An Optimal Exit Time Problem

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Introduction

Optimisation of mixing for a passive scalar subject to advection-diffusion by an incompressible flow has a long and celebrated story. The question of its measure is a major issue and various metrics have been adopted in order to quantify the mixing efficiency in terms of suppression of the scalar variances by a stirring flow ([13], [2] for a quantification of mixing efficiency on multiple scales, [7], [5], and the review [15]). Using these different metrics, various related problems have been addressed - for instance, optimising either the flow ([8], for an unsteady problem) or a source-sink distribution ([14], steady case) given a fixed energy budget, and deriving sharp asymptotic bounds on the mixing enhancement due to advection ([13], [9], [12]). In the present report we will address the following question: given a fixed kinetic energy constraint and an initial, uniform distribution of particles, what steady, incompressible flow achieves minimum exit time (in L^1-norm) from a cavity with absorbing boundary conditions?

The equation governing the advection-diffusion of a passive scalar c by an autonomous flow with velocity field u and constant, homogeneous molecular diffusivity κ is:

$$\frac{\partial c^\dagger}{\partial t^\dagger} = -u^\dagger \cdot \nabla^\dagger c^\dagger + \kappa \triangle^\dagger c^\dagger. \tag{1}$$

In what follows, the flow is incompressible ($\nabla^\dagger \cdot u^\dagger = 0$) and the magnitude of the velocity field is tuned through a fixed kinetic energy constraint:

$$\left\langle |u^\dagger|^2 \right\rangle = 2E^\dagger, \tag{2}$$

where $\langle \cdot \rangle = \int_\Omega \cdot \, d\Omega$, so that $\langle | \cdot | \rangle$ denotes the L^1-norm. Using the typical size of the domain as the unit length and a diffusive timescale, (1) then becomes in dimensionless variables:

$$\frac{\partial c}{\partial t} = -u \cdot \nabla c + \triangle c. \tag{3}$$

Note that with this choice of non-dimensionalisation, the magnitude of the dimensionless velocity u can be regarded as a Péclet number $Pe = \frac{UL}{\kappa}$, which quantifies the ratio between diffusive and advective timescales. All the scalings in terms of dimensionless, kinetic energy E can thus be regarded as equivalent scalings in terms of squared Péclet number Pe^2.

1 The Exit Time Problem

A flow with velocity u is prescribed in a cavity where the concentration of released particles satisfies the advection diffusion equation (3) with absorbing (homogeneous Dirichlet) boundary conditions, meaning that particles are removed from the system as soon as they first hit the boundary. Suppose we initially release particles at a location x_0:

$$c(x,0) = \delta(x - x_0), \tag{4}$$

and let $f(t, x_0)$ the probability density for a particle released in x_0 at $t = 0$ to hit the wall (and therefore exit the domain) at time t. The fraction $S(t, x_0)$ of surviving particles after a time t is given by integrating the probability density for particles to exit the domain at later times, and we have:

$$S(t, x_0) = \langle c(x, t, x_0) \rangle = \int_t^\infty f(s, x_0) ds, \tag{5}$$

implying

$$\frac{dS(t, x_0)}{dt} = -f(t, x_0). \tag{6}$$

The expected exit time for particles issued from x_0 at $t = 0$ is then, using the fact that no particle is expected to survive at infinite times ($S(t, x_0) \underset{t \to \infty}{\to} 0$):

$$T(x_0) = \int_0^\infty t f(t, x_0) dt \tag{7}$$

$$= -\int_0^\infty t \frac{dS(t, x_0)}{dt} dt \tag{8}$$

$$= \int_0^\infty S(t, x_0) dt \tag{9}$$

$$= \left\langle \int_0^\infty c(x, t, x_0) dt \right\rangle. \tag{10}$$

Let us introduce the linear operator $\mathscr{L} = u \cdot \nabla - \triangle$ and prove that

$$\mathscr{L}^* T = 1, \tag{11}$$

where $\mathscr{L}^*$ is the adjoint of $\mathscr{L}$, which we shall determine to be $\mathscr{L}^* = -u \cdot \nabla - \triangle$. The particles which are released on the boundary are expected to instantly leave the domain, therefore T satisfies the homogeneous Dirichlet boundary condition:

$$T = 0 \qquad \text{on} \quad \partial \Omega. \tag{12}$$

Let us apply $\mathscr{L}$ to $\int_0^\infty c(x, t, x_0) dt$. Using (3) we get:

$$\mathscr{L} \int_0^\infty c(x, t, x_0) dt = \int_0^\infty \mathscr{L} c(x, t, x_0) dt \tag{13}$$

$$= \int_0^\infty -\frac{\partial c}{\partial t} dt \tag{14}$$

$$= c(x, 0) \tag{15}$$

$$= \delta(x - x_0). \tag{16}$$

From now on we write $\mathscr{C} = \int_0^\infty c(x, t, x_0)dt$, so that (16) reduces to $\mathscr{L}\mathscr{C} = \delta(x - x_0)$, and we assume that the initial distribution of particles is spatially homogeneous. Let U be the unique solution of the equation $\mathscr{L}^*X = 1$ with $X = 0$ on $\partial\Omega$. Then for every x_0,

$$U(x_0) = \langle \delta(x - x_0)U \rangle = \langle \mathscr{L}\mathscr{C}U \rangle = \langle \mathscr{C}\mathscr{L}^*U \rangle = \langle \mathscr{C} \rangle = T(x_0), \tag{17}$$

therefore $U = T$.

The boundary terms in the calculation of the adjoint vanish due to impermeable boundary condition ($u \cdot n = 0$ on $\partial\Omega$), (12) and the fact that $\mathscr{C} = 0$ on the boundary:

$$\langle T\mathscr{L}\mathscr{C} \rangle = \langle T(u \cdot \nabla - \triangle)\mathscr{C} \rangle \tag{18}$$

$$= \langle \mathscr{C}(-u \cdot \nabla - \triangle)T \rangle + \int_{\partial\Omega} (uT\mathscr{C} - T\nabla\mathscr{C} + \mathscr{C}\nabla T) \cdot nd(\partial\Omega) \tag{19}$$

$$= \langle \mathscr{C}\mathscr{L}^*T \rangle. \tag{20}$$

2 Formulation of the Variational Problem

2.1 ... for the minimum exit time

The problem considered here is the minimisation of the particles expected exit time from a cavity with absorbing boundary conditions on the periphery and uniform initial distribution of particles, where we will use the L^1-norm as a metric. In the absence of stirring ($u = 0$), the transport is purely conductive and the mean expected exit time solely depends on the fluid molecular diffusivity κ. As will be seen in the next section, stirring always results in lowering the mean exit time for the particles with our choice of a metric. Note that this result is true for the L^1-norm but not, for example, for the L^∞-norm as demonstrated by [5], who proved that for any 2D, simply connected domain different from a disc, there always exists a flow that enhances the largest exist time compared to the pure conduction case (see Theorem 1.1 in [5]). In the following we will also consider a 2D domain with incompressible flow ($\nabla \cdot u = 0$), and therefore we introduce the stream-function Ψ such that:

$$u_x = -\frac{\partial\Psi}{\partial y} \qquad \text{and} \qquad u_y = \frac{\partial\Psi}{\partial x}. \tag{21}$$

We aim at determining the structure of the flow that realizes optimally efficient stirring, under a given energy constraint. The expected exit time and stream-function have to satisfy:

$$\mathscr{L}^*T = 1, \tag{22}$$

$$\langle |\nabla\Psi|^2 \rangle = 2E, \tag{23}$$

where we write $\mathscr{L}^*T = -J(\Psi, T) - \triangle T$, with the Jacobian $J(a, b) = \partial_x a \partial_y b - \partial_y a \partial_x b$. We thus define the following functional, to be minimised in order to achieve minimal mean exit time under the above constraints:

$$\mathscr{F}(T, \Psi, \Theta, \mu) = \langle T \rangle - \langle \Theta (\mathscr{L}^*T - 1) \rangle + \frac{\mu}{2} \left(\langle |\nabla\Psi|^2 \rangle - 2E \right), \tag{24}$$

where Θ, μ are Lagrange multipliers. With impermeability boundary conditions ($\Psi = 0$ on the wall), we get the first variations of $\mathscr{F}$:

$$\delta_T \mathscr{F} = \langle \delta T \rangle - \langle \Theta (\mathscr{L}^* \delta T) \rangle = \langle \delta T \rangle - \langle (\mathscr{L}\Theta) \, \delta T \rangle, \tag{25}$$

$$\delta_\Psi \mathscr{F} = \langle \Theta J(\delta \Psi, T) \rangle + \mu \langle \nabla(\delta \Psi) \cdot \nabla \Psi \rangle = \langle \delta \Psi J(T, \Theta) \rangle - \mu \langle \delta \Psi \triangle \Psi \rangle, \tag{26}$$

$$\delta_\Theta \mathscr{F} = \langle \delta \Theta (\mathscr{L}^* T - 1) \rangle. \tag{27}$$

We look for an extremum by setting the functional derivative of $\mathscr{F}$ to 0, which gives the Euler-Lagrange equations:

$$\mathscr{L}\Theta = 1, \tag{28}$$

$$J(T, \Theta) - \mu \triangle \Psi = 0, \tag{29}$$

$$\mathscr{L}^* T = 1. \tag{30}$$

2.2 ... for an internal heating problem

Let us temporarily drop our previous notations and consider now the steady-state of temperature $\Theta^\dagger$ in an internally heated flow with heat source $Q^\dagger(x)$, divergence-free velocity field $u^\dagger$ and constant, homogeneous molecular diffusivity κ:

$$0 = -u^\dagger \cdot \nabla^\dagger \Theta^\dagger + \kappa \triangle^\dagger \Theta^\dagger + Q^\dagger \tag{31}$$

or equivalently, in dimensionless variables and diffusive timescale:

$$\mathscr{L}\Theta = Q. \tag{32}$$

Let us consider a uniform heating source and fix $Q = 1$. We look for optimum cooling by a 2D flow using the L^1-norm of the temperature as a metric, under incompressibility and fixed energy constraints. Let the functional:

$$\mathscr{F}(\Theta, \Psi, T, \mu) = \langle \theta \rangle - \langle T (\mathscr{L}\Theta - 1) \rangle + \frac{\mu}{2} \left(\langle |\nabla \Psi|^2 \rangle - 2E \right), \tag{33}$$

where T, μ are Lagrange multipliers (again, let us forget about last paragraph notations!). With impermeability boundary conditions ($\Psi = 0$ on the wall), the first variations of $\mathscr{F}$ are:

$$\delta_\Theta \mathscr{F} = \langle \delta \Theta \rangle - \langle T (\mathscr{L} \delta \Theta) \rangle = \langle \delta \Theta \rangle - \langle (\mathscr{L}^* T) \, \delta \Theta \rangle, \tag{34}$$

$$\delta_\Psi \mathscr{F} = - \langle T J(\delta \Psi, \Theta) \rangle + \mu \langle \nabla(\delta \Psi) \cdot \nabla \Psi \rangle = \langle \delta \Psi J(T, \Theta) \rangle - \mu \langle \delta \Psi \triangle \Psi \rangle, \tag{35}$$

$$\delta_T \mathscr{F} = \langle \delta T (\mathscr{L}\Theta - 1) \rangle. \tag{36}$$

Looking for an extremum implies setting the functional derivative of $\mathscr{F}$ to 0, which now yields:

$$\mathscr{L}^* T = 1, \tag{37}$$

$$J(T, \Theta) - \mu \triangle \Psi = 0, \tag{38}$$

$$\mathscr{L}\Theta = 1. \tag{39}$$

Remarkable here is the conjugacy between both systems: under fixed energy constraint, optimization of cooling in the internal heating problem and minimisation of expected time in the exit time problem require solving the same set of Euler-Lagrange equations, the Lagrange-multiplier in the first problem (28)-(30) satisfying the same equation as the passive scalar in the second one (37)-(39) and vice versa. Solving for the exit time problem therefore provides a solution for the internal heating problem as well. We will focus on the former in the next sections, although the later remains an underlying motivation as it may be relevant for many engineering purposes.

3   The System

This section is dedicated to the introduction of the various notations and relationships that will be used in undertaking an analytical solution of the Euler-Lagrange equations. The master equations are expressed in terms of the new variables and the geometry of the domain is then restricted to the unit disc. Finally, the introduction of a particular *Ansatz* results in recasting the variational problem in a 1D, non-linear eigenvalue problem to be solved under fixed energy constraint.

Re-writing (28)-(30) in terms of Jacobians, we now have:

$$-J(\Psi, T) - \triangle T = 1, \tag{40}$$

$$J(\Psi, \Theta) - \triangle \Theta = 1, \tag{41}$$

$$-J(T, \Theta) + \mu \triangle \Psi = 0, \tag{42}$$

to be solved under the energy constraint:

$$\left\langle |\nabla \Psi|^2 \right\rangle = 2E, \tag{43}$$

and homogeneous Dirichlet boundary conditions for T, Θ and Ψ on $\partial \Omega$. Let

$$T = C + \tau \qquad \text{and} \qquad \Theta = C + \sigma, \tag{44}$$

where C is the pure conduction solution in the domain:

$$-\triangle C = 1. \tag{45}$$

Since C verifies (45), the system (40)-(42) becomes after subtraction:

$$-J(\Psi, C) - J(\Psi, \tau) - \triangle \tau = 0, \tag{46}$$

$$J(\Psi, C) + J(\Psi, \sigma) - \triangle \sigma = 0, \tag{47}$$

$$J(\sigma, \tau) + \mu \triangle \Psi = J(\tau - \sigma, C). \tag{48}$$

Introducing the new variables

$$\xi = \tau - \sigma \qquad \text{and} \qquad \eta = \tau + \sigma, \tag{49}$$

in order to simplify the calculations and noting that $J(\tau, \sigma) = \frac{1}{2} J(\xi, \eta)$, we obtain the following system, whose solutions correspond to the extrema of the cost function $\mathscr{F}$:

$$-J(\Psi, \xi) - \triangle \eta = 0, \tag{50}$$

$$-J(\Psi, \eta) - \triangle \xi = 2J(\Psi, C), \tag{51}$$

$$\frac{1}{2} J(\eta, \xi) + \mu \triangle \Psi = J(\xi, C). \tag{52}$$

3.1 A few useful identities

Before we attempt to solve for the system (50)-(52) in terms of ξ, η and Ψ, it will prove useful to write down a few integral relationships obtained from taking the scalar product of the master equations (under their various forms) against the different variables. In this paragraph, we justify that stirring effectively improves the mixing compared to the case where the flow is at rest ($u = 0$) and derive two different expressions for the mean expected exit time which we will need later on.

Let us prove first that stirring enhances mixing with our choice of metric. Starting back from the original equation (40) and multiplying it by T, integration over the domain leads to:

$$\boxed{\left\langle |\nabla T|^2 \right\rangle = \left\langle T \right\rangle}. \tag{53}$$

A similar operation on (45) yields

$$\boxed{\left\langle |\nabla C^2 \right\rangle = \left\langle C \right\rangle}. \tag{54}$$

As the expected exit time can be decomposed as $T = C + \tau$, we can re-write:

$$\left\langle T \right\rangle = \left\langle |\nabla C|^2 \right\rangle + 2 \left\langle \nabla \tau \cdot \nabla C \right\rangle + \left\langle |\nabla \tau|^2 \right\rangle. \tag{55}$$

Multiplying (46) by respectively C and τ and integrating over the domain leads to:

$$0 = \quad - \left\langle CJ(\Psi, C) \right\rangle - \left\langle CJ(\Psi, \tau) \right\rangle - \left\langle C\triangle\tau \right\rangle = \quad - \left\langle CJ(\Psi, \tau) \right\rangle + \left\langle \nabla C \nabla \tau \right\rangle, \tag{56}$$

$$0 = \quad - \left\langle \tau J(\Psi, C) \right\rangle - \left\langle \tau J(\Psi, \tau) \right\rangle - \left\langle \tau\triangle\tau \right\rangle = \quad + \left\langle CJ(\Psi, \tau) \right\rangle + \left\langle (\nabla \tau)^2 \right\rangle. \tag{57}$$

Thus

$$\left\langle \nabla C \cdot \nabla \tau \right\rangle = - \left\langle |\nabla \tau|^2 \right\rangle \tag{58}$$

and

$$\left\langle |\nabla T|^2 \right\rangle = \left\langle |\nabla C|^2 \right\rangle - \left\langle |\nabla \tau|^2 \right\rangle \qquad \text{i.e.} \qquad \boxed{\left\langle T \right\rangle = \left\langle C \right\rangle - \left\langle |\nabla \tau|^2 \right\rangle}, \tag{59}$$

which shows that stirring can only lower the L^1-norm for the expected exit time (compared to the purely conductive case).

Multiplying now (50), (51), (52) by, respectively, η, ξ and Ψ and integrating over the domain yields:

$$\left\langle |\nabla \eta|^2 \right\rangle = \left\langle \eta J(\Psi, \xi) \right\rangle = \left\langle \Psi J(\xi, \eta) \right\rangle, \tag{60}$$

$$\left\langle \eta J(\Psi, \xi) \right\rangle + \left\langle |\nabla \xi|^2 \right\rangle = 2 \left\langle \xi J(\Psi, C) \right\rangle, \tag{61}$$

$$-\mu \left\langle |\nabla \Psi|^2 \right\rangle = \frac{1}{2} \left\langle \Psi J(\xi, \eta) \right\rangle + \left\langle \Psi J(\xi, C) \right\rangle. \tag{62}$$

It follows that

$$\left\langle |\nabla \eta|^2 \right\rangle + \left\langle |\nabla \xi|^2 \right\rangle = -2 \left\langle \Psi J(\xi, C) \right\rangle, \tag{63}$$

and

$$2E\mu = -\frac{1}{2}\left\langle|\nabla\eta|^2\right\rangle + \frac{1}{2}\left\langle|\nabla\eta|^2\right\rangle + \frac{1}{2}\left\langle|\nabla\xi|^2\right\rangle. \tag{64}$$

We also have

$$4|\nabla\tau|^2 = |\nabla\xi|^2 + |\nabla\eta|^2 + 2\nabla\xi\cdot\nabla\eta, \tag{65}$$

$$4\left\langle|\nabla\tau|^2\right\rangle = \left\langle|\nabla\xi|^2\right\rangle + \left\langle|\nabla\eta|^2\right\rangle, \tag{66}$$

as multiplying (50) by ξ and integrating over the domain yields:

$$\left\langle\nabla\xi\cdot\nabla\eta\right\rangle = 0. \tag{67}$$

Therefore

$$\left\langle|\nabla\tau|^2\right\rangle = E\mu + \frac{1}{4}\left\langle|\nabla\eta|^2\right\rangle \qquad \text{i.e.} \qquad \boxed{\langle T\rangle = \langle C\rangle - E\mu - \frac{1}{4}\left\langle|\nabla\eta|^2\right\rangle}. \tag{68}$$

Moreover, we can also write:

$$\langle T\rangle = \langle C\rangle + \frac{1}{2}\langle\eta\rangle + \frac{1}{2}\langle\xi\rangle. \tag{69}$$

Multiplication of (40) and (41) by ξ and integration over the domain yields:

$$-\langle\xi J(\Psi,T)\rangle - \langle\xi\triangle T\rangle = \langle\xi\rangle \qquad \text{and} \quad \langle\xi J(\Psi,\Theta)\rangle - \langle\xi\triangle\Theta\rangle = \langle\xi\rangle. \tag{70}$$

Hence

$$2\langle\xi\rangle = -\langle\xi\triangle\eta\rangle - \langle\xi J(\Psi,\xi)\rangle = \langle\nabla\xi\cdot\nabla\eta\rangle = 0, \tag{71}$$

which, once combined with (69), provides an alternative expression for $\langle T\rangle$:

$$\boxed{\langle T\rangle = \langle C\rangle + \frac{1}{2}\langle\eta\rangle}. \tag{72}$$

3.2 The disc case

From now on the domain is assumed to be a disc of radius 1 and cylindrical coordinates are adopted. The conduction solution in the unit disc is given by $C = \frac{1-r^2}{4}$ and the system above simplifies to:

$$-J(\Psi,\xi) - \triangle\eta = 0, \tag{73}$$

$$-J(\Psi,\eta) - \triangle\xi = \frac{\partial\Psi}{\partial\theta}, \tag{74}$$

$$\mu\triangle\Psi = \frac{1}{2}J(\xi,\eta) + \frac{1}{2}\frac{\partial\xi}{\partial\theta}, \tag{75}$$

with the kinetic energy constraint

$$2E = \left\langle|\nabla\Psi|^2\right\rangle. \tag{76}$$

3.2.1 Preliminary remark

If we consider small-amplitude perturbation from the pure conduction problem (say $E = \epsilon$), with $\Psi = O(\epsilon)$ and, accordingly, $\tau = O(\epsilon)$, $\sigma = O(\epsilon)$, then the system (46)-(47) becomes at order $O(\epsilon)$:

$$-J(\Psi, C) - \triangle \tau = 0, \tag{77}$$
$$J(\Psi, C) - \triangle \sigma = 0, \tag{78}$$

which implies $\tau + \sigma = o(\epsilon)$ since τ and σ verify the same boundary conditions. Therefore $\eta = o(\epsilon)$ and from (51)-(52) we get at order $O(\epsilon)$:

$$-\triangle \xi = \frac{\partial \Psi}{\partial \theta}, \qquad \text{and} \qquad \mu \triangle \Psi = \frac{1}{2} \frac{\partial \xi}{\partial \theta}. \tag{79}$$

Combining both equation in order to eliminate one variable leads to the same equation for ξ and Ψ: [1]

$$\mu \triangle^2 \xi = -\frac{1}{2} \frac{\partial^2 \xi}{\partial \theta^2}, \qquad \mu \triangle^2 \Psi = -\frac{1}{2} \frac{\partial^2 \Psi}{\partial \theta^2}. \tag{80}$$

As ξ and Ψ also satisfy the same homogeneous Dirichlet BCs, we can look for $\Psi = \lambda \xi$ with λ a proportionality constant, to be determined. For the two equations of (79) to be consistent we need $\boxed{\lambda = (2\mu)^{-\frac{1}{2}}}$. Assuming continuation of the linear solution into the non-linear regime, this observation provides a guess for a class of solutions satisfying the following *Ansatz*.

3.2.2 The *Ansatz*

Let's look for a variable-separated solution of the form:

$$\xi = A(r) \cos(m\theta), \qquad \text{and} \qquad \Psi = B(r) \sin(m\theta), \tag{81}$$

with m an integer. Consistency in (74) requires retaining only $\cos(m\theta)$ terms, hence $\frac{\partial \eta}{\partial \theta} = 0$. Plugging these forms into (73)-(75), we get

$$\frac{m}{r} B' A \sin^2(m\theta) + \frac{m}{r} BA' \cos^2(m\theta) - \frac{1}{r} \frac{d}{dr}(r\eta') = 0, \tag{82}$$

$$\frac{m}{r} B\eta' - \frac{1}{r}(rA'' + A') + \frac{m^2}{r^2} A - mB = 0, \tag{83}$$

$$\frac{\mu}{r}(rB'' + B') - \frac{\mu m^2}{r^2} B - \frac{m}{2r} A\eta' + \frac{m}{2} A = 0, \tag{84}$$

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[1] Assuming now separation of variables with $\Psi = \hat{\Psi}(r) \sin(m\theta)$ (or $\Psi = \hat{\Psi}(r) \cos(m\theta)$), we notice that Ψ necessarily verifies (80) if it verifies Helmholtz' equation

$$\triangle \Psi = -\frac{m}{\sqrt{2\mu}} \Psi,$$

as then

$$\triangle^2 \Psi = \frac{m^2}{2\mu} \Psi = -\frac{1}{2\mu} \frac{\partial^2 \Psi}{\partial \theta^2}.$$

Based on the previous analysis for small-energy stirring, we require A and B be proportional (let $B = \lambda A$) such that

$$B'A = A'B. \tag{85}$$

Then we can re-write (82) as:

$$B'B\lambda^{-1}m = \frac{d}{dr}(r\eta'), \tag{86}$$

hence

$$r\eta' = \frac{m}{2\lambda}B^2, \tag{87}$$

where the integration constant has been set to 0 so as to ensure regularity of η at $r = 0$ (regularity of the velocity field across the disc center requires $B(0) = 0$ for $m \neq 0$). Using $A = B\lambda^{-1}$, (83) and (84) become

$$(r^2 B'' + rB') - m^2 B - \lambda mr\eta'B + \lambda mr^2 B = 0, \tag{88}$$

$$(r^2 B'' + rB') - m^2 B - \lambda \frac{m}{2\mu}r\eta'B + \lambda \frac{m}{2\mu}r^2 B = 0, \tag{89}$$

with $\lambda = (2\mu)^{-\frac{1}{2}}$ for (88) and (89) to be equivalent. Then using (87) the ODE on B becomes:

$$\boxed{r^2 B'' + rB' + \left(r^2\lambda m - m^2\right)B = \frac{m^2}{2}B^3}. \tag{90}$$

This eigenproblem, with eigenvalue $m\lambda$, has to be solved under homogeneous Dirichlet boundary conditions $B(0) = 0$ and $B(1) = 0$ and the energy constraint:

$$2E = \langle |\nabla\Psi|^2 \rangle = \iint \left(B'^2 \sin^2(m\theta) + \frac{B^2 m^2}{r^2}\cos^2(m\theta) \right) r dr d\theta \tag{91}$$

$$\boxed{2E = \pi \int_0^1 \left(B'^2 + \frac{B^2 m^2}{r^2} \right) r dr}. \tag{92}$$

An analytical solution of (90) under the energy constraint and boundary conditions mentioned above will be undertaken asymptotically in the two limiting cases of small ($E \to 0$, or equivalently $Pe \to 0$) and large-energy flows ($E \to \infty$, $Pe \to \infty$).

4   Small E Case: The Linear Problem

In the limit of small-energy flow, the cubic term is omitted from (90) and we recover a Bessel equation:

$$r^2 B'' + rB' + \left(r^2\lambda m - m^2\right)B = 0. \tag{93}$$

Rescaling $\tilde{r} = \alpha r$ with $\tilde{B}(\tilde{r}) = B(r)$, and choosing $\alpha = \sqrt{\lambda m}$ yields

$$\tilde{r}^2 \tilde{B}'' + \tilde{r}\tilde{B}' + \left(\tilde{r}^2 - m^2\right)\tilde{B} = 0. \tag{94}$$

We look for non-axisymmetric solutions as the $m = 0$ mode evidently has poor mixing properties, the flow only sweeping particles along the iso-contours of C in the purely diffusive case. We expect the optimum flow pattern to favor radial transport so as to quickly expel particles toward the boundary.

For a given mode m, non-singular solutions of (94) are proportional to the Bessel function $J_m(\tilde{r})$ whose (stricly) positive roots determine the constant λ (and therefore μ) so as to meet the homogeneous Dirichlet boundary condition $B(1) = 0$. This requires, for a given m:

$$\sqrt{m\lambda} = j_{m,n} = \sqrt{\frac{m}{\sqrt{2\mu}}} \tag{95}$$

where $j_{m,n}$ is the n-th positive root of the Bessel function $J_m(\tilde{r})$.

Coming back to the linearized ODE, we divide (93) by r and rewrite it as:

$$(rB')' - \frac{m^2}{r}B = -\lambda m r B \tag{96}$$

Multiply by B and integrate in r leads in

$$\int_0^1 \lambda m r B^2 dr = \int_0^1 \left(-B(rB')' + \frac{m^2}{r}B^2\right) dr$$

$$= \int_0^1 rB'^2 + \frac{m^2}{r}B^2 dr$$

since $B = 0$ on $r = 0$ and $r = 1$. Thus with $\tilde{B} = \beta J_m$ we have

$$\frac{2E}{\pi} = \lambda m \int_0^1 rB^2 dr = j_{m,n}^2 \int_0^1 r\tilde{B}(j_{m,n}r)^2 dr = j_{m,n}^2 \beta^2 \int_0^1 rJ_m(j_{m,n}r)^2 dr \tag{97}$$

and using the result (see [6]) that $\int_0^1 rJ_m(j_{m,n}r)^2 dr = \frac{1}{2}J_m'(j_{m,n})^2$, we finally obtain the amplitude of the solution:

$$\beta^2 = \frac{4E}{\pi j_{m,n}^2} \frac{1}{J_m'(j_{m,n})^2}. \tag{98}$$

The mean expected exit time is given by (68), where in the unit disc case $\langle C \rangle = \frac{\pi}{8}$. Since $B = O(E)$ and $\langle |\nabla \eta|^2 \rangle = O(E^4)$ (following (87)), this last contribution can be neglected in the calculation of $\langle T \rangle$ and we find:

$$\frac{\langle T \rangle}{\langle C \rangle} = 1 - \mu \frac{8E}{\pi}. \tag{99}$$

Optimal stirring efficiency therefore requires μ be as large as possible. Since

$$\mu = \frac{1}{2}\frac{m^2}{j_{m,n}^4}, \tag{100}$$

this yields

$$\boxed{\langle T \rangle = \frac{\pi}{8} - \frac{m^2}{2j_{m,n}^4}E,} \tag{101}$$

and the optimal streamlines pattern necessarily displays a single cell in the radial direction ($n = 1$). For $m = 1$, $n = 1$, we find $\mu = 0.00232$; for $m = 2$, $n = 1$: $\mu = 0.00287$; for $m = 3$, $n = 1$: $\mu = 0.00271$. Thus optimal stirring at low energy budget (or low Péclet number) is realised for $\boxed{(m,n) = (2,1)}$ (see pattern on figure 1).
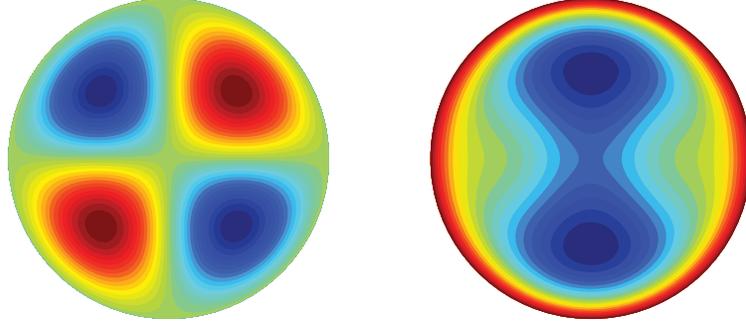
Figure 1: *Left:* Streamlines pattern and *Right:* τ-isocontours for the optimal stirring flow at small energy E, where $\tau = T - C$ is the correction to the purely conductive exit time (dark blue denotes "large", negative values, with $\tau = 0$ on the boundary).

5   The Large Energy Limit

5.1   Fixed m, large E case

For a given mode m, let us first consider the asymptotic behavior of (90) at large energy (the details of the calculations are left to the appendix).

Dominant balance in the bulk of the flow occurs between the eigenvalue term and the cubic one provided $\lambda \sim B^2$, resulting in a linear leading term for the outer solution in the limit of infinitely large E. A peripheral boundary layer, of thickness ϵ, accommodates for the homogeneous Dirichlet boundary condition in $r = 1$, while local analysis reveals a behavior of $B \sim r^m$ near the origin, in a region whose typical thickness goes to 0 as E tends to infinity (this internal layer existing for $m > 1$ only). Neglecting this region in a first approximation, we thus form the following composite solution:

$$B \approx \sqrt{\frac{2\lambda}{m}} r \tanh\left(\sqrt{\frac{m\lambda}{2}}(r - 1)\right). \tag{102}$$

The energy constraint on (102) requires $B = O(E^{\frac{1}{3}}) = \epsilon^{-1}$ and determines the value of λ at leading order:

$$\lambda = \left(\frac{9m}{2\pi^2}\right)^{\frac{1}{3}} E^{\frac{2}{3}} = \frac{1}{\sqrt{2\mu}}. \tag{103}$$

We then compute the L^1-norm of the expected exit time at infinitely large E:

$$\langle T \rangle = \langle C \rangle + \frac{1}{2}\langle \eta \rangle = \frac{\pi}{8} - \frac{\pi}{2}\int_0^1 \eta' r^2 dr = \frac{\pi}{8} - \frac{\pi}{4}\int_0^1 \lambda^{-1} m B^2 r dr. \tag{104}$$

Replacing B by the composite solution (102) and splitting the integral at $1 - \delta$ with $\epsilon \ll$

$\delta \ll 1$, we roughly get after introducing $x = \sqrt{\frac{m\lambda}{2}}(r-1)$:

$$\langle T \rangle \approx \frac{\pi}{8} - \frac{\pi}{2} \int_0^{1-\delta} r^3 dr - \frac{\pi}{2}\sqrt{\frac{2}{m\lambda}} \int_0^{\sqrt{\frac{m\lambda}{2}}\delta} \tanh^2(x) dx \tag{105}$$

$$\approx \frac{\pi}{8} - \frac{\pi}{8}(1-\delta)^4 - \frac{\pi}{2}\delta + \frac{\pi}{2}\sqrt{\frac{2}{m\lambda}}. \tag{106}$$

Finally

$$\langle T \rangle = \frac{\pi}{2}\sqrt{\frac{2}{m\lambda}} + o(\delta) + o(\epsilon), \tag{107}$$

and, replacing λ by (103), we find at leading order:

$$\boxed{\langle T \rangle = \left(\frac{\pi^4}{6}\right)^{\frac{1}{3}} m^{-\frac{2}{3}} E^{-\frac{1}{3}}.} \tag{108}$$

The calculation of the mean exit time in (105) shows that the contribution of the conductive exit time $\langle C \rangle = \frac{\pi}{8} = O(1)$ is exactly canceled by the flow in the bulk (outer region) provided we assume linear behavior for B everywhere except in the wall boundary layer. On the other hand, the remaining, leading-order mean exit time $O(\epsilon)$ is solely determined by this peripheral boundary layer profile.

The large E, fixed m expected exit time (108) appears to correctly describe the asymptotic behavior of the solution for a given mode as will be seen in the next paragraph. However, asymptotics at fixed m do not provide any evidence for the existence of an optimum flow pattern as the mean exit time goes to 0 if m is chosen arbitrarily large. The existence of an optimum, if any, must result from a "penalty" on large wavenumbers m, which will arise from taking the distinguished limit for large m and large E.

5.2 Some numerics

For different values of the wavenumber m, we solve for the non-linear eigenvalue problem (90) by means of a continuation method, using the dedicated **bvp5c** function ([11]) in MATLAB with λ as a parameter: starting from the Bessel function solution as an initial in the quasi-linear regime (typically $E = 0.001 - 0.1$ depending on m), E is gradually increased up to $E = 10^8$, the output of each computation providing a "first guess" for the next one. [2]

The numerical results are in excellent agreement with the asymptotics and reproduce the scaling (103) of $\lambda = O(E^{\frac{2}{3}})$. As can be seen on figure 2 *(top)*, the numerical mean exit time for a given m perfectly superimposes with the large-energy asymptotics provided E is large enough, with a decay $\sim E^{-\frac{1}{3}}$.

---

[2]The function bvp5c (as its older version bvp4c) relies on a collocation method where the non-linear eigenvalue problem is solved iteratively by linearizing around the (user-provided) initial solution. This unfortunately prevents one from capturing all the possibly existing solutions (and in particular, flows with more cellular structures in the radial direction) for numerical convergence requires a relatively accurate first guess.
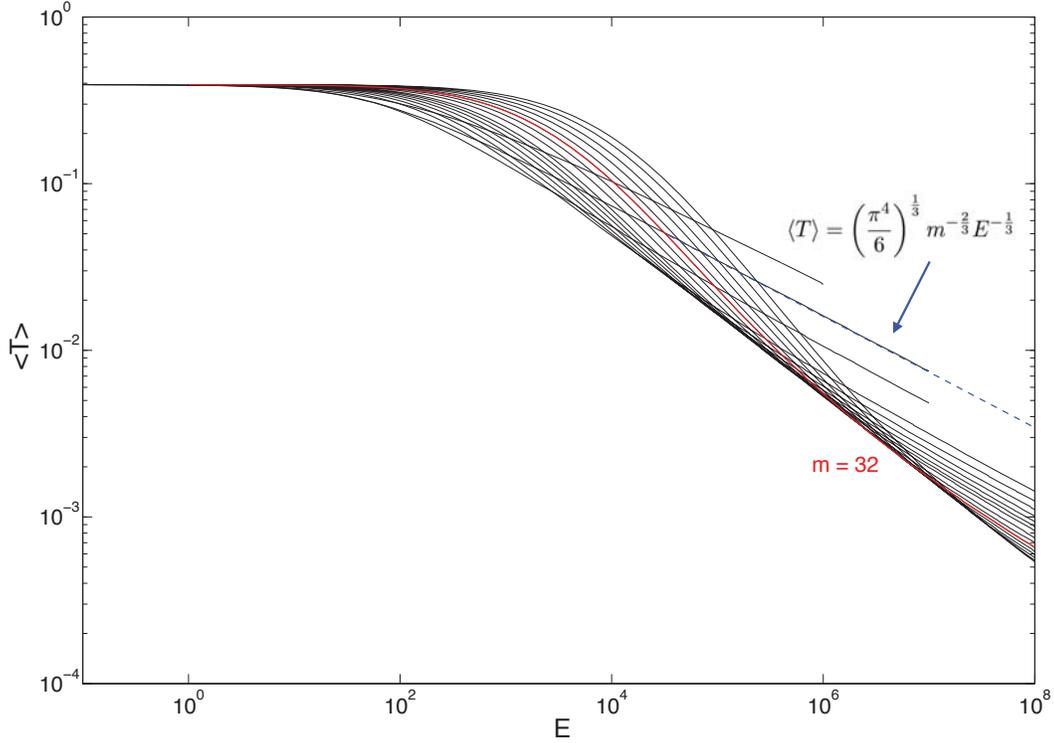
Figure 2: Solid lines: mean expected exit time versus energy (numerical result from bvp5c), for the wavenumber: $m = 1, 2, 4, 8, 10, 12, 14, 16, 18, 20, 24, 28, 32, 36, 42, 48, 56, 64$. Dashed, blue line: large E, fixed m asymptotics, for $m = 2$. As an example, the result obtained for mode $m = 32$ is highlighted in red.

The superposition of mean exit times for various wavenumbers on figure 2 clearly indicates the existence of an optimal m at a given energy E. The minimal mean exit time corresponds to the lower envelope of the various m graphs, whose equation seems to satisfy a power law in E (namely $\sim E^{-\frac{1}{2}}$, as will be shown in the following).

5.3   Large m, large E case

The "penalty" on large wavenumbers actually results from the presence of an internal layer (for $m > 1$): indeed for large m, the streamlines of the divergence-free flow strongly tighten near the center of the disc where diffusion is likely to overpower radial transport. This creates a very-low velocity region (a "stagnant" zone) which widen at fixed E with increasing m - and shrinks at increasing E for fixed m. In this region, the flow is nearly ineffective and the exit time corresponds to the purely conductive one. Hence for a given energy budget, an optimum flow has to be found in combining a large number of cells that efficiently expel particles toward the wall with a stagnation area of limited extent at the center of the disc.

5.3.1 A composite solution for the large-energy flow

The outer solution (bulk):   If we consider large wavenumbers and assume that m scales as a power of E, (90) at leading order degenerates into:

$$\left(m\lambda r^2 - m^2\right)B = \frac{m^2}{2}B^3. \tag{109}$$

As in the fixed m case, balancing the eigenvalue and the cubic terms yields $\lambda \sim mB^2$. The difference in the solution arises from the second term in the LHS of (109), whose weight becomes comparable to the one of the first term for a typical radius $r_\times$ such that:

$$r_\times = \sqrt{\frac{m}{\lambda}}. \tag{110}$$

The solution of (109) (outer solution) is then

$$B = \sqrt{\frac{2\lambda}{m}(r^2 - r_\times^2)}. \tag{111}$$

This solution breaks down for a radius of $r \approx r_\times$, which is the typical thickness of the stagnation zone. (Note that we have then $r_\times \sim B^{-1}$, implying that the stagnation zone shrinks as the energy budget increases.) As r approaches 1, we have

$$B \approx \sqrt{\frac{2\lambda}{m}}r\left(1 - \frac{1}{2}\frac{r_\times^2}{r^2}\right) \tag{112}$$

$$\approx \sqrt{\frac{2\lambda}{m}}r \qquad \text{at leading order since } r_\times^2 \sim B^{-2}. \tag{113}$$

The leading terms in (109) as the wall is approached are therefore the first and the third ones, which have to be retained while solving for B near $r = 1$.

The inner solution ($r \to 1$):   A boundary layer develops on the wall so as to accommodate for the Dirichlet homogeneous condition on B. Writing ε the typical thickness of this peripheral layer, we rescale the radial coordinate as $r = 1 - \varepsilon\rho$. Expressing (90) in the fast variable ρ yields:

$$(1 - \varepsilon\rho)^2\varepsilon^{-2}B'' + (1 - \varepsilon\rho)\varepsilon^{-1}B' + m\lambda(1 - \varepsilon\rho)^2 B - m^2 B = \frac{m^2}{2}B^3, \tag{114}$$

and retaining the higher order derivative in the dominant balance implies $\lambda m \sim \varepsilon^{-2}$. At leading order, since $\lambda \sim mB^2$ we have

$$\varepsilon^{-2}B'' + (m\lambda - m^2)B = \frac{m^2}{2}B^3. \tag{115}$$

The solution of (115) (inner solution) satisfying the boundary condition in $r = 1$ (or $\rho = 0$) is

$$B = \sqrt{\frac{2\lambda}{m}\left(1 - \frac{m}{\lambda}\right)}\tanh\left(k\rho\right), \tag{116}$$

where $k = \varepsilon\sqrt{\frac{m\lambda - m^2}{2}}$. This inner solution clearly satisfies asymptotic matching with the outer solution (using (111)).

The stagnation zone (internal layer): Local analysis in the vicinity of the center reveals that $B \sim r^m$ as r goes to zero - there the stirring is largely ineffective. For this reason the composite solution for B which is proposed in the next paragraph, where $B = 0$ is assumed everywhere in the stagnation zone, turns out to provide sufficient accuracy for the calculations to come. However, a complete description of the asymptotic solution is much more satisfactory and we aim at capturing the region where the outer solution (111) breaks down as well as its reconnection with the stagnant zone, which seems hard to reconcile with the infinite slope of the bulk solution in $r = r_\times$. Introducing the change of variables $t = \ln\left(\frac{r}{r_\times}\right)$, (90) becomes:

$$B_{tt} + \left(\lambda m r_\times^2 e^{2t} - m^2\right) B = \frac{m^2}{2} B^3. \tag{117}$$

Linearizing around $t = 0$ (or equivalently $r = r_\times$) yields

$$B_{tt} + \left(\lambda m r_\times^2 (1 + 2t) - m^2\right) B = \frac{m^2}{2} B^3 \qquad \text{at leading order,} \tag{118}$$

$$B_{tt} + 2m^2 t B = \frac{m^2}{2} B^3 \qquad \text{as } r_\times = \sqrt{\frac{m}{\lambda}}. \tag{119}$$

Let us rescale the variable $s = \alpha t$ and the unknown function $\beta b(s) = B(t)$. The choice of $\alpha = (2m^2)^{\frac{1}{3}}$ and $\beta = \left(\frac{16}{m}\right)^{\frac{1}{3}}$ results in recovering a second Painlevé's transcendent equation with constant zero:

$$b'' = 2b^3 - sb. \tag{120}$$

Fortunately, this equation does admit a particular solution - namely the Hastings-McLeod solution (see [4], up to a change of sign $x \leftarrow -x$) - which asymptotically satisfies:

$$b_{HM}(s) \underset{s \to -\infty}{\to} 0 \qquad \text{and} \qquad b_{HM}(s) \underset{s \to +\infty}{\sim} \sqrt{\frac{s}{2}}. \tag{121}$$

Moreover, the Hastings-McLeod solution decays toward 0 as a Airy function as $s \to -\infty$, thus displaying the correct behavior for $r \to 0$, and its asymptotic matching with the bulk solution (111) takes care of itself as $r > r_\times$: there

$$B_{HM}(r) \sim \left(\frac{16}{m}\right)^{\frac{1}{3}} \sqrt{\frac{(2m^2)^{\frac{1}{3}}}{2} \ln\left(1 + \frac{r - r_\times}{r_\times}\right)}, \tag{122}$$

$$\sim 2\sqrt{\frac{(r - r_\times)}{r_\times}}, \tag{123}$$

while the bulk solution (111) in the vicinity of $r_\times$ is equivalent to:

$$B(r) = \sqrt{\frac{2}{r_\times^2}(r^2 - r_\times^2)} \underset{r \to r_\times^+}{\sim} 2\sqrt{\frac{(r - r_\times)}{r_\times}}. \tag{124}$$

Nevertheless, the calculation of the asymptotic mean exit time in the next paragraphs will be made considerably simpler by ignoring this last refinement and adopting the expression (125)!
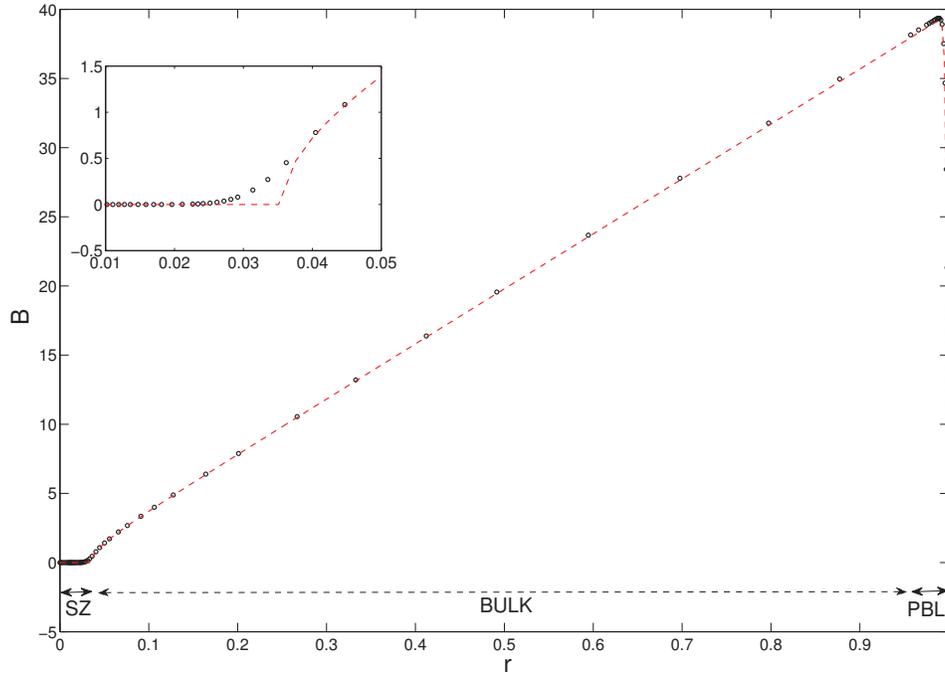
Figure 3: Black circles: numerical solution for $m = 16$, $E = 8.4e5$, computed with bvp5c by a continuation method. Red, dashed line: approximated composite (following (125)) for the corresponding parameters, with the eigenvalue $\lambda(m, E)$ provided by the dispersion relation (142) (see next section). *Inset:* Blow-up on the vicinity of $r_\times$. The different flow regions are indicated on the graph: SZ = internal boundary layer or "stagnation zone", PBL = peripheral boundary layer, BULK = main flow.

A composite solution   In the following, let us approximate the full solution by the composite:

$$B \approx \begin{cases} \sqrt{\frac{2\lambda}{m}(r^2 - r_\times^2)} \tanh\left[k\varepsilon^{-1}(1-r)\right], & \text{if } r_\times < r; \\ 0, & \text{if } 0 < r < r_\times. \end{cases} \tag{125}$$

Indeed, superposition with the numerical solution of (90) on figure 3 (obtained with the function bvp5c using our continuation method) shows excellent agreement except for the small region restricted to the vicinity of $r_\times$, which we should neglect in the next calculations.

5.3.2   Dispersion relation and optimal exit time

So far the eigenvalue λ is still unknown and its relationship with m and E will be determined by the fixed energy budget, and will in turn be used in the minimisation of the mean expected exit time. Multiplying (90) by B and integrating yields a useful, alternative expression for the energy constraint:

$$rB'' + B' + (m\lambda r - \frac{m^2}{r})B = \frac{m^2}{2}B^3 \tag{126}$$

$$\implies \int_0^1 \left(B(rB')' + (m\lambda r - \frac{m^2}{r})B^2\right) dr = \int_0^1 \frac{m^2}{2}B^4 dr \tag{127}$$

$$\int_0^1 \left(rB'^2 + \frac{m^2}{r}B^2\right) dr = \int_0^1 \left(m\lambda r B^2 - \frac{m^2}{2}B^4\right) dr \tag{128}$$

$$\frac{2E}{\pi} = \int_0^1 \left(m\lambda r B^2 - \frac{m^2}{2}B^4\right) dr. \tag{129}$$

Injecting (125) into the last expression, we find:

$$\frac{2E}{\pi} = m^2 \int_0^1 \left(\frac{\lambda}{m}rB^2 - \frac{1}{2}\frac{B^4}{r}\right) dr = m^2 \int_0^1 rB^2 \left(\frac{1}{r_\times^2} - \frac{1}{2}\frac{B^2}{r^2}\right) dr. \tag{130}$$

Noting that the term inside the brackets can be rewritten as

$$\frac{1}{r_\times^2} - \frac{1}{2}\frac{B^2}{r^2} = \frac{1}{r_\times^2} - \frac{1}{r^2}\left(\frac{r^2}{r_\times^2} - 1\right)\tanh^2(k\rho) \tag{131}$$

$$= \frac{1}{r_\times^2} - \frac{1}{r_\times^2}\tanh^2(k\rho) + \frac{1}{r^2}\tanh^2(k\rho) \tag{132}$$

$$= \frac{1}{r_\times^2}\cosh^{-2}(k\rho) + \frac{1}{r^2}\tanh^2(k\rho), \tag{133}$$

we obtain:

$$\frac{2E}{\pi} = m^2 \int_{r_\times}^1 2r\left(\frac{r^2}{r_\times^2} - 1\right)\tanh^2(k\rho)\left(\frac{\cosh^{-2}(k\rho)}{r_\times^2} + \frac{\tanh^2(k\rho)}{r^2}\right)dr \tag{134}$$

$$= \frac{2m^2}{r_\times^2}\int_{r_\times}^1 (r^3 - rr_\times^2)\left(1 - \cosh^{-2}(k\rho)\right)\left(\frac{\cosh^{-2}(k\rho)}{r_\times^2} + \frac{\tanh^2(k\rho)}{r^2}\right)dr \tag{135}$$

$$= \frac{2m^2}{r_\times^2}\int_{r_\times}^1 (r^3 - rr_\times^2)\left(\frac{\cosh^{-2}(k\rho)}{r_\times^2} - \frac{\cosh^{-4}(k\rho)}{r_\times^2} + \frac{1}{r^2} - \frac{2\cosh^{-2}(k\rho)}{r^2} + \frac{\cosh^{-4}(k\rho)}{r^2}\right)dr \tag{136}$$

$$\frac{E}{\pi}\frac{r_\times^2}{m^2} = \int_{r_\times}^1 \left(\underbrace{(r - \frac{r_\times^2}{r})}_{(a)} + \underbrace{\frac{1}{\cosh^2}(k\rho)\left(\frac{r^3}{r_\times^2} - 3r + \frac{2r_\times^2}{r}\right)}_{(b)} + \underbrace{\frac{1}{\cosh^4}(k\rho)(k\rho)\left(-\frac{r^3}{r_\times^2} + 2r - \frac{r_\times^2}{r}\right)}_{(c)}\right)dr. \tag{137}$$

Retaining only the leading term in each contribution (a),(b),(c) yields:

$$\frac{E}{\pi}\frac{r_\times^2}{m^2} = \int_{r_\times}^1 rdr + \frac{\varepsilon}{k}\int_0^{\frac{k}{\varepsilon}r_\times}\frac{(1 - \frac{\varepsilon}{k}x)^3}{r_\times^2\cosh^2(x)}dx - \frac{\varepsilon}{k}\int_0^{\frac{k}{\varepsilon}r_\times}\frac{(1 - \frac{\varepsilon}{k}x)^3}{r_\times^2\cosh^4(x)}dx \tag{138}$$

$$\approx \frac{1}{2} + \frac{\varepsilon}{k}\int_0^{\frac{k}{\varepsilon}r_\times}\frac{1}{r_\times^2\cosh^2(x)}dx - \frac{\varepsilon}{k}\int_0^{\frac{k}{\varepsilon}r_\times}\frac{1}{r_\times^2\cosh^4(x)}dx \tag{139}$$

$$\approx \frac{1}{2} + \frac{1}{r_\times^2}\frac{\varepsilon}{k} - \frac{2}{3r_\times^2}\frac{\varepsilon}{k} \tag{140}$$

$$\approx \frac{1}{2} + \frac{1}{3r_\times^2}\frac{\varepsilon}{k}. \tag{141}$$

Finally, recalling that at leading order $k = \varepsilon\sqrt{\frac{m\lambda}{2}}$ and $r_\times = \sqrt{\frac{m}{\lambda}}$, we obtain a compact expression for the energy constraint as a function of m and λ:

$$\boxed{\frac{2E}{\pi} = m\lambda + \frac{2\sqrt{2}}{3}\sqrt{\frac{\lambda^3}{m}}.} \tag{142}$$

Similarly, we compute the mean exit time at leading order:

$$\langle T \rangle = \frac{\pi}{8} - \frac{\pi}{4}\int_0^1 \gamma m B^2 rdr \tag{143}$$

$$= \frac{\pi}{8} - \frac{\pi}{2}\int_{r_\times}^1 r^3(1 - \frac{r_\times^2}{r^2})\tanh^2(k\rho)dr \tag{144}$$

$$= \frac{\pi}{8} - \frac{\pi}{2}\frac{\varepsilon}{k}\int_0^{\frac{k}{\varepsilon}r_\times}(1 - \frac{\varepsilon}{k}x)^3\left(1 - \frac{r_\times^2}{(1 - \frac{\varepsilon}{k}x)^2}\right)\tanh^2(k\rho)dr \tag{145}$$

$$\approx \frac{\pi}{8} - \frac{\pi}{2}\left(\frac{1}{4} - \frac{r_\times^2}{2} + \frac{\delta_{SZ}^4}{4} + \frac{\varepsilon}{k}\right) \tag{146}$$

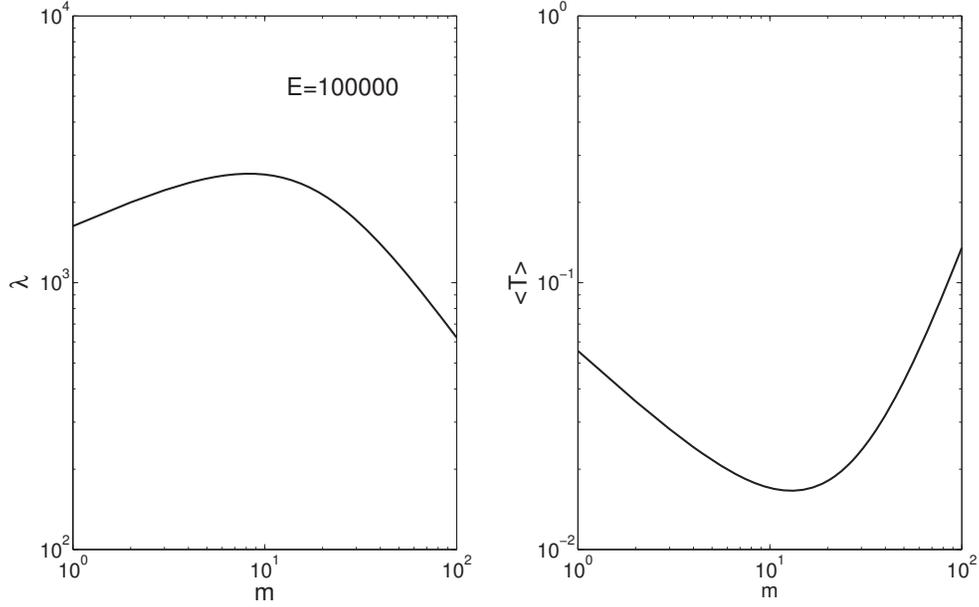$$\approx \frac{\pi r_\times^2}{4} + \frac{\pi}{2}\frac{\varepsilon}{k}, \tag{147}$$

Figure 4: *Left:* eigenvalue λ against azimuthal mode m, for a given kinetic energy budget $E = 10^6$. *Right:* expected exit time (in L^1-norm) against azimuthal mode m.

which in terms of m and λ gives:

$$\langle T \rangle = \frac{\pi}{4} \frac{m}{\lambda} + \frac{\pi}{\sqrt{2}} (m\lambda)^{-\frac{1}{2}}. \tag{148}$$

For a given E, we can already draw - numerically - the graph of λ and $\langle T \rangle$ as a function of m as illustrated on figure 5.3.2.

Considering the energy constraint, we can now guess the scaling for m and λ requiring all the terms to be of the same order. If $m = O(E^\nu)$ and $\lambda = O(E^\beta)$, we get from (142) that $1 = \nu + \beta = \frac{3}{2}\beta - \frac{1}{2}\nu$ i.e.

$$\nu = \frac{1}{4} \qquad \text{and} \qquad \beta = \frac{3}{4}. \tag{149}$$

As a result, we also know the stagnation zone and the peripheral typical thicknesses:

$$r_\times = O(E^{-\frac{1}{4}}) \qquad \text{and} \qquad \varepsilon = O(E^{-\frac{1}{2}}). \tag{150}$$

Accordingly, let us write $m = \left(\frac{2E}{\pi}\right)^{\frac{1}{4}} \tilde{m}$, $\lambda = \left(\frac{2E}{\pi}\right)^{\frac{3}{4}} \tilde{\lambda}$ and the conduction-normalised mean exit time $\frac{8}{\pi} \langle T \rangle = \left(\frac{2E}{\pi}\right)^{-\frac{1}{2}} \tilde{T}$. The energy constraint then becomes a dispersion relation between m and λ:

$$1 = \tilde{m}\tilde{\lambda} + \frac{2\sqrt{2}}{3} \frac{\tilde{\lambda}^{\frac{3}{2}}}{\tilde{m}^{\frac{1}{2}}}, \tag{151}$$

and the exit time estimate is now:

$$\tilde{T} = 2\tilde{m}\tilde{\lambda}^{-1} + \frac{8}{\sqrt{2}}(\tilde{m}\tilde{\lambda})^{-\frac{1}{2}}. \tag{152}$$

For the sake of simplification we introduce a new variable $Z = \tilde{\lambda}^{-\frac{1}{2}}\tilde{m}^{\frac{3}{2}}$ and re-write both equations in terms of Z, thus eliminating $\tilde{\lambda}$. This yields respectively

$$1 = \tilde{m}^4 \left(\frac{1}{Z^2} + \frac{2\sqrt{2}}{3Z^3} \right) \quad \text{and} \quad \tilde{T} = \left(2Z + 4\sqrt{2} \right) \left(1 + \frac{2\sqrt{2}}{3Z} \right)^{\frac{1}{2}}. \tag{153}$$

The energy constraint provides an expression for $\tilde{m}$ as a function of Z and, plugging this into the second expression in (153), we derive the conduction-normalized mean exit time as a function of Z solely. The asymptotic behavior of $\tilde{T}(Z)$ as $Z \to 0$ and $Z \to \infty$ clearly indicates the existence of a global minimum on R_+, which is obtained for

$$\frac{\partial \tilde{T}}{\partial Z} = 0 \quad \implies \quad 0 = Z^2 + \frac{\sqrt{2}}{3}Z - \frac{4}{3} \quad \implies \quad Z_{>0} = \frac{2\sqrt{2}}{3}. \tag{154}$$

Therefore the value of $\tilde{m}^*$ and $\tilde{\lambda}^*$ that correspond to optimal flow efficiency are

$$\tilde{m}^* = \sqrt{\frac{2}{3}} \quad \text{and} \quad \tilde{\lambda}^* = \sqrt{\frac{3}{8}} \quad \text{i.e.} \quad m^* = \sqrt{\frac{2}{3}}\left(\frac{2E}{\pi}\right)^{\frac{1}{4}} \quad \text{and} \quad \lambda^* = \sqrt{\frac{3}{8}}\left(\frac{2E}{\pi}\right)^{\frac{3}{4}}. \tag{155}$$

For this optimal mode m^*, the achieved, conduction-normalized mean exit time is then:

$$\boxed{\frac{\langle T \rangle}{\langle C \rangle} = \frac{32}{3}\left(\frac{2E}{\pi}\right)^{-\frac{1}{2}}.} \tag{156}$$

As can be seen on figure 5, (156) is in excellent agreement with the numerical solutions of the eigenvalue problem (90) found with bvp5c: the straight (dashed) line corresponding to the asymptotic optimal exit time perfectly matches the lower envelope of the different modes m for E larger than 10^4 approximately.

5.4   A fast mixer

The result in (156) is expressed in terms of dimensionless energy. Using the fact that the master equations were made dimensionless in diffusive timescale, we have in dimensional variables:

$$E^{\dagger} = \left\langle |u^{\dagger}|^2 \right\rangle \propto \kappa^2 E \quad \text{and} \quad \left\langle C^{\dagger} \right\rangle \propto \frac{L^4}{\kappa} \quad \implies \quad \left\langle T^{\dagger} \right\rangle \propto \frac{L^4}{\kappa} \frac{\kappa}{E^{\dagger\frac{1}{2}}} \tag{157}$$

i.e.

$$\boxed{\left\langle T^{\dagger} \right\rangle \propto \frac{L^3}{U}.} \tag{158}$$

It is important to emphasize that due to the very particular scaling of (156), the dimensional mean exit time (158) is completely independent of the molecular diffusivity: κ may be
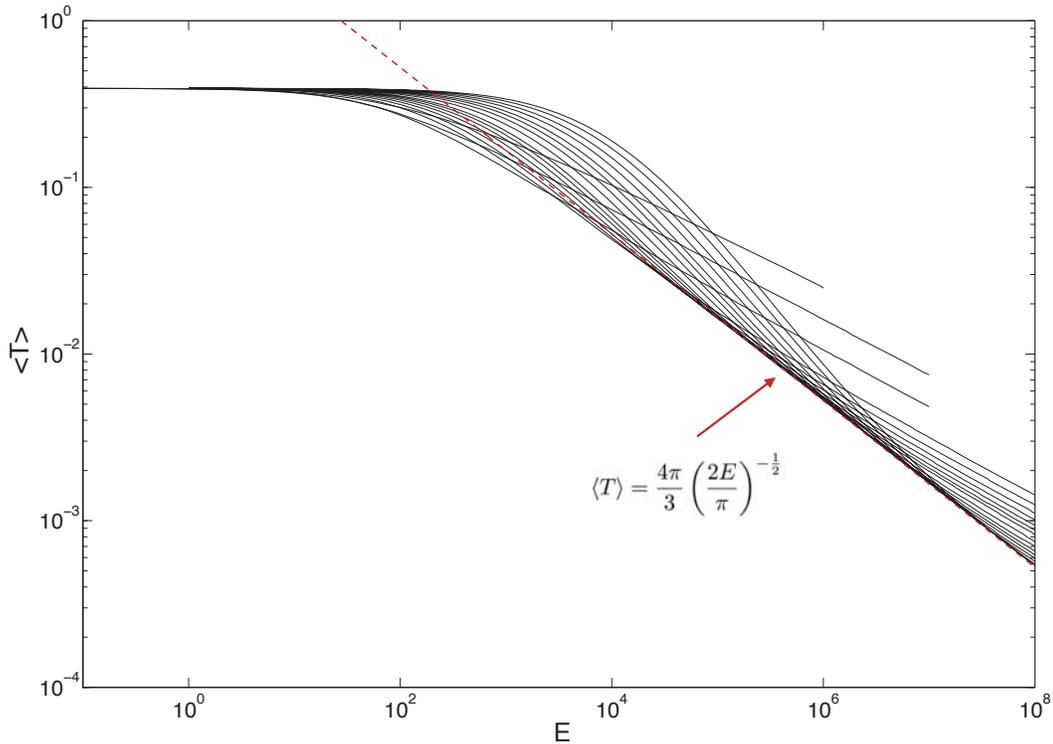
Figure 5: Blue, solid lines: mean expected exit time (from bvp5c) as a function of the kinetic energy, for different wavenumbers (as in figure 2). The black, dashed line represents the asymptotic, optimal exit time as computed from (156).

chosen arbitrarily small (but non-zero) and the particles will be expelled from the domain in purely mechanical time (once they have reached the external boundary layer, diffusivity is still needed for the particles to exit the domain despite impermeability boundary condition). This result is consistent with the bound on mixing efficiency derived in [13] at large Péclet number, which turns out to be independent of the molecular diffusivity (and is expected to hold under turbulent or chaotic mixing). A comparison can be drawn between this physical result and the existence of the so-called fast dynamos (as, for instance, the Ponomarenko ([10]) or the ABC flows dynamo ([1], [3]), whose growth-rate becomes independent of the magnetic diffusivity in the limit of large magnetic Reynolds number.

   Although the problem we consider here is quite different from the one addressed for instance by [14] (source optimization), and even though we use a different metric for quantifying the mixing efficiency (they consider the ratio of the L^2-norms for the scalar concentration without and with stirring, see the case $p = 0$ in [14]), we could recast (156) in terms of a "mixing enhancement factor"

$$\mathscr{E} = \frac{\langle C \rangle}{\langle T \rangle} = \frac{3}{32} \left(\frac{2E}{\pi} \right)^{\frac{1}{2}}, \tag{159}$$

and recover a similar linear dependency of the enhancement factor with Péclet number in the asymptotic $Pe \to \infty$ regime.

Conclusion

Although neither our analytical approach (due to the choice of the *Ansatz* (85)), nor the numerical one (due to the use of a continuation method, the routine bvp5c being highly sensitive to the initial) enables us to capture any existing bifurcation in the large-E regime, we exhibit a solution which at least achieves mechanical exit time. Direct numerical simulation would be necessary to ensure that the solution we find with our choice of an *Ansatz*, which is suggested by the small-energy case, truly corresponds to the general optimum over all possible flows, including small-scale, cellular patterns. However, assuming a stronger exponent for E in (156) - say $E^{-\alpha}$ with $\alpha > \frac{1}{2}$ - would yield

$$\left\langle T^{\dagger} \right\rangle \propto \left(\frac{L^4}{E^{\dagger \alpha}} \right) \kappa^{2\alpha - 1}, \tag{160}$$

meaning that the L^1-norm of the expected exit time corresponding to the optimal flow would be enhanced by increasing molecular diffusivity, which seems unconvincing as a higher diffusivity tends to improve mixing. We therefore expect any improvement compared to the solution we find with (85) to deal only with the prefactor in (156). An important source for improvement would presumably reside in considering the mixing enhancement achieved by a time-dependent flow, which we have not attempted, but would be of considerable interest also from an engineering point of view.

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References

[1] CHILDRESS, S. & GILBERT, A.D. 1995 *Stretch, Twist, Fold: The Fast Dynamo* Springer, Berlin

[2] DOERING, C.R. & THIFFEAULT, J.-L. 2006 Multiscale mixing efficiencies for steady sources, *Phys. Rev. E*, **74 (2)** 025301(R).

[3] GALLOWAY, D. & FRISCH, U. 1986 Dynamo action in a family of flows with chaotic streamlines, *Geophysical & Astrophysical Fluid Dynamics* **36**, 53–83.

[4] Huang, M., Xu, S.-X. & Zhang, L. 2015 Location of Poles for the Hastings-McLeod Solution to the Second Painlev Equation, *arXiv:1410.3338v2 [math.CA]*

[5] Iyer, G., Novikov, A., Ryzhik, L. & Zlatos, A. 2010 Exit times of diffusions with incompressible drift, *SIAM J. Math. Anal.* **42** 2484–98.

[6] Lebedev, N. N. 1972 *Special functions and their applications* Dover Publications, Inc., New-York

[7] Lin, Z., Bodova, K. & Doering, C.R. 2010 Models and measures of mixing and effective diffusion, *Discrete Cont. Dyn. Syst.* **28** 259–74.

[8] Lin, Z., Thiffeault, J.-L. & Doering, C.R. 2011 Optimal stirring strategies for passive scalar mixing, *J. Fluid Mech.* **675** 465–76.

[9] Plasting, S., & Young, W.R. 2006 A bound on scalar variance for the advection-diffusion equation, *J. Fluid Mech.* **552** 289–298.

[10] Ponomarenko, Y. B. 1973 On the theory of the hydromagnetic dynamo, *J. App. Mech. & Techn. Physics* **14** 775

[11] Shampine, L.F., Reichelt, M.W. & Kierzenka, J. 2000 Solving Boundary Value Problems for Ordinary Differential Equations in MATLAB with bvp4c, available at ftp://ftp.mathworks.com/pub/doc/papers/bvp/

[12] Shaw, T. A., Thiffeault, J.-L. & Doering, C.R. 2007 Stirring up trouble: Multi-scale mixing measures for steady scalar sources, *Physica D*, **231 (2)** 143–164.

[13] Thiffeault, J.-L., Doering, C.R. & Gibbon, J.D. 2004 A bound on mixing efficiency for the advection-diffusion equation, *J. Fluid Mech.* **521**, 05–114.

[14] Thiffeault, J.-L. & Pavliotis, G.A. 2008 Optimizing the source distribution in fluid mixing, *Physica D* **237** 918–29.

[15] Thiffeault, J.-L. 2012 Using multiscale norms to quantify mixing and transport, *Nonlinearity* **25** 1–44.

A   Large E, fixed m case

In the case of large E, we look at the dominant balance taking place in the bulk of the flow. Introducing the rescaled variables $B = E^\alpha \tilde{B}$ and $\lambda = E^\beta \tilde{\lambda}$, (90) becomes:

$$r^2 \tilde{B}'' E^\alpha + r \tilde{B}' E^\alpha + r^2 \tilde{\lambda} m \tilde{B} E^{\alpha+\beta} - m^2 \tilde{B} E^\alpha = \frac{m^2}{2} \tilde{B}^3 E^{3\alpha}. \tag{161}$$

The only contribution likely to balance the cubic term is the one containing the eigenvalue, requiring $\beta = 2\alpha$. Then at leading order (90) degenerates into:

$$r^2 \tilde{\lambda} m \tilde{B} = \frac{m^2}{2} \tilde{B}^3, \tag{162}$$

which directly gives the outer solution

$$B_o = \tilde{B} E^\alpha = \pm \sqrt{\frac{2}{m} \tilde{\lambda}} E^\alpha r \tag{163}$$

This solution does not satisfy $B(1) = 0$. Proper rescaling of the radial coordinate is required so as to accommodate for the boundary condition: we introduce a new scale $r = 1 - \epsilon \rho$. The inner solution has to satisfy asymptotic matching with the outer solution, which suggests using $B = E^\alpha \bar{B}$ to get

$$\frac{(1-\epsilon\rho)^2}{\epsilon^2} \bar{B}'' E^\alpha + \frac{(1-\epsilon\rho)}{\epsilon} \bar{B}' E^\alpha + (1-\epsilon\rho)^2 \tilde{\lambda} m \bar{B} E^{3\alpha} - m^2 \bar{B} E^\alpha = \frac{m^2}{2} \bar{B}^3 E^{3\alpha}. \tag{164}$$

Dominant balance and the requirement that the higher-order derivative be retained yield $\epsilon = O(E^{-\alpha})$. At leading order, (164) then becomes:

$$\bar{B}'' + \tilde{\lambda} m \bar{B} = \frac{m^2}{2} \bar{B}^3, \tag{165}$$

a solution of which can be found under the form $\bar{B} = \bar{B}_\infty \tanh(k\rho)$ where asymptotic matching with the outer solution as $\rho \to \infty$ leads to $\bar{B}_\infty = \pm\sqrt{\frac{2\tilde{\lambda}}{m}}$ and $k = \pm\sqrt{\frac{m\tilde{\lambda}}{2}}$. At leading order the inner solution is then, up to a change of sign:

$$B_i = \sqrt{\frac{2\tilde{\lambda}}{m}} E^\alpha \tanh\left(\sqrt{\frac{m\tilde{\lambda}}{2}} \rho\right). \tag{166}$$

The energy constraint determines α:

$$\frac{2E}{\pi} = \int_0^1 r B'^2 + \frac{m^2}{r} B^2 dr = \int_0^{1-\delta} (r B_o'^2 + \frac{m^2}{r} B_o^2) dr + \int_{1-\delta}^1 (B_i'^2 + m^2 B_i^2) dr \tag{167}$$

$$\frac{2E}{\pi} = \int_0^{1-\delta} \underbrace{B_\infty^2 r(1+m^2) dr}_{=O(E^{2\alpha})} + \int_{''-\infty''}^0 \left(\underbrace{\frac{B_\infty^2 k^2}{\epsilon} \left(1 - \tanh^2(k\rho)\right)^2}_{=O(E^{2\alpha\epsilon^{-1}})} + \underbrace{m^2 B_\infty^2 \epsilon \tanh^2(k\rho))}_{=O(E^{2\alpha\epsilon})} \right) d\rho$$

$$\tag{168}$$

Dominant balance requires $\epsilon = E^{-\alpha}$ and $\alpha = \frac{1}{3}$, hence $\beta = \frac{2}{3}$. The leading terms in the energy constraint then determine the factor $\tilde{\lambda}$:

$$\frac{2}{\pi} = \int_{"-\infty"}^{0} \frac{\tilde{B}_\infty^2 k^2}{\epsilon} \left(1 - \tanh^2(k\rho)\right)^2 d\rho = \frac{\tilde{B}_\infty^2 k^2}{\epsilon} \int_{"-\infty"}^{0} \left(1 - 2\tanh^2(k\rho) + \tanh^4(k\rho)\right) d\rho \tag{169}$$

and since $\tanh^2 - \tanh^4 = \tanh^2 \tanh'$, we finally get:

$$\frac{2}{\pi} = \frac{\tilde{B}_\infty^2 k^2}{\epsilon} \left[\frac{\tanh(k\rho)}{k} - \frac{\tanh^3(k\rho)}{3k}\right]_{-\infty}^{0} = \frac{2\tilde{B}_\infty^2 k}{3\epsilon} = \frac{2\sqrt{2}\tilde{\lambda}^{\frac{3}{2}}}{3\sqrt{m}} \tag{170}$$