Nonlinear Waves:
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The Great Wave at Kanagawa

The Great Wave at Kanagawa (from a Series of Thirty-Six Views of Mount Fuji), Edo period (1615–1868), ca. 1831–33
Katsushika Hokusai (Japanese, 1760–1849); Published by Eijudo
Polychrome ink and color on paper

10.2 Lagrangian formulation

Part 2: General theory

Suppose that the physical system is governed by a **Lagrangian** $L(\phi, \phi_t, \phi_{x_i}; t, x)$ where the field variables are the vector-valued $\phi(t, x_i)$. Here t is time, and $x_i, i = 1, 2, \cdots$ are the spatial variables. The governing equations are then the Euler-Lagrange equations

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \phi_t}\right) + \frac{\partial}{\partial x_i}\left(\frac{\partial L}{\partial \phi_{x_i}}\right) - \frac{\partial L}{\partial \phi} = 0\,. \tag{1}$$

Here we use the summation convention over the index "i".

From this formulation we can recover the **conservation laws** corresponding to the symmetries of the Lagrangian. Thus, for instance, energy conservation corresponds to time symmetry, that is when $\partial L/\partial t = 0$, and momentum conservation corresponds to space symmetry, that is when $\partial L/\partial x_i = 0$. Wave action conservation corresponds to a **phase symmetry**.

10.2 Waves and mean flow

We now suppose that the solution being sought consists of **waves** and a **mean flow**. To describe the waves, we introduce a phase parameter θ, such that

$$\phi(t, x_i, \theta + 2\pi) = \phi(t, x_i, \theta)\,. \tag{2}$$

For example, for small-amplitude sinusoidal waves,

$$\phi(t, x_i) \approx a \sin\left(k_i x_i - \omega t + \theta\right),$$

where k_i is the wavenumber vector, ω is the wave frequency, a is the wave amplitude. Define a phase average by

$$< \cdot >= \frac{1}{2\pi}\int_0^{2\pi} (\cdot) d\theta\,. \tag{3}$$

We then denote $\bar{\phi} =< \phi >$ and write

$$\phi = \bar{\phi} + \hat{\phi}\,, \tag{4}$$

so that by definition $\bar{\phi}$ is the **mean flow** and $\hat{\phi}$ is the **wave field**.

10.3 Wave action

Since the Lagrangian has no intrinsic dependence on the parameter θ, it can be shown that the corresponding symmetry is

$$\frac{\partial \mathbf{A}}{\partial t} + \frac{\partial \mathbf{B_i}}{\partial x_i} = 0\,, \tag{5}$$

$$\mathbf{A} = <\hat{\phi}_\theta \frac{\partial L}{\partial \phi_t}>\,, \quad \mathbf{B}_i = <\hat{\phi}_\theta \frac{\partial L}{\partial \phi_{x_i}}>\ . \tag{6}$$

Then **A is the wave action density, $\mathbf{B}_i$ is the wave action flux**. Importantly it is a wave quantity, being zero if there are no waves, and so is a good measure of wave activity. Equation (6) is a conservation law in **all** physical systems, assuming (as here) that there is no forcing and no dissipation. Note that formally this law is valid without restriction on amplitude and without restriction on the relative time and space scales of the waves *vis-a-vis* the mean flow.

But we shall now proceed to implement it for **small-amplitude waves in a slowly-varying medium**.

10.4 Slowly varying medium

We now suppose that the mean flow, the background medium and the wave parameters are slowly varying, and write

$$\hat{\phi} \sim \hat{\phi}(S(t, x_i) + \theta; t, x_i) \,, \quad \omega = -\frac{\partial S}{\partial t} \,, \ \kappa_i = \frac{\partial S}{\partial x_i} \,. \qquad (7)$$

Here ω is the local frequency, and κ_i is the local wavenumber. Then the expressions (6) reduce to

$$\mathbf{A} \sim -\frac{\partial \bar{L}}{\partial \omega} \,, \quad \mathbf{B}_i \sim \frac{\partial \bar{L}}{\partial \kappa_i} \,. \qquad (8)$$

Here $\bar{L} =< L >$ is the **averaged Lagrangian** and like ω, κ_i is a slowly varying function of t, x_i. But, recall that ϕ is rapidly varying in the phase itself. Note that (8) is formally valid without any amplitude restriction. Also, for slowly varying waves, we must add the equation for **conservation of waves**

$$\frac{\partial \kappa_i}{\partial t} + \frac{\partial \omega}{\partial x_i} = 0 \,. \qquad (9)$$

10.5 Linearized waves

Next we can decompose L as follows

$$L = L_0(\bar{\phi}_t, \bar{\phi}_{xi}, \bar{\phi}; t, x_i) + L_1(\hat{\phi}_t, \hat{\phi}_{xi}, \hat{\phi}; t, x_i)\,, \tag{10}$$

where $L_0 = L(\bar{\phi}, \bar{\phi}_t, \bar{\phi}_{x_i}; t, x_i)$ is the Lagrangian for the mean flow and L_1 is then Lagrangian for the wave field. The dependence of L_1 on the mean fields is here temporarily suppressed into the explicit t, x_i dependence.

In the small amplitude approximation the wave field is

$$\hat{\phi}(t, x_i) \approx a(t, x_i) \sin\left(S(t, x_i) + \theta\right). \tag{11}$$

Then in this small amplitude approximation (11)

$$\bar{L}_1 \approx D(\omega^*, \kappa_i; t, x_i) a^2\,, \quad \omega^* = \omega - U_i \kappa_i\,. \tag{12}$$

where we have extracted the dependence on the mean velocity field, U_i and ω^* is the **intrinsic frequency**, while the remaining mean fields remain suppressed into the explicit x_i, t dependence. In this linearized approximation, the mean fields are known. The result that $\bar{L}_1$ depends only on the mean velocity U_i through ω^* follows from Galilean invariance.

10.6 Wave action for linearized waves

Then, from the expressions (8) we get that

$$\mathbf{A} = -\frac{\partial \bar{L}_1}{\partial \omega} = -\frac{\partial D}{\partial \omega^*} a^2\,, \qquad \mathbf{B}_i = \frac{\partial \bar{L}_1}{\partial \kappa_i} = \frac{\partial D}{\partial \kappa_i} a^2\,. \tag{13}$$

But we also have that $\partial \bar{L}_1 / \partial a = 0$, so that we get the dispersion relation

$$D(\omega^*, \kappa_i; t, x_i) = 0\,. \tag{14}$$

This defines $\omega = U_i(t, x_i)\kappa_i + \omega^*, \omega^* = \Omega(\kappa_i; x_i, t)$. Then substitution into (14) and differentiation with respect to the wavenumber κ_i yields

$$\frac{\partial D}{\partial \omega^*} c_{gi} + \frac{\partial D}{\partial \kappa_i} = 0\,, \tag{15}$$

$$\text{where} \quad c_{gi} = \frac{\partial \omega}{\partial \kappa_i} = U_i + c_{gi}^*\,, \quad c_{gi}^* = \frac{\partial \omega^*}{\partial \kappa_i}\,, \tag{16}$$

defines the **group velocity**. Hence we finally get that $\mathbf{B}_i = c_{gi}\mathbf{A}$ and the wave action equation (5) is

$$\frac{\partial \mathbf{A}}{\partial t} + \frac{\partial (c_{gi}\mathbf{A})}{\partial x_i} = 0\,. \tag{17}$$

10.7 Energy and momentum

We now need to provide a more physical interpretation of wave action, and for this we shall consider the conservation laws for energy and momentum for the full Lagrangian system (1), obtained from the Lagrangian $L(\phi, \phi_{x_s}; x_s), s = 0, 1, 2, 3$ where $x_0 = t$. It is useful to note the general expression, valid for any $\psi = \psi(x_s)$,

$$\frac{\partial}{\partial x_s}(\psi \frac{\partial L}{\partial \phi_{x_s}}) = \psi_{x_s} \frac{\partial L}{\partial \phi_{x_r}} + \psi \frac{\partial L}{\partial \phi} \, . \tag{18}$$

Then putting $\psi = \phi_{x_r}$ generates the conservation laws

$$\frac{\partial T_{rs}}{\partial x_s} = -\frac{\partial L}{\partial x_s} \, , \quad \text{where} \quad T_{rs} = \phi_{x_r} \frac{\partial L}{\partial \phi_{x_s}} - L\delta_{rs} \tag{19}$$

T_{rs} is the energy-momentum tensor of classical physics. We can identify T_{00} with **energy density**, T_{0i} with energy flux, T_{i0} with **momentum density**, and T_{ij} with momentum flux.

We can now apply the averaging operator to the conservation laws (19) and so obtain the equations governing the averaged **total energy** $< T_{00} >$ and **averaged total momentum** $< \mathbf{T_{i0}} >$. But these are not particularly useful as they contain both the mean fields and the waves.

10.8 Pseudoenergy and pseudomomentum

Instead of the total energy and momentum, we define the **pseudoenergy** and **pseudomomentum** (see Andrews and McIntyre (1978)) by putting $\psi = \hat{\phi}_{x_s}$ in (18) and then averaging, so that

$$\frac{\partial \mathbf{T}_{rs}}{\partial x_s} = -\frac{\partial \bar{L}_1}{\partial x_r}, \quad \text{where} \quad \mathbf{T}_{rs} = < \hat{\phi}_{x_r} \frac{\partial L_1}{\partial \hat{\phi}_{x_s}} - L_1 \delta_{rs} > \tag{20}$$

Here recall L_1 is the "Lagrangian" (10), defined by $L = L_0 + L_1$ where L_0 is the "mean" Lagrangian which depends only on the mean flow. Hence $\mathbf{T}_{rs}$ is an $O(a^2)$ wave property. We can identify $\mathbf{T}_{00}$ with **pseudoenergy density**, $\mathbf{T}_{0i}$ with pseudoenergy flux, $\mathbf{T}_{i0}$ with **pseudomomentum density**, and $\mathbf{T}_{ij}$ with pseudomomentum flux. But (20) is not a conservation law unless L_1 is independent of x_s, and this *inter alia* requires that the mean flow is independent of x_s. Note that putting $\psi = \hat{\phi}_\theta$ yields the wave action equation (6). Thus, if phase averaging corresponds to averaging over a particular coordinate, $\theta = x_s$, then $\mathbf{T}_{s0} = \mathbf{A}$ and $\mathbf{T}_{si} = \mathbf{B}_i$, noting that in this case the diagonal term $\mathbf{T}_{ss}$ is then absent from the conservation law (20). Thus the wave action is pseudoenergy for time averaging, and is pseudomomentum for space averaging.

10.9 Wave energy

In general wave energy is not as useful a quantity as wave action, as in general it is not a conserved quantity. Suppose that the mean flow consists of a mean velocity U_i, and a vector-valued mean field Λ (mean depth, etc.), which for convenience will be required to satisfy the equation

$$\frac{d\lambda}{dt} + \Lambda_{ij}\lambda\frac{\partial U_i}{\partial x_j} = 0\,. \tag{21}$$

Then, we follow Bretherton and Garrett (1968) and define the wave energy $\mathbf{E}$ as the pseudoenergy in a reference frame moving with the mean flow.

$$\mathbf{E} = \mathbf{T}_{00} + U_i\mathbf{T}_{i0} =< \frac{d\hat{\phi}}{dt}\frac{\partial L_1}{\partial\hat{\phi}_t} - L_1 >\,, \quad \frac{d}{dt} = \frac{\partial}{\partial t} + U_i\frac{\partial}{\partial x_i}\,. \tag{22}$$

The corresponding wave energy flux is

$$\mathbf{F}_i =< \frac{d\hat{\phi}}{dt}\frac{\partial L_1}{\partial\hat{\phi}_{x_i}} - U_iL_1 > \tag{23}$$

Here $L_1 = L_1(\hat{\phi}, \hat{\phi}_t, \hat{\phi}_{x_i}; U_i, \lambda; x_i, t)$. Further we suppose that the dependence on $\hat{\phi}_t$ is only through $d\hat{\phi}/dt = \hat{\phi}_t + U_i\hat{\phi}_{x_i}$.

The wave energy equation then becomes

$$\frac{\partial \mathbf{E}}{\partial t}+\frac{\partial \mathbf{F}}{\partial x_i}=-\mathbf{R}_{ij}\frac{\partial U_i}{\partial x_j}-(\frac{d\bar{L}_1}{dt})_e\,, \tag{24}$$

$$\mathbf{R}_{ij}=-\mathbf{T}_{ij}+U_j\mathbf{T}_{i0}-\Lambda_{ij}\lambda\frac{\partial \bar{L}_1}{\partial \lambda}\,, \tag{25}$$

is the **radiation stress tensor**. Here the subscript "e" denotes the explicit derivative with respect to t, x_i when the wave field and the mean fields U_i, λ are held fixed. Finally we need an equation for the mean flow, which is obtained by variation of the mean fields, subject to the constraint (21). The mean momentum equation is

$$\frac{\partial}{\partial t}(\frac{\partial L_0}{\partial U_i})+\frac{\partial}{\partial x_j}(U_j\frac{\partial L_0}{\partial U_i}-\Lambda_{ij}\lambda\frac{\partial L_0}{\partial \lambda}+L_0\delta_{ij})-(\frac{\partial L_0}{\partial x_i})_e= -\frac{\partial R_{ij}}{\partial x_j}+(\frac{\partial \bar{L}_1}{\partial x_i})_e\,. \tag{26}$$

When $\Lambda_{ij}=M\delta_{ij}$ is isotropic, there is a mean pressure $\mathbf{Q}$ such that

$$-\frac{\partial \mathbf{R}_{ij}}{\partial x_j}=-\frac{\partial \mathbf{T}_{i0}}{\partial t}-\frac{\partial (U_j\mathbf{T}_{i0})}{\partial x_j}+\frac{\partial \mathbf{Q}}{\partial x_i}\,. \tag{27}$$

10.11 Slowly varying waves

$$\hat{\phi} \sim \hat{\phi}(S(t, x_i) + \theta; t, x_i)\,, \quad \omega = -\frac{\partial S}{\partial t}\,,\ \kappa_i = \frac{\partial S}{\partial x_i}\,.$$

In this case we get the reductions

$$\textbf{Pseudoenergy:} \quad \mathbf{T}_{00} \approx \omega \mathbf{A} - \bar{L}_1\,, \quad \mathbf{T}_{0i} \approx \omega \mathbf{B}_i\,, \tag{28}$$

$$\textbf{Pseudomomemtum:} \quad \mathbf{T}_{i0} \approx -\kappa_i \mathbf{A},\,, \quad \mathbf{T}_{ij} \approx -\kappa_i \mathbf{B}_j - \bar{L}_1 \delta_{ij}\,, \tag{29}$$

$$\textbf{Wave energy:} \quad \mathbf{E} \approx \omega^* \mathbf{A} - \bar{L}_1\,, \quad \mathbf{F} \approx \omega^* (\mathbf{B}_i - U_i \mathbf{A})\,, \tag{30}$$

$$\textbf{Radiation Stress:} \quad \mathbf{R}_{ij} \approx \kappa_i (\mathbf{B}_j - U_j \mathbf{A}) + \bar{L}_1 \delta_{ij} - \Lambda_{ij} \lambda \frac{\partial \bar{L}_1}{\partial \lambda}\,. \tag{31}$$

Recall that $\omega^* = \omega - U_i \kappa_i$ is the intrinsic frequency.

Now, we have a dispersion relation (14) so that $\omega^* = \Omega(\kappa_i; \lambda; x_i, t)$, $\omega = \kappa_i U_i + \omega^*$, and the wave action equation is (17)

$$\frac{\partial \mathbf{A}}{\partial t} + \frac{\partial (c_{gi}\mathbf{A})}{\partial x_i} = 0\,,$$

where $c_{gi} = U_i + \partial\Omega/\partial\kappa_i$ is the group velocity. For linearized waves $\bar{L}_1 = 0$ so **wave energy** $\mathbf{E} = \omega^* \mathbf{A}$, and the **pseudoenergy** $\mathbf{T}_{00} = \omega \mathbf{A}$. The wave energy equation (24) and the radiation stress (25) become

$$\frac{\partial \mathbf{E}}{\partial t} + \frac{\partial}{\partial x_i}([U_i + c^*_{gi}]\mathbf{E}) = -\mathbf{R}_{ij}\frac{\partial U_i}{\partial x_j} + \frac{\mathbf{E}}{\omega^*}(\frac{d\Omega}{dt})_e\,, \qquad (32)$$

$$\mathbf{R}_{ij} = \mathbf{A}(\kappa_i c_{gj} + \Lambda_{ij}\lambda\frac{\Omega^*}{\partial\lambda})\,. \qquad (33)$$

10.13 Extensions

Modal waves: Often waves are confined to **waveguides** such that there is propagation in only a reduced set of spatial dimensions, and a modal structure in the remaining spatial dimensions. Examples are water waves and internal waves. This general theory can still be used, but we need to combine the Lagrangian averaging with integration across the waveguide.

Generalized Lagrangian mean (GLM) theory: For applications to fluid flows we need to identify a suitable Lagrangian. Although these can sometimes be found in the Eulerian formulation, it is usually best for our present purposes to use a Lagrangian formulation of the equations of motion. But to be able to consider finite amplitude waves, we define the particle displacements from a mean position that moves with the mean velocity, U_i. This is the GLM theory developed by Andrews and McIntyre (1978). Thus, x_i are Lagrangian variables moving with the **Lagrangian mean velocity** U_i, relative to which we define the particle displacements ξ_i. The Eulerian variables are then $x_i' = x_i + \xi_i$, and the Eulerian velocity is $u_i' = U_i + d\xi_i/dt$ $(d/dt = \partial/\partial t + U_i \partial/\partial x_i)$ and $< \xi_i >= 0$.

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