

Length and Shape of a Lava Tube

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Abstract

We study a model of a viscous melted substance flowing in a cold circular tube. As the fluid flows, it cools and solidifies at the tube radius, and we investigate the question ‘how far can the fluid flow and remain liquid?’ A theoretical solution is derived for the radius of the liquid tube and the temperature profiles in the liquid and the solid. It is shown that if the fluid is maintained at constant flux, the tube can be infinitely long, but if it is maintained at constant pressure difference across the length of the tube, then there is a maximum length which depends on the Peclet number and a dimensionless temperature. The stability of the steady-state profiles are investigated, and it is shown that the linear stability of the tube radius can be determined from the functional relationship between pressure and flux. Numerical simulations are performed to test these predictions.

1 Introduction

Lava tubes are a common feature in basaltic lava flows. When a long, slow eruption supplies a steady stream of low-viscosity lava, the flow tends to concentrate in channels. If the channels roof over and become encased in solid material, then the tube of fluid is thermally insulated and can transport hot lava a long way with little loss of heat. An insulated tube such as this can also form in pahoehoe (sheet) flows, without first flowing in a channel, when the sheet cools and gradually restricts the fluid to narrow regions in the interior. These lava tubes can feed flows that are far away from their source, making the extent of a volcanic flow much greater than if the lava were to flow as a slab. [10]

It is common to find lava tubes with lengths of 10-30km, but some tubes are much longer. The Mauna Loa flow tube, in Hawaii, is over 50km long, and the Toomba and Undara flows in Queensland, Australia are 123km and 160km long respectively. The longest known tube is over 200km long, on the volcanic region of Alba Patera, Mars.

We would like to address the question: how long can a lava tube be? If the geometry of the environment were not a factor, how far could a flow of liquid, which is embedded in a solid of the same material, transport hot fluid before cooling and solidification arrests the flow? Previous studies have looked at the

temperature distribution within a tube of constant radius ([13], [4]), the velocity profile of flow in a constant-width channel ([11]), the driving pressure required to keep open a short, constant radius tube of pillow lava ([7]), the cooling processes operating along the length of a tube ([6]), the effect of temperature-dependent viscosity on flow localization ([16], [5], [17]), and the time-dependent melting or solidification of flow through a two-dimensional slot with cooling at infinity ([2], [3], [9]). We will expand on these studies by allowing the radius of the tube to vary, and by providing the appropriate boundary conditions so that we can look at the problem in steady-state. We will ignore the temperature dependence of viscosity, so as to isolate the effect of melting and solidification processes at the boundary of the tube, and their relationship to the heat advected through the tube and conducted radially outward.

We will construct an idealized model of a tube, find a solution for the shape of the tube and conditions for its existence, investigate the stability of the tube, explain the results of our numerical simulations, and finally discuss extensions of the model and ideas for further research.

2 The Model

2.1 Setup and equations

We will model a lava tube as a tube of a fixed length L with a perfectly circular cross-section, whose radius $a(x)$ may depend on the distance down the tube. Liquid enters the tube at a uniform initial hot temperature T_i and it flows through the tube with a velocity profile \vec{u} , to be determined. We suppose that the tube is embedded in a solid, made of the same material as the liquid, which is a large cylinder of radius r_0 . The boundary of this cylinder is maintained at a constant temperature T_0 , which is colder than the melting temperature. The temperature varies continuously from $T = T(0, x) > T_m$ at the center of the tube, which is liquid, to T_0 at the edge of the cylinder, which is solid. The radius of the liquid tube is exactly the isotherm $T = T_m$. (See Figure 1).

Some justification needs to be made for our assumptions and choice of boundary conditions. On the tube boundary, we have chosen to have a clear distinction between solid material and liquid material, rather than to vary the viscosity and so to have a transition region. While we acknowledge that the viscosity change with temperature may be a factor, particularly in the formation of a tube or channel, we assume that once the tube has been formed, the increase in viscosity from its solid-like state to its liquid-like state is rapid enough that the fluid can be modelled as a two-state material with a simple cutoff solidification temperature. ([9]). The melting temperature is chosen to be the temperature at which the amount of crystallization in the lava exceeds 55%. This is based on observations that the lava behaves as a fluid until the amount of crystallization exceeds a threshold, at which point its crystalline network is strong enough that it behaves as a brittle solid, and is not susceptible to erosion by the shear forces in the flow. ([13])

L	=	length of tube
r_0	=	radius of cylinder
T_s	=	melting temperature
T_i	=	temperature of fluid entering tube
T_0	=	temperature of outer boundary
ΔP	=	pressure difference across tube
Q	=	flux through tube
μ	=	viscosity of fluid
κ	=	thermal diffusivity of material (assumed equal for solid and liquid)
ρ	=	density of liquid
c_p	=	heat capacity of liquid
L_H	=	latent heat of fusion

Table 1: Parameters used in the model

The boundary condition on the edge of the cylinder of constant temperature was chosen for two reasons: (i) The difference between the eruption temperature T_i and the melting temperature T_m is an order of magnitude smaller than the difference between T_m and the ambient conditions, represented by T_0 . ([13]). This implies that it is the processes within the tube, and not the environment, which are limiting the cooling, so any heat lost to the walls of the cylinder can be conducted away as fast as it is generated; and (ii) It corresponds to the experiment set up by Jack Whitehead. It is not intended to reproduce the conditions of a real tube, which lives in a much more inhomogeneous environment, and it generates a singularity when combined with the constant temperature boundary condition at the entrance, but it allows us to investigate the properties of such an ideal tube before adding in the complications of a more detailed heat flux function or a model of the source of lava.

We choose the lava tube to have a fixed length L , rather than to vary it dynamically, because often the length of a lava tube is set by factors external to the flow. For example, the slope of the terrain may increase abruptly, causing the lava to pour out of the tube and begin a new sheet flow, or the tube may reach the ocean and drain as a pillow lava flow.

We now introduce the equations of the model.

The variables in the system are:

$$\begin{aligned}
 \vec{u}(x, r) &= \text{velocity field in tube} \\
 T(x, r) &= \text{temperature profile in liquid} \\
 T_e(x, r) &= \text{temperature profile in solid} \\
 a(x) &= \text{radius of tube}
 \end{aligned}$$

The parameters used in the model are shown in Table 1.

When solving for these variables, we are looking for steady-state, axisymmetric solutions, so time derivatives and azimuthal derivatives will be ignored.

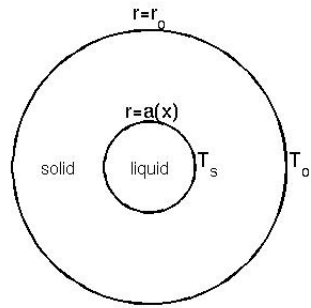
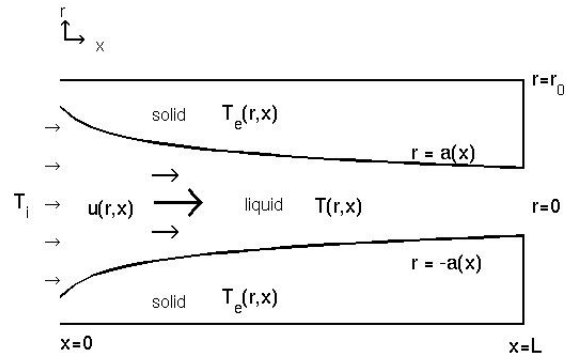


Figure 1: Top: Cross-section of the lava tube through its center line. Bottom: Cross section of the tube at constant x

We will also assume that changes in the radial direction are much greater than changes in the axial direction, so x -derivatives are ignored when they are of the same order as r -derivatives. In particular, we will ignore conduction in the x -direction.

We solve the equations for each of the velocity, temperatures, and radius in turn.

Velocity The viscosity of lava is high enough that we expect to be able to neglect the nonlinear terms in the Navier-Stokes equations. We also expect laminar flow because the Reynolds number of lava flowing in a tube is order 100, much less than the transition to turbulence in a circular pipe, which occurs at $Re = 2000$ to 4000 . ([13],[7]) Indeed, the velocity of lava in a channel has been shown to conform very well to the parabolic profile predicted by assuming

a laminar Newtonian fluid ([11]), and we expect the same to hold true for a tube.

Thus, the along-tube component of velocity satisfies

$$P_x = \mu \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right), \quad u = 0 \text{ at } r = \pm a(x) \quad (1)$$

Solving this gives

$$u = \frac{-P_x}{4\mu} (a^2(x) - r^2)$$

Integrating u from $r = 0$ to $r = a(x)$ gives a relationship between the flux and the pressure:

$$P_x = -\frac{8\mu Q}{\pi a^4(x)} \quad (2)$$

We can use this to express u as

$$u = \frac{2Q}{\pi a^2} (1 - (r/a)^2) \quad (3)$$

The radial component of velocity can be solved for using the condition of non-divergence:

$$v = \frac{2Qa'r}{\pi a^3} (1 - (r/a)^2)$$

In non-dimensional form (see (9)), u is

$$u = \frac{Pe\kappa L}{a_0^2 a^2} (1 - h^2) \quad (4)$$

External Temperature Field The temperature field in the solid, neglecting x -derivatives, satisfies a diffusion equation:

$$\kappa \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T_e}{\partial r} \right) = 0, \quad T_e|_{r=r_0} = T_0, \quad T_e|_{r=a(x)} = T_s \quad (5)$$

This can be solved to give

$$T_e = \frac{T_0 - T_s}{\ln \frac{r_0}{a(x)}} \ln \frac{r}{a(x)} + T_s \quad (6)$$

Or, in non-dimensional form (see (9)),

$$T_e = \frac{K}{\ln \frac{r_0}{a(x)}} \ln \frac{r}{a(x)} \quad (7)$$

where $K = \frac{T_0 - T_s}{T_i - T_s}$ is a non-dimensional constant relating the amount of cooling by the boundary condition to the amount of heating from the incoming lava.

Internal Temperature Field The internal temperature field is given by a balance between advection and diffusion:

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial r} = \kappa \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right)$$

$$T|_{r=a(x)} = T_s, \quad T|_{x=0} = T_i, \quad \frac{\partial T}{\partial r}|_{r=0} = 0 \quad (8)$$

Before solving for $T(x, r)$, we first do a change of variables:

$$h = \frac{r}{a(x)}$$

This will prove to be very convenient, as the streamlines of the flow are lines of constant h , so we will end up with only one partial derivative in the advection term. After the change of variables, and substituting $u = \frac{2Q}{\pi a^2}(1 - h^2)$, the equation becomes

$$\frac{2Q}{\kappa \pi a(x)^2} (1 - h^2) \frac{\partial T}{\partial x} = \frac{1}{a(x)^2} \frac{1}{h} \frac{\partial}{\partial h} \left(h \frac{\partial T}{\partial h} \right)$$

We will non-dimensionalize with the following:

$$\begin{aligned} x &= Lx' \\ a(x) &= a_0 a'(x) \\ \frac{T - T_s}{T_i - T_s} &= T' \\ Q &= Pe \frac{\kappa L \pi}{2} q' \\ P &= \Delta P P' \end{aligned} \quad (9)$$

Here, a_0 is a typical scale for $a(x)$, usually chosen to be r_0 . The pressure was non-dimensionalized with a typical pressure difference across the length of the tube, and the flux was non-dimensionalized so that it has a nice form when related to P - the scale is likely not representative. The non-dimensional parameter Pe is a modified Peclet number, which we will simply call the Peclet number, and is defined to be

$$Pe = \frac{\Delta P a_0^4}{4 \kappa \mu L^2} \quad (10)$$

After dropping the primes, we can write the equations in terms of either P or q as

$$\frac{Pe P}{\int_0^1 \frac{1}{a^4}} (1 - h^2) \frac{\partial T}{\partial x} = \frac{1}{h} \frac{\partial}{\partial h} \left(h \frac{\partial T}{\partial h} \right) \quad (11)$$

$$q(1 - h^2) \frac{\partial T}{\partial x} = \frac{1}{h} \frac{\partial}{\partial h} \left(h \frac{\partial T}{\partial h} \right) \quad (12)$$

with boundary conditions

$$T|_{h=1} = 0, \quad \frac{\partial T}{\partial h}|_{h=0} = 0, \quad T|_{x=0} = 1.$$

The pressure difference across the tube, P , is related to the flux by

$$P = \frac{q}{Pe} \int_0^1 \frac{1}{a^4} dx \quad (13)$$

We can solve (11) or (12) by separation of variables. We show the solution for (12) because it is simpler; to obtain the solution for (11) we simply substitute $PeP/\int_0^1 \frac{1}{a^4}$ for q .

The temperature field that solves the equation is

$$T(x, h) = \sum_n A_n e^{-\lambda_n^2 x/q} \phi_n(h) \quad (14)$$

where λ_n, ϕ_n are the eigenvalues and eigenvectors of the problem and A_n are constants that are determined from the initial temperature distribution. See the Appendix for a discussion of the eigenfunctions.

Radius of Tube The rate of change of the radius of the tube is proportional to the difference in heat flux at the boundary of the tube. This heat flux should be the heat flux in the normal direction, but using our slowly-varying-in- x assumption, we take it to be the flux in the radial direction only. The time-dependent equation for the radius is a standard Stefan equation (see [15]):

$$\frac{L_H}{c_p} \frac{da}{dt} = \kappa \left(\frac{\partial T_e}{\partial r} \Big|_{r=a(x)} - \frac{\partial T}{\partial r} \Big|_{r=a(x)} \right)$$

where L_H is the latent heat of solidification, and c_p is the heat capacity.

In steady-state, and with our change of variables and non-dimensionalization, this becomes

$$\frac{\partial T}{\partial h} \Big|_{h=1} = \frac{\partial T_e}{\partial h} \Big|_{h=1} \quad (15)$$

From (6) and (14), we can calculate

$$\begin{aligned} \frac{\partial T}{\partial h} \Big|_{h=1} &= \sum_n G_n e^{-\lambda_n^2 x/q}, \quad G_n = A_n \phi_n'(1) \\ \frac{\partial T_e}{\partial h} \Big|_{h=1} &= \frac{K}{\ln r_0/a(x)} = \frac{-K}{\ln a(x)} \end{aligned}$$

Substituting into (15) and solving for $a(x)$ gives

$$\begin{aligned}
a(x) &= \exp\left(\frac{-K}{\frac{\partial T}{\partial h}\big|_{h=1}}\right) \\
&= \exp\left(\frac{-K}{\sum G_n e^{-\lambda_n^2 x/q}}\right)
\end{aligned} \tag{16}$$

Before proceeding, we remark that our steady-state model, defined by (4), (7), (11), and (15), depends on only two non-dimensional parameters:

$$Pe = \frac{\Delta P a_0^4}{4\kappa\mu L^2}, \quad K = \frac{T_0 - T_s}{T_i - T_s}$$

The first, the Peclet number, gives the ratio of the horizontal advection of temperature to the vertical conduction of heat. The greater the Peclet number, the more heat is being advected through the tube, so we expect the tube to be more open. The second, which we will call the temperature constant, gives the ratio between the amount of cooling and the amount of heating. A greater $|K|$ means stronger cooling, so we expect the tube to be more closed.

2.2 Existence of a Solution

Notice from (16) that $a(x)$ depends on q . If we set q , then we can solve explicitly for $a(x)$. However, if we choose to set P instead, then $q = PeP / \int \frac{1}{a^4}$, so we must solve a transcendental equation for a . This may or may not have a solution. We will show that if we choose to hold ΔP constant, so that $P = 1$, then the existence of a solution depends on our choice of Peclet number.

Claim There exists a critical number depending on K , call it $Pe_c(K)$, such that

$$\begin{aligned}
Pe > Pe_c(K) &\Rightarrow \text{There are 2 solutions for } a(x) \\
Pe = Pe_c(K) &\Rightarrow \text{There is 1 solution for } a(x) \\
Pe < Pe_c(K) &\Rightarrow \text{There are no solutions for } a(x)
\end{aligned} \tag{17}$$

Proof Consider K to be fixed. We consider the solution (16) to (12), and use this to plot P as a function of q . We define

$$f(q) \equiv q \int_0^1 \frac{1}{a^4(x, q)} dx$$

so that

$$P(q) = \frac{1}{Pe} f(q) \tag{18}$$

We must find a q that solves (18) for $p = 1$.

By calculating $f'(q)$, it is possible to show (see Appendix) that there is a value of q , call it q_c , such that $f(q)$ has a minimum at q_c , is monotone decreasing for $q < q_c$ and monotone increasing for $q > q_c$. (See Figure 2 for an example.)

This implies that (18) has two solutions whenever $Pe > f(q_c)$ and no solutions if $Pe < f(q_c)$. Defining $Pe_c(K) = f(q_c)$ implies the result.

A plot of the critical Peclet number versus K is shown in figure 3. As expected, when $|K|$ increases the critical Peclet number does too, because there is stronger cooling.

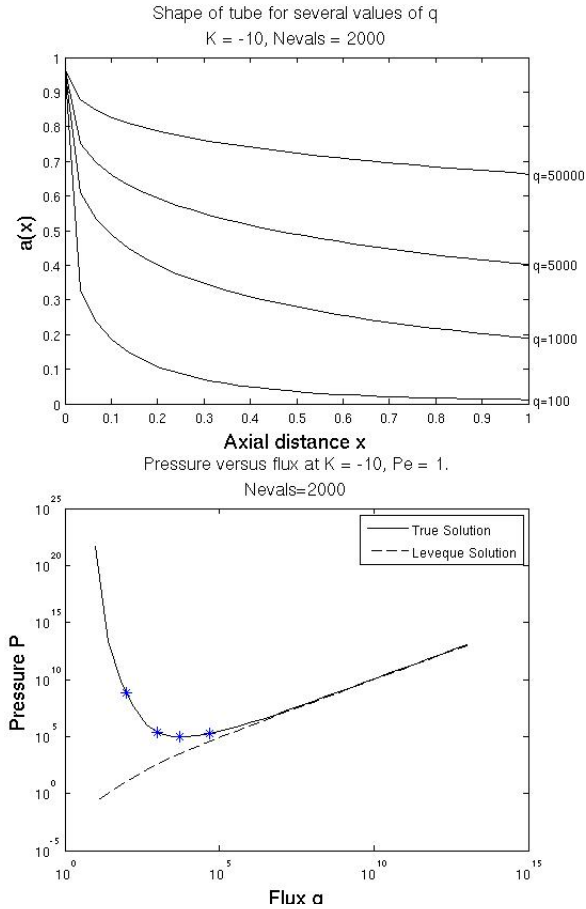


Figure 2: Top: Steady-state $a(x)$ calculated for several different values of q . Bottom: Pressure difference across the tube as a function of flux. The pressure differences for the tubes shown on the left are plotted as starred points. The pressure calculated from the Leveque approximation is shown with a dashed line.

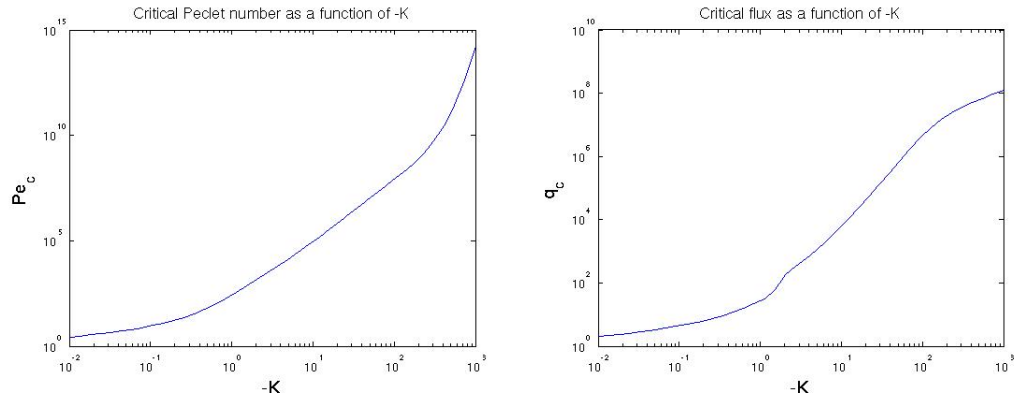


Figure 3: Left: Critical Peclet number versus K . Right: Critical flux versus K .

2.3 Implication for length of a lava tube

From (17), we can calculate the maximum possible length of a lava tube. We must have

$$L_{\max} < \sqrt{\frac{\Delta P a_0^4}{4\kappa\mu Pe_c(K)}} \quad (19)$$

We try to estimate this maximum length in two ways:

1. Assuming the flow is gravity-driven
2. Assuming the flow is pressurized at its source

Variable	Range in Literature	Value Used
κ	1e-7 - 1e-6 m ² /s	1e-7m ² /s
μ	30 - 200 Pa·s	100 Pa·s
ρ	1560-2600kg/m ³	2600 kg/m ³
a_0	1-100m	10m
T_i	1150 – 1180°C	
T_s	1077-1130°C	
T_0	30-100°C	
K	-10 to -40	
$Pe_c(k)$	9.2e4 to 6.4e6	

Table 2: Values of some parameters for basaltic lava, and the values used in our calculations. K is calculated from the temperature data, and Pe_c is calculated from K .

Gravity-Driven flow If the flow is gravity-driven, then the pressure difference is given by the hydrostatic pressure difference calculated from the change in height H :

$$\Delta P = \rho g H$$

We use the parameters shown in Table 2, and take the radius of the tube to be 10m. This may be slightly larger than tubes that are most commonly observed, but many of the longest tubes also have very large radii, sometimes exceeding 50m ([12]). For a flow that drops 1km, with an initial radius of 10m, we find that for $K = -10$ to -40 ,

$$L \approx 30 - 250\text{km}$$

This is certainly in the range expected, although it varies a lot with a change in the parameters. It is therefore likely that a flow is not entirely gravity-driven.

Pressure-Driven Flow There is much evidence that the flow along a lava tube has a pressure-driven component, particularly in very long tubes over shallow slopes. Features along the tube such as tumuli, and dome-shaped fountaining at tube breakout points, are commonly found along long tubes and indicate that such tubes are probably fed by a source under high pressure. ([12]) The source could become pressurized in many ways; for example, if there is a thick layer of heavy, solid material floating on top of the lava source, or if the lava originates from an elastic chamber.

It therefore makes sense to consider the pressure to be limited by a certain value, which is the maximum value that the source can sustain. We suppose that our source is surrounded by solid basalt, and choose this value to be the tensile failure strength of solid basalt:

$$\Delta P_{\text{crit}} = 0.1 - 2.5\text{MPa}$$

If $\Delta P > \Delta P_{\text{crit}}$, we expect the magma at the source to break out of its solid chamber and to develop a new system of tubes and flows.

Using ΔP_{crit} as the value of ΔP in our calculation of Pe and using the given range of K gives an upper bound on the length of a tube of initial radius 10m as

$$L_{\text{max}} \leq 200 - 900\text{km}$$

These values are slightly larger than the longest observed lava flow (of ~ 200 km, on Mars), and an order of magnitude larger than a typical lava tube. However, real lava tubes have much more complicated cooling terms and geometry, have time-dependent parameters, and are likely not operating at their maximal capacity, so this shows that our theory is a candidate for a model of the processes important in restricting the length of a tube. To properly evaluate the accuracy of the length predictions, however, one would need to look at data and values of the parameters for a particular tube.

2.4 Shape of $a(x)$

It is interesting to examine a plot of $P(q)$ and compare it to plots of $a(x)$. Figure 2 shows plots of $a(x)$ for several values of q . When q is on the branch where $\frac{dP}{dq} > 0$, the tube is quite large and has radius comparable to the radius of the cylinder. As $q \rightarrow q_c$, the tube gets much smaller, and when q is on the branch $\frac{dP}{dq} < 0$, the tube becomes tiny very quickly. In this regime, the pressure difference required to maintain such a flux increases very rapidly with a small decrease in flux.

This feature is analogous to that of a fluid whose viscosity changes with temperature. The size of the tube determines its resistance to the flow, and can be thought of as its ‘viscosity’. The smaller the tube, the higher its viscosity. In a high flux regime, the tube’s size, or viscosity, changes little with flux, so the pressure is determined mainly from the shear forces induced by the flux. As the flux decreases, a point is reached where the ‘viscosity’ starts to increase rapidly, dominating the contribution from the flux, so the pressure begins to rise.

Whitehead and Helfrich ([16]) found a curve similar to ours, relating the flux through a fissure to the pressure drop across it, for a fluid whose viscosity changes with temperature. Their curve had the same shape except for very small fluxes, where the pressure reached a maximum and then came back down to 0 at $q = 0$.

2.5 Leveque Approximation

When q is large, x/q is small, so we will need to sum up a large number of eigenfunctions to get a good approximation of the temperature field and tube radius. Leveque ([8]) found an approximate solution of (12) that is valid when x/q is very small. He assumed that the change in temperature happened only within a thin boundary layer of size $\delta(x/q)$ near the edge of the tube. He neglected the curvature term, approximating the flow in this boundary layer as planar, and used a linear approximation for the velocity field. After introducing a similarity variable $\eta = (1 - h)/\delta(x/q)$, he found a similarity solution $F(\eta) = T(x/q, h)$ for the temperature in the boundary layer:

$$T(x/q, h) = F(\eta) = \frac{1}{\Gamma(4/3)} \int_0^\eta e^{-\gamma^3} d\gamma, \quad \eta = \frac{1 - h}{\delta(x/q)} \quad (20)$$

$$\delta(x/q) = \left(\frac{9}{2}h\right)^{1/3} \quad (21)$$

We can combine this with (16) to obtain an approximation for $a(x)$ near $x = 0$. Since $a(x) = r_0 \exp(-K/\frac{\partial T}{\partial h}|_{h=1})$, we find that

$$a(x) \doteq r_0 e^{K\Gamma(\frac{4}{3})(\frac{9x}{2q})^{1/3}}, \quad x/q \ll 1. \quad (22)$$

When q is large enough, this gives almost perfect agreement with the exact solution. Even when q is not so large, the shape still agrees quite well. The

pressure, however, is not so easily approximated with the Leveque solution. When q is large, the pressure agrees very well (see Figure 2), but it starts to diverge as q approaches the critical q_c , and fails completely when $q < q_c$ as the Leveque pressure is monotonically increasing.

3 Stability

Once we know the radius of the tube, we would like to know whether this shape is stable. If we perturb it a little, will it return to steady-state or will it continue to melt back the walls or to solidify until it plugs up?

To answer these questions we must reintroduce time into the equations. Let us assume that the velocity field and external temperature adjust instantly to the radius, and introduce time only into the equations for internal temperature and radius. If we non-dimensionalize time with the diffusive timescale, so that

$$t = \frac{a_0^2}{\kappa} t'$$

then the equations become

$$T_t + q(1 - h^2) \frac{\partial T}{\partial x} - aa_t h T_h = \frac{1}{h} \frac{\partial}{\partial h} \left(h \frac{\partial T}{\partial h} \right) \quad (23)$$

$$S \frac{da}{dt} = \frac{1}{a} \left(\frac{\partial T_e}{\partial h} \Big|_{h=1} - \frac{\partial T}{\partial h} \Big|_{h=1} \right) \quad (24)$$

with the same boundary conditions as before.

The non-dimensional Stefan number is

$$S = \frac{L_H}{c_p(T_i - T_s)}$$

Let us assume that S is large, so we can ignore a_t in (23), and also that q is large enough that T_t becomes negligible. Then we only have one time-dependent equation, (24).

Suppose we start with the radius $a(x)$ that solves the steady problem, and then we change it a little bit. Will it go back to the original radius? If we maintain a constant flux q through the tube, then we see that the answer must be yes. If we increase $a(x)$, then we decrease $\frac{\partial T_e}{\partial h}$ and we increase $\frac{\partial T}{\partial h}$, (which are both negative), so $\frac{da}{dt} < 0$ and $a(x)$ relaxes to its original value. If we decrease $a(x)$, the signs of the fluxes are reversed and the walls melt back to their original configuration. This argument is valid at each point x since the heat fluxes in (24) are local, for constant flux.

However, if we keep the pressure difference across the tube constant, then a heuristic argument shows that there is a potential for instabilities. Suppose we increase the radius of the tube a little bit. The conductive heat flux acts to resolidify it. However, since the pressure difference is fixed, the flux through

the tube also increases, by (2). Thus, more heat is being advected through the tube, and this might cause enough melting to offset the conduction terms.

We now investigate the linear stability quantitatively. Let $\tilde{a}(x) = a(x) + \epsilon a_1(x)$, where $a(x)$ is the steady profile for a given q and ϵ is small. We expand all relevant terms and variables up to $O(\epsilon)$. Thus

$$\begin{aligned}\frac{\partial T_e}{\partial h} &= \frac{K}{\ln r_0/a} + \epsilon \left(\frac{K a_1}{a \ln^2 r_0/a} \right) \\ \tilde{q} &= q + \epsilon q \frac{4 \int \frac{a_1}{a^5}}{\int \frac{1}{a^4}} \\ \frac{\partial T}{\partial h} \Big|_{h=1} &= \sum G_n e^{-\frac{\lambda_n^2}{q} x} \left(1 + \epsilon \lambda_n^2 \frac{x}{q} \frac{4 \int \frac{a_1}{a^5}}{\int \frac{1}{a^4}} \right)\end{aligned}$$

Keeping terms of $O(\epsilon)$ in (24) (and ignoring S , or re-defining time by it) gives

$$a \frac{da_1}{dt} = \frac{K a_1}{a \ln^2 r_0/a} - \frac{x}{q} \frac{\int \frac{4a_1}{a^5}}{\int \frac{1}{a^4}} \sum \lambda_n^2 G_n e^{-\frac{\lambda_n^2}{q} x} \quad (25)$$

Using

$$\sum \lambda_n^2 G_n e^{-\frac{\lambda_n^2}{q} x} = -q \frac{\partial}{\partial x} \left(\frac{\partial T}{\partial h} \Big|_{h=1} \right) = -q \frac{K a'}{a \ln^2 1/a}$$

we can write this as

$$\frac{da_1}{dt} = \frac{K}{a^2 \ln^2 1/a} \left(a_1 + x a'(x) \frac{\int \frac{4a_1}{a^5}}{\int \frac{1}{a^4}} \right) \quad (26)$$

This has the form

$$\frac{da_1}{dt} = c(x) a_1 + k(x) \int_0^1 j(y) a_1(y) dy \quad (27)$$

where $c(x) = \frac{K}{a^2 \ln^2 a} < 0$, $k(x) = x a'(x) c(x) > 0$, $j(x) = 4/a^5(x) > 0$.

The RHS can be turned into a self-adjoint operator by a simple change of variables. Let

$$v(x) = \sqrt{\frac{j(x)}{k(x)}} a_1(x)$$

Then the equation becomes

$$\frac{dv}{dt} = Lv \equiv c(x)v + h(x) \int_0^1 h(y)v(y)dy \quad (28)$$

where $h(x) = \sqrt{k(x)j(x)}$.

We note the following:

- $\text{Range}(c(x)) = (-\infty, c_m]$, where $c_m = \sup(c(x)) < 0$. This is because $a^2 \ln^2 a = 0$ at $a = 0, 1$ so it has a maximum on $a \in (0, 1]$, and $K < 0$.
- $0 < h(x) \leq h_m$. This still needs to be proved - we must show that $\lim_{x \rightarrow 0} h(x) < \infty$. Numerical solutions indicate that it should be true.

Thus, although L is an unbounded operator on $L_2(\mathbb{R})$, the Fredholm component $h(x) \int h(y)v$ is bounded. We could truncate $c(x)$ at some finite value to obtain a bounded operator, and then we could apply the spectral theorem to represent the solution in spectral space. We won't need the boundedness of L to calculate its spectrum, however, so we leave it in its unbounded form for now.

We make the following claims about the spectrum of L .

Claim

- The point spectrum of L consists of one eigenvalue λ_0 , with corresponding eigenfunction $\frac{h(x)}{\lambda_0 - c(x)}$. λ_0 is the solution to $\int \frac{h^2(y)}{\lambda - c(y)} = 1$, and $\lambda_0 > c_m$.
- The continuous spectrum is equal to $\text{Range}(c(x))$.
- There are no other points in the spectrum.

Proof To see that $\frac{h(x)}{\lambda_0 - c(x)}$ is an eigenfunction, simply calculate: $Lv = \frac{c(x)h(x)}{\lambda_0 - c(x)} + h(x) \int \frac{h^2(y)}{\lambda_0 - c(y)} = \frac{\lambda_0 h(x)}{\lambda_0 - c(x)}$.

To see that there is only one such $\lambda_0 > c_m$ solving $f(\lambda) = \int \frac{h^2(y)}{\lambda - c(y)} = 1$, note that $\lim_{\lambda \rightarrow c_m} f(\lambda) = \infty$, $\lim_{\lambda \rightarrow \infty} f(\lambda) = 0$, and $f'(\lambda) = - \int \frac{h^2(y)}{(\lambda - c(y))^2} < 0$.

To show that $\lambda = c(x_0)$ is in the continuous spectrum, we show that $\lambda - L$ cannot have a bounded inverse. We do this by providing a sequence $\{v_n\} \in L_2$ such that $\|v_n\| = 1$ and $\|(\lambda - L)v_n\| \rightarrow 0$. Letting $v_n = b_n \mathbf{1}_{(x_0 - \delta_n, x_0 + \delta_n)}$, where $2\delta_n b_n = 1$, $\delta_n \rightarrow 0$, and using the continuity of $c(x)$ does the job.

To show that this is the entire spectrum, we show that if $\lambda \notin \{c(x)\} \cup \{\lambda_0\}$, then $\lambda - L$ has a bounded inverse.

Let

$$Tv \equiv \frac{v + \int \frac{h(y)v}{\lambda - c(y)} / \left(1 - \int \frac{h^2(y)}{\lambda - c(y)}\right)}{\lambda - c(x)}$$

Then $(\lambda - L)T = T(\lambda - L) = I$, and

$$\|Tv\| \leq \frac{2\|v\|}{(\lambda - c_m)^2} \left(1 + \frac{h_m^2 / (\lambda - c_m)^2}{1 - \int \frac{h^2(y)}{\lambda - c(y)}}\right)$$

Thus, appealing to the spectral theorem for unbounded operators, we can express the solution to (28) as

$$v(x, t) = A_0 e^{\lambda_0 t} \frac{h(x)}{\lambda_0 - c(x)} + \int_{\lambda \in \text{Range}(c(x))} A_\lambda e^{\lambda t} \phi(\lambda, x) d\lambda \quad (29)$$

where $\phi(\lambda, x)$ are the distributions corresponding to the continuous spectrum.

Since the continuous spectrum is always negative, the stability of the problem is determined entirely by the sign of the largest eigenvalue, λ_0 . It is possible to show that

Claim

$$\begin{aligned} \frac{dP}{dq} > 0 &\Rightarrow \lambda_0 < 0 && \text{(stable)} \\ \frac{dP}{dq} < 0 &\Rightarrow \lambda_0 > 0 && \text{(unstable)} \end{aligned}$$

Proof See Section 8.3 in the Appendix for a proof given general heat flux and pressure relationships.

It is helpful to change back to our original variables to see the structure of the discrete eigenfunction. In the original variables, it becomes

$$xa'(x) \frac{c(x)}{\lambda_0 - c(x)}$$

Since $xa_x + qa_q = 0$, it is also proportional to

$$a_q \frac{c(x)}{\lambda_0 - c(x)} = \frac{a_q}{\frac{\lambda_0}{c(x)} - 1}$$

Thus, if $\lambda_0/c(x) \approx \text{const}$, which can happen if $\lambda_0 \gg c_m$, for example, (since $c(x)$ changes slowly over most of its range), then the most slowly-decaying perturbation is almost in the direction of the nearest steady profile.

4 Numerical Simulations

Numerical simulations were performed to test the stability predictions. The pressure difference was kept constant, and the tube radius was stepped forward in time using (24). Time derivatives were calculated using forward Euler, the trapezoidal rule was used for integration, and 1000 eigenfunctions were used to calculate the heat flux and steady profiles. 40 points were used to represent the tube in the horizontal. The simulations were stopped if the tube ‘plugged up’ - defined to be when $a(x) = 0$ for some x .

The numerical simulations confirm the theoretical predictions. If we start with a profile that is linearly stable and perturb it a little, it returns to its original state. We can even perturb it a lot, provided the perturbation is not

too negative, and it will return to the steady state. If the perturbation is too much in the direction towards 0, however, then the tube plugs up - this seems to happen when $\int \frac{1}{a^4}$ for the perturbed profile is too large. It is hypothesized, though not shown, that if the initial flux q_0 , calculated from $q_0 = P / \int \frac{1}{a_0^4}$, where a_0 is the initial profile, is smaller than the flux q_{unstable} corresponding to the unstable profile for a given P , then the tube will plug up.

If we start with a profile that is predicted to be linearly unstable and perturb it a little, it moves away from the unstable state. Which way it moves depends on how we perturb it. If the perturbation is mostly positive, in the direction of the stable profile corresponding to the same value of P , then it opens up, and moves to the stable profile. If the perturbation is mostly negative, away from the stable profile, then the tube plugs up.

As the tube moves from one profile to another, its shape is always close to that of a steady profile. Any localized disturbances to the profile are rapidly ironed out. This is consistent with the linear theory, which predicts large negative eigenvalues in the continuous spectrum, which appears to be associated with highly localized eigenfunctions.

5 Extensions of the model

5.1 Chamber Dynamics

We can introduce more dynamics into the problem by allowing the pressure to change. One simple modification is to assume the pressure is given by the height of lava in a lake, which is fed by a fixed flux q_0 , or that the lava fills an elastic chamber whose pressure depends on how much lava is inside. In both cases, we can write the equation for the change in pressure as

$$\gamma \frac{dP}{dt} = q_0 - q(t) \quad (30)$$

where γ is a non-dimensional constant related to the characteristics of the magma chamber, such as the area of the lake or the elasticity of the chamber. The ratio S/γ tells us the rate of change of pressure compared to the rate of change of the radius.

Numerical simulations show that if the lake is fed with a flux q_0 that is in the stable regime, $q_0 > q_c$, then the tube converges to a tube with a radius $a(x, q_0)$, as long as it starts off with a great enough radius. If the lake is fed with a flux that is too small, $q_0 < q_c$, then every tube plugs up no matter how great its initial radius. Although the system showed growing or decaying oscillations for certain choices of γ/S , no limit cycles were observed, as was the case in the simulations performed by Whitehead and Helfrich ([16]). This is probably because, unlike their pressure-flux relationship, which had an extra stable regime near $q = 0$, our pressure-flux relationship lacks a second stable branch that can help to sustain oscillations.

We also tried setting $q = 0$ to see how fast a lake can be drained by a lava tube. It was never possible to drain the lake; the tube plugged up rapidly as soon as the pressure dropped below the critical value.

5.2 Branching Tubes

Whitehead and Helfrich ([16]) performed an experiment in which they let hot paraffin flow radially outward from a source. After a certain time, the paraffin was cool and viscous enough to be considered a solid, except in a few locations where it flowed rapidly in channels. Initially there were several channels, but as time progressed they all closed up except for one, which continued to flow indefinitely.

Wylie et al ([17]) performed a similar experiment, in which they showed that flow of liquid wax tends to concentrate in a single, narrow finger, with the rest of the flow almost stagnant.

In real lava flows, networks of tubes are occasionally observed instead of a single tube. We are interested in the processes that allow or inhibit several tubes to exist simultaneously. Under which combinations of parameters is it more favourable to feed a flow with several tubes, rather than a single big one? Is there an optimal density of tubes that we should expect? Answers to these questions would help us not only to understand the size and emplacement of lava tubes, but also the location and spacings of volcanoes themselves, as these are formed when a localized tube of lava flows up a fissure in a dike.

We have constructed a simple model in an attempt to answer these questions. Since this work is in its beginning phases, we outline the ideas only briefly so that they can be pursued later in more depth.

We suppose we have a series of n identical tubes, all parallel to the x -direction, which are located at points $\{y_i\}$ along the y -axis. We can non-dimensionalize y so that the points lie between 0 and 1. The tubes are fed by flow through a pipe which lies along the y -axis. There is a uniform flux per unit length q_0 into the pipe. The pressure difference across each tube is given by the pressure P_i at its corresponding point on the pipe. The flux through each tube q_i is determined by the pressure through some relationship, $P_i = f_i(q_i)$, such as (13). This in turn determines the flow through the pipe, which determines the pressure in the pipe.

We need a relationship between the pressure in the pipe and the flux through the pipe. Let us assume the flow in the pipe is Poiseuille flow. We then have

$$P_y = -\gamma q(y)$$

where γ is a non-dimensional parameter related to the size of the pipe compared to the sizes of the tubes.

We can calculate that the flux in the pipe is given by

$$q(y) = q_0 y - \sum q_i 1_{y > y_i}$$

(assuming that r_0 is small enough that we can neglect the modification to q near the entrance of a pipe).

Thus

$$P(y) = P_0 - \gamma \left(q_0 \frac{y^2}{2} + \sum q_i (y - y_i) 1_{y > y_i} \right)$$

where P_0 is the pressure at 0. Let P_f be the pressure at $y = 1$. Our full set of variables to solve for are

$$P_0, \quad \{P_i\}_{i=1..n}, \quad P_f, \quad \{q_i\}_{i=1..n}$$

The equations we have to solve for them are

$$\begin{aligned} \{P_i &= P_0 - \gamma \left(q_0 \frac{y_i^2}{2} + \sum q_k (y_i - y_k) 1_{y_i > y_k} \right)\} \\ P_f &= P_0 - \frac{q_0}{2} + \sum q_k (y_i - y_k) \\ \{P_i &= f_i(q_i)\} \end{aligned}$$

We can add one more equation, so we add conservation of mass:

$$\sum q_i = q_0$$

These equations were modelled on Matlab, and a solution can be found given the locations of the tubes $\{y_i\}$.

We next want to add time into the system. One way to do this is to let the relationship $P_i = f_i(q_i)$ depend on time, so that $f_i = f_i(q_i, t)$. In our simulations, we included time in the pressure-flux relationship by simulating the dynamics of each individual tube, and calculating the pressure as $P_i = q_i \int 1/a_i^4(x, q_i, t) dx$. Our procedure was to solve the system given initial tube profiles, use the calculated values of pressure to step the tube radii forward in time, and repeat using the new radii.

Figure (4) shows some of our results for $\gamma = 1$. We started with 50 identical tubes at locations chosen randomly from a uniform distribution in $(0, 1)$, and provided a flux per unit length that was enough to sustain 6 tubes in a stable configuration. Most of the tubes plugged up, and we ended up with 5, or occasionally 4, tubes in a steady flow. The model showed strong localization: almost all of the time the 5 tubes were sequential points, and only in a very small number of runs did they split into 2 groups.

5.3 2D Planar Flow

There are several situations in which we may be interested in a type of lava transport which could be modelled as a 2-dimensional ‘tube’. One is when lava flows down a slope as a sheet, and is homogeneous in the cross-sheet direction. Another is when a volcano erupts and send lava up a long, narrow fissure. The

shape and thermal properties of the 2-dimensional system should be susceptible to an analysis similar to the 3-D case. We looked at the equations for the 2D system in the hopes that they would have similar results to the 3-D. We were surprised to find that they were actually harder to analyze than the 3-D equations, but preliminary calculations suggest that they may share the same qualitative characteristics.

The 2-D equations use the same non-dimensionalization, except for a slight change. The Peclet number is modified to be

$$Pe = \frac{2\Delta Pa_0^4}{3\kappa\mu L^2}$$

The external temperature equation solves $\frac{\partial^2 T_e}{\partial h^2} = 0$, $T_e|_{h=r_0/a} = K$, $T_e|_{h=1} = 0$, and has solution and consequent flux

$$T_e = \frac{Ka(x)(h-1)}{r_0 - a(x)}, \quad \frac{\partial T_e}{\partial h} = \frac{Ka(x)}{r_0 - a(x)}$$

The internal temperature equation and the relationship between P and q are

$$\begin{aligned} a(x)q(1-h^2)T_x &= \frac{\partial^2 T}{\partial h^2} \\ q &= \frac{PeP}{\int \frac{1}{a^3} dx} \end{aligned}$$

The internal temperature is thus

$$T(x, h) = \sum_n A_n \exp\left(-\frac{\lambda_n^2}{q} \int^x \frac{1}{a(s)} ds\right) \phi_n(h) \quad (31)$$

where λ_n , ϕ_n are the eigenvalues and eigenvectors of the problem, solving

$$\phi'' - \lambda^2(1-h^2)\phi = 0, \quad \phi(1) = 0, \quad \phi'(0) = 0$$

and A_n are determined from the temperature distribution at $x = 0$. These eigenfunctions are discussed in Shah and London ([14]).

Unfortunately, $a(x)$ appears in the solution for T . This means that even if we are given the flux, we may not be able to solve for $a(x)$. We find that

$$\begin{aligned} \frac{Ka(x)}{r_0 - a(x)} &= \sum_n G_n \exp\left(-\frac{\lambda_n^2}{q} \int^x \frac{1}{a(s)} ds\right), \\ G_n &= A_n \phi'_n(1) \\ \Rightarrow a(x) &= \frac{r_0 \sum A_n R_n \exp\left(-\frac{\lambda_n^2}{q} \int^x \frac{1}{a(s)} ds\right)}{Kr_0 + \sum A_n R_n \exp\left(-\frac{\lambda_n^2}{q} \int^x \frac{1}{a(s)} ds\right)} \end{aligned} \quad (32)$$

This is a transcendental equation that defines $a(x)$. It may or may not have a solution, and the solution may or may not be unique. A simple argument shows that it probably does not have a solution in all of parameter space.

Suppose the initial temperature distribution is such that it is made up of only one eigenfunction. (This would also be the case if we start the experiment far down the tube, where the other eigenfunctions have decayed exponentially.)

We must solve

$$G_1 \exp\left(-\lambda_1^2 \int_0^x \frac{1}{qa(s)} ds\right) = \frac{Ka(x)}{r_0 - a(x)} \quad (33)$$

for $a(x)$. Taking the derivative of both sides and substituting for the exponential gives

$$\frac{da}{dx} - \frac{\lambda_1^2}{qr_0}a = -\frac{\lambda_1^2}{q}$$

Solving and using initial condition $a(0) = \frac{G_1}{K+G_1}\bar{r}$, obtained by setting $x = 0$ in (33), gives

$$a(x) = r_0 \left(1 - \frac{K}{K+G_1} e^{\frac{\lambda_1^2}{q}x}\right)$$

In order for this to be greater than 0, we need

$$\frac{G_1}{K} > e^{\frac{\lambda_1^2}{qr_0}}$$

Thus, if q is too small or $|K|$ is too large, we expect there to be no solution.

6 Conclusions

We have created a simplified model of the heat transport in a lava tube, and used this to investigate the existence, shape, and maximal possible length of a lava tube. This model predicts a functional relationship between the pressure difference across the length of the tube, and the flux of fluid through the tube in steady-state, such that for large values of the flux, $\frac{dP}{dq}$ is positive, for small values it is negative, and P has a minimum at $q = q_c$, which depends on a non-dimensional temperature constant. This curve tells us whether or not, for a given non-dimensional Peclet number, a steady-state lava tube can exist. It further tells us when such a solution is stable to small linear perturbations: when $q > q_c$ the tube is stable and when $q < q_c$ the tube is unstable. These linear stability predictions were confirmed with numerical simulations. Unstable tube shapes either went to the stable state corresponding to the same pressure difference, or plugged up and ceased to exist. The maximal length of a lava tube was estimated for typical values of the parameters, and was found to be approximately 30-900km, depending on the assumptions.

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8 Appendix

8.1 Eigenvalues of the temperature problem

The eigenfunctions are solutions of

$$\phi'' + \frac{1}{h}\phi' + \lambda^2(1 - h^2)\phi = 0, \quad \phi(1) = 0, \quad \phi'(0) = 0.$$

They are given by

$$\phi_n = e^{-\lambda_n^2 h^2 / 2} M\left(\frac{1}{2} - \frac{\lambda_n}{4}, 1, \lambda_n h^2\right)$$

where $M(a, b, z)$ is the confluent hypergeometric function:

$$\begin{aligned} M(a, b, z) &= 1 + \frac{a}{b}z + \frac{(a)_2}{(b)_2} \frac{z^2}{2!} + \dots + \frac{(a)_n}{(b)_n} \frac{z^n}{n!} + \dots \\ (a)_n &= a(a+1)(a+2)\dots(a+n-1), (a)_0 = 1 \end{aligned}$$

and λ_n are solutions of the transcendental equation

$$M\left(\frac{1}{2} - \frac{\lambda}{4}, 1, \lambda\right) = 0.$$

It can be shown that

$$M(a, b, z) = \frac{\Gamma(b)}{\Gamma(a)} e^z z^{a-b} (1 + O(|z|^{-1}))$$

so

$$\phi_n(h) = \frac{e^{\lambda_n/2}}{\lambda_n \Gamma\left(\frac{1}{2} - \frac{\lambda_n}{4}\right)} \frac{e^{-\frac{\lambda_n}{2}(h-1)^2}}{h^{(1+\frac{\lambda_n}{2})}} \left(1 + O\left(\frac{1}{\lambda_n h^2}\right)\right)$$

The eigenfunctions are orthogonal with respect to weighting function $h(1 - h^2)$.

See Chapter 13 of Abramowitz and Stegun for more information about the confluent hypergeometric function. [1]

Shah and London ([14]) show that the eigenvalues and initial conditions G_n can be approximated as

$$\begin{aligned}\lambda_n &= \lambda + S_1\lambda^{-4/3} + S_2\lambda^{-8/3} + S_3\lambda^{-10/3} + S_4\lambda^{-11/3} + O(\lambda^{-14/3}) \\ G_n &= \frac{C}{\lambda_n^{1/3}} \left(1 + B_1\lambda^{-4/3} + B_2\lambda^{-6/3} + B_3\lambda^{-7/3} + B_4\lambda^{-10/3} + B_5\lambda^{-11/3} + O(\lambda^{-4}) \right)\end{aligned}$$

where

$$\begin{aligned}\lambda &= 4n + 8/3 \\ S_1 &= 0.159152288 \\ S_2 &= 0.011486354 \\ S_3 &= -0.224731440 \\ S_4 &= -0.033772601 \\ C &= 1.012787288 \\ B_1 &= 0.144335160 \\ B_2 &= 0.115555556 \\ B_3 &= -0.21220305 \\ B_4 &= -0.187130142 \\ B_5 &= 0.0918850832\end{aligned}$$

They also list more accurate values of these numbers for the first few eigenvalues.

8.2 Calculation of $f'(q)$

We have that

$$f(q) = q \int_0^1 \frac{1}{a^4(x, q)}$$

Thus

$$\begin{aligned}f'(q) &= \int_0^1 \frac{1}{a^4} \left[1 - 4q \frac{\partial(\ln a)}{\partial q} \right] \\ &= \int_0^{1/q} \frac{1}{a^4} \left[1 - 4K \frac{\sum G_n \lambda_n^2 s e^{-\lambda_n^2 s}}{(\sum G_n e^{-\lambda_n^2 s})^2} \right] ds\end{aligned}$$

after change of variables $s = x/q$. Now the only dependence of $f'(q)$ on q is in the limit of the integral.

Let

$$g(s) \equiv \frac{K \sum G_n \lambda_n^2 s e^{-\lambda_n^2 s}}{(\sum G_n e^{-\lambda_n^2 s})^2}$$

Then

$$g'(s) = \frac{(\sum G_n e^{\lambda_n^2 s})(\sum -K G_n \lambda_n^2 (1 - \lambda_n^2) s e^{-\lambda_n^2 s}) + 2K (\sum G_n \lambda_n^2 e^{-\lambda_n^2 s})(\sum G_n \lambda_n^2 s e^{-\lambda_n^2 s})}{(\sum G_n e^{-\lambda_n^2 s})^3}$$

Using $K, G_n < 0$ we find that $g(0) = 0$, $g'(s) > 0$ ($s > 0$), $g(s) \rightarrow \infty$ as $s \rightarrow \infty$. Thus, we are integrating a quantity which is positive for small s and negative, going to $-\infty$, for large s , and changes sign only once, so $\exists q_c$ s.t. $f'(q) < 0$ for $q < q_c$, $f'(q) > 0$ for $q > q_c$.

8.3 Derivation of the linear stability equation - general case

We derive the linear stability problem for a general external heat flux, internal heat flux, and pressure relationship. We show that if these functions have a particular form, then the tube radius is linearly stable whenever $\frac{dP}{dq} > 0$ and linearly unstable whenever $\frac{dP}{dq} < 0$.

Let us have the following functions:

$$\begin{aligned} a(x, q) : \quad \mathbf{R} \times \mathbf{R} &\rightarrow \mathbf{R} &= \text{tube radius} \\ P(q, a) : \quad \mathbf{R} \times C(\mathbf{R}) &\rightarrow \mathbf{R} &= \text{Pressure} \\ E(a) : \quad \mathbf{R} &\rightarrow \mathbf{R} &= \text{External heat flux} \\ I(x, q) : \quad \mathbf{R} \times \mathbf{R} &\rightarrow \mathbf{R} &= \text{Internal heat flux} \end{aligned}$$

We make the following assumptions:

- $\frac{\delta P}{\delta a}$ is a linear, positive definite operator from $C(\mathbf{R}) \rightarrow \mathbf{R}$.
- $E' a < 0$
- $\frac{\partial a}{\partial q} > 0$.

We also suppose that any arbitrary profile evolves in time according to

$$\frac{da}{dt} = E(a) - I(x, q), \quad \frac{dP}{dt} = 0.$$

Suppose we start with a steady profile a , with corresponding flux q , and perturb it by ϵa_1 . Let the flux change by an amount ϵq_1 . We derive the $O(\epsilon)$ equation for the evolution of a_1 . We have that

$$\begin{aligned} \frac{dP}{dt} = 0 &\Rightarrow \frac{\partial}{\partial \epsilon} \Big|_{\epsilon=0} P(q_0 + \epsilon q_1, a + \epsilon a_1) = 0 \\ &\Rightarrow q_1 \frac{\partial P}{\partial q} + \frac{\delta P}{\delta a} [a_1] = 0 \\ &\Rightarrow q_1 = \frac{-\frac{\delta P}{\delta a} [a_1]}{\frac{\partial P}{\partial q}} \end{aligned}$$

Thus, a_1 evolves according to

$$\begin{aligned} \frac{da_1}{dt} &= \frac{\partial}{\partial \epsilon} \Big|_{\epsilon=0} (E(a + \epsilon a_1) - I(x, q + \epsilon q_1)) \\ &= a_1 \frac{\partial E}{\partial a} - q_1 \frac{\partial I}{\partial q} \\ &= a_1 \frac{\partial E}{\partial a} + \frac{\frac{\delta P}{\delta a} [a_1]}{\frac{\partial P}{\partial q}} \frac{\partial I}{\partial q} \end{aligned}$$

Since a satisfies $E(a) = I(x, q)$, we can take the q -derivative of this to get $\frac{dE}{da} \frac{\partial a}{\partial q} = \frac{\partial I}{\partial q}$. We also know that $\frac{\partial P}{\partial q} = \frac{dP}{dq} - \frac{\delta P}{\delta a} [\frac{\partial a}{\partial q}]$. Substituting these into the last equation gives

$$\frac{da_1}{dt} = \frac{dE}{da} \left(a_1 + \frac{\frac{\partial a}{\partial q} \frac{\delta P}{\delta a} [a_1]}{\frac{dP}{dq} - \frac{\delta P}{\delta a} [\frac{\partial a}{\partial q}]} \right) \quad (34)$$

Set the RHS equal to λa_1 and solve for a_1 to get

$$a_1 = \frac{\frac{dE}{da} \frac{\delta P}{\delta a} [a_1] \frac{\partial a}{\partial q}}{\left(\frac{dP}{dq} - \frac{\delta P}{\delta a} [\frac{\partial a}{\partial q}] \right) (\lambda - \frac{dE}{da})}$$

Let $\frac{\delta P}{\delta a} [a_1] = D$. Take $\frac{\delta P}{\delta a}$ of the above equation to get

$$\frac{\delta P}{\delta a} \left[\frac{-\frac{dE}{da} \frac{\partial a}{\partial q}}{\left(\frac{dP}{dq} - \frac{\delta P}{\delta a} [\frac{\partial a}{\partial q}] \right) (\lambda - \frac{\partial E}{\partial a})} \right] = 1 \quad (35)$$

If $\frac{dP}{dq} = 0$ and $\lambda = 0$, then LHS = 1. If $\frac{dP}{dq} \uparrow$ or $\lambda \uparrow$, then LHS \downarrow , and if $\frac{dP}{dq} \downarrow$ or $\lambda \downarrow$, then LHS \uparrow . Therefore

$$\begin{aligned} \frac{dP}{dq} > 0 &\Rightarrow \lambda < 0 && \text{(stable)} \\ \frac{dP}{dq} < 0 &\Rightarrow \lambda > 0 && \text{(unstable)} \end{aligned}$$

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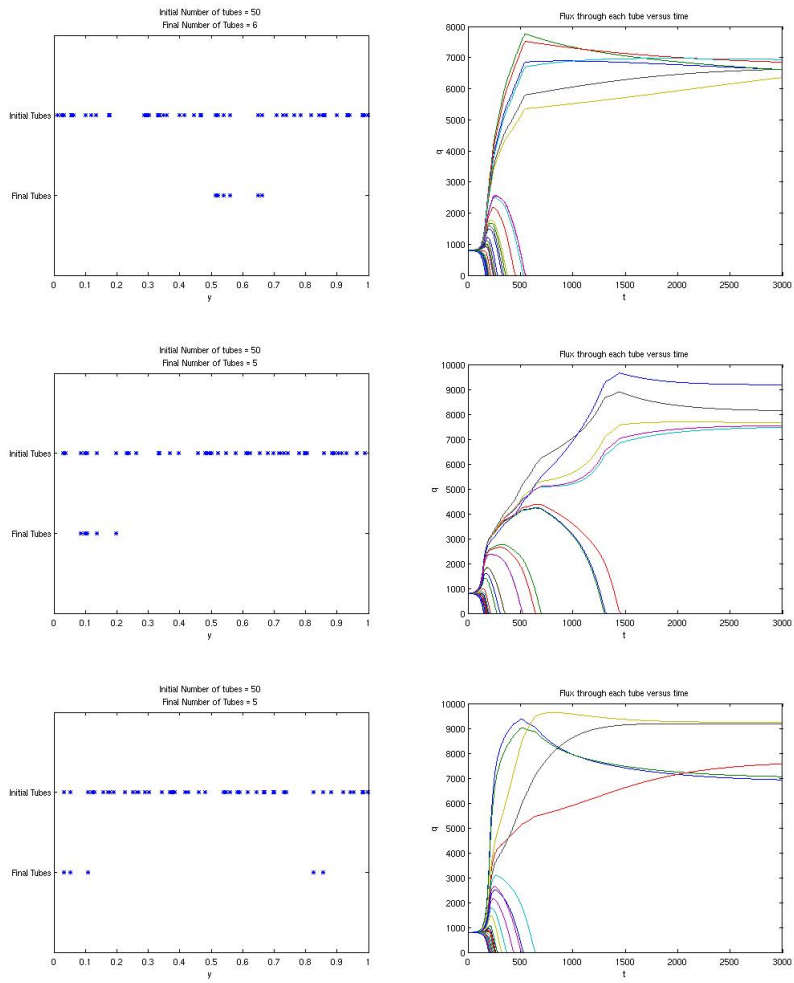


Figure 4: Simulations of systems of tubes. Left: Plot showing initial location of tubes (top row) and tubes that remained open after a long time (bottom row). Right: Flux through each tube as a function of time.