \langle w\theta\rangle)^{5/3} b_N^{*-1/3} \right\} + b_1^2 \langle w\theta\rangle . \quad (102)$$

Minimizing F_N with respect to b_n and b_n^* yields

$$\frac{\partial D}{\partial b_1} = 0 \Rightarrow 2b_1 \langle w\theta\rangle = \beta \left[(h_1 + \langle w\theta\rangle) \left(\frac{b_2}{b_1}\right)^{4/3} + (h_0 + \langle w\theta\rangle) \left(\frac{b_2^*}{b_1}\right)^{4/3} \right], \quad (103)$$

$$\frac{\partial D}{\partial b_n} = 0 \Rightarrow \left[\frac{b_{n+1}}{b_n}\right]^{4/3} = 4 \left[\frac{b_n}{b_{n-1}}\right]^{1/3}, \quad (104)$$

$$\frac{\partial D}{\partial b_n^*} = 0 \Rightarrow \left[\frac{b_{n+1}^*}{b_n^*}\right]^{4/3} = 4 \left[\frac{b_n^*}{b_{n-1}^*}\right]^{1/3}, \quad (105)$$

$$\frac{\partial F_N}{\partial b_N} = 0 \Rightarrow \left[\frac{b_{N+1}}{b_N}\right]^{4/3} = 4 \left[\frac{b_N}{b_{N-1}}\right]^{1/3}, \quad (106)$$

$$\frac{\partial D}{\partial b_N^*} = 0 \Rightarrow \left[\frac{b_{N+1}^*}{b_N^*}\right]^{4/3} = 4 \left[\frac{b_N^*}{b_{N-1}^*}\right]^{1/3}, \quad (107)$$

where

$$b_{N+1} = \left(\frac{\sigma}{\beta}\right)^{4/3} \left(\frac{(h_1 + \langle w\theta\rangle) \langle w\theta\rangle}{D}\right)^{1/2}, \quad (108)$$

$$b_{N+1}^* = \left(\frac{\sigma}{\beta}\right)^{4/3} \left(\frac{(h_0 + \langle w\theta\rangle) \langle w\theta\rangle}{D}\right)^{1/2}. \quad (109)$$

From the above relations, b_n can be solved

$$b_{n+1} = 4^{n-1} \left[\left(\frac{b_{N+1}}{4^{N-1}}\right)^{1-4^{-n}} \cdot (4b_1)^{4^{-n}-4^{-N}} \right]^{\frac{1}{1-4^{-N}}} \quad (110)$$

b_{n+1}^* has a similar form

$$b_{n+1}^* = 4^{n-1} \left[\left(\frac{b_{N+1}^*}{4^{N-1}}\right)^{1-4^{-n}} \cdot (4b_1)^{4^{-n}-4^{-N}} \right]^{\frac{1}{1-4^{-N}}} \quad (111)$$

It is clear from the above expressions that $b_n \neq b_n^*$ for $n \neq 1$ since b_N (equation (108) is different from b_N^* (equation (109). b_1 can be solved from equation (103) and the recursion relation

$$b_1 = \left\{ \frac{\beta}{2^{5/3} \langle w\theta\rangle} \left(\frac{\sigma}{\beta}\right)^{\frac{3}{4\left(1-4^{-N}\right)}} \left[(h_1 + \langle w\theta\rangle)^{\frac{4\left(1-4^{-N}\right)}{3-4^{-N}}} + (h_0 + \langle w\theta\rangle)^{\frac{4\left(1-4^{-N}\right)}{3-4^{-N}}} \right] \right\}^{\frac{1-4^{-N}}{3-4^{-N}}} \quad (112)$$

Putting all these together, the prefactor F_N is a function of $\langle w\theta \rangle$ only:

$$\begin{aligned} F_N &= \frac{D^2}{< w\theta >}\frac{3-4^{-N}}{1-4^{-N}} \\ &= (3-4^{-N})(1-4^{-N})2^{\frac{-4N4^{-N}}{3-4^{-N}}} \cdot \left(2^{5/3}\beta \times \left(\frac{\sigma}{\beta}\right)^{\frac{3}{4(1-4^{-N})}}\right)^{\frac{4(1-4^{-N})}{3-4^{-N}}} \\ &\times \left[\frac{(h_1+\langle w\theta\rangle)^{\frac{3-2\cdot 4^{-N}}{2(1-4^{-N})}} + (h_0+\langle w\theta\rangle)^{\frac{3-2\cdot 4^{-N}}{2(1-4^{-N})}}}{\langle w\theta\rangle^{\frac{1-3\cdot 4^{-N}}{4(1-4^{-N})}}}\right]^{\frac{4\left(1-4^{-N}\right)}{3-4^{-N}}} \end{aligned} \tag{113}$$

Now the value of $\langle w\theta \rangle$ can be determined by setting $\frac{F_N}{\langle w\theta\rangle}$ to zero. The resulting equation for $\langle w\theta \rangle$ is

$$(\alpha-1)x^{\frac{3-2c}{2(1-c)}} - \alpha x - \alpha x^{\frac{1}{1-c}} + (\alpha - 1) = 0 \tag{114a}$$

where

$$x = \frac{\sqrt{3} + < w\theta >}{\sqrt{3} - < w\theta >}, \qquad \alpha = \frac{3-2c}{1-3c}, \qquad c = 4^{-N}. \tag{114b}$$

For general values of N, the above equation has to be solved numerically:

$$\begin{aligned} N &= 1, \qquad \langle w\theta\rangle = 0.4831 \\ N &= 2, \qquad \langle w\theta\rangle = 0.9259 \\ N &= 3, \qquad \langle w\theta\rangle = 1.0120 \\ &\vdots \end{aligned}$$

When $N \to \infty$, the above equation can be solved exactly:

$$\langle w\theta\rangle_\infty = \frac{3\sqrt{3}}{5} = 1.039. \tag{115}$$

This shows that there indeed is a boundary layer at $z = 0$ since all $\langle w\theta \rangle$'s are less than $h_0 = \sqrt{3}$. Now we can write down the scaling of $\langle T\rangle$ as $N \to \infty$:

$$\langle T\rangle = \frac{1}{\sqrt{12}} F_\infty \mu^{2/3} = 10.285\mu^{2/3}. \tag{116}$$

Recalling the identity (47) with $T_0 = 0$:

$$\langle T\rangle = -\left\langle \left(z - \frac{1}{2}\right) w\theta \right\rangle + \frac{1}{12}R,$$

we know that as $\mu \to \infty$

$$\mu \sim \frac{1}{\sqrt{12}}R. \tag{117}$$

This leads to the scaling of $\langle T \rangle$ with respect to R:

$$\langle T \rangle \sim 4.421 R^{2/3}. \tag{118}$$

The profiles of $\tilde{w}_1$ and $\tilde{\theta}_1$ can be determined from the fact that in the interior of the interval $0 < z < 1$,

$$\tilde{w}_1 \tilde{\theta}_1 \approx h + \langle w\theta \rangle, \quad \text{and} \quad \tilde{w}_1 = \tilde{\theta}_1. \tag{119}$$

In the case $N \to \infty$, $h = 2\sqrt{3}z - \frac{2\sqrt{3}}{5}$. And then

$$\tilde{w}_1 = \sqrt{\left| 2\sqrt{3}z - \frac{2\sqrt{3}}{5} \right|}, \qquad \tilde{\theta}_1 = \pm \sqrt{\left| 2\sqrt{3}z - \frac{2\sqrt{3}}{5} \right|} \tag{120}$$

However, whether θ changes sign in $0 < z < 1$ can not be inferred from the variational problem since only the product of w and θ appears in the funcional $\mathcal{F}$. Thus the possibility of w changing its sign can not be excluded.

6 Conclusion

In this project, the scaling of the min $\langle T \rangle$ has been studied for an internally heated fluid layer with both background method and multi-α solution approach. For the case when two plates are held at the same temperature these two methods yield the same scaling: $\langle T \rangle \sim R^{2/3}$. The prefactor given by the background method is about a quarter of that from the other appoach. By adjusting the background field we expect the prefactor to be closer to that predicted by the multi-α approach. However, The scaling of the minimum average temperature when two plates are at different temperatures is not clear yet. It is part of our future work to investicate the scaling in this case.

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