Nonlinear Waves: Woods Hole GFD Program 2009

Roger Grimshaw

Loughborough University, UK

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Lecture 4: Internal solitary waves in the ocean

The aim here is to describe internal solitary waves in the coastal ocean, where the bottom topography may vary from the deep ocean to the shallow seas of the coastal oceans, and also the background hydrography can also vary along the path of the wave. Hence the asymptotic models must incorporate a **variable background state**. On the assumption that this is **slowly varying** relative to the waves, the outcome is a KdV-type equation, but with **variable coefficients**, namely the variable-coefficient extended Korteweg-de Vries (veKdV) equation. This is, in dimensional un-scaled coordinates,

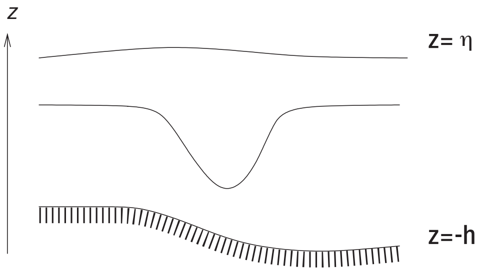
$$A_\tau + \alpha A A_\xi + \alpha_1 A^2 A_\xi + \lambda A_{\xi\xi\xi} = 0\,. \tag{1}$$

$$\tau = \int^x \frac{dx}{c}\,, \quad \xi = t - \tau\,, \tag{2}$$

Here the original amplitude A has been replaced by $\sqrt{Q}\,A$, where Q is the linear magnification factor, defined so that QA^2 **is the wave action flux**. The linear long-wave speed c, and the coefficients $\alpha, \alpha_1, \lambda$ depend on x, and hence on the evolution variable τ. These equations are not integrable in general, and so we must then seek a combination of asymptotic and numerical solutions.

4.1 Derivation of the veKdV equation

In the basic state the fluid has **density** $\rho_0(z)$, stably stratified so that $\mathbf{N^2 = -g\rho_{0z}/\rho_0 > 0}$, a corresponding pressure $p_0(z)$ ($p_{0z} = -g\rho_0$), and a horizontal shear flow $u_0(z)$ in the $x-$direction. Here we shall consider only the case when the bottom is flat, that is h is a constant. Extensions to variable depth are possible and lead to a variable-coefficient extended KdV equation.



4.2 Linear long waves

Then to describe internal solitary waves we seek solutions whose horizontal length scales are much greater than h, and whose time scales are much greater than N^{-1}. We shall also assume that the waves have small amplitude. Then the dominant balance is linear long wave theory. We will use the **vertical particle displacement** ζ as the primary fluid variable, so that,

$$\zeta = A(x - ct)\phi(z)\,, \tag{3}$$

Here c is the **linear long wave speed**, and the **modal functions** $\phi(z)$ are defined by the boundary-value problem,

$$\{\rho_0(c - u_0)^2\phi_z\}_z + \rho_0 N^2\phi = 0\,, \quad \text{for} \quad -h < z < 0\,, \tag{4}$$

$$\phi = 0 \quad \text{at} \quad z = -h\,, \quad (c - u_0)^2\phi_z = g\phi \quad \text{at} \quad z = 0\,, \tag{5}$$

4.3 Linear long-wave modes

The boundary-value problem (4, 5) defines an infinite sequence of **linear long-wave modes**, $\phi_n(z)$, $n = 0, 1, 2, \ldots$, with corresponding speeds c_n. Within the context of linear long wave theory, any localized initial disturbance will evolve into a set of outwardly propagating modes (3), each propagating with its own distinctive speed. Hence for **large times we can consider just a single mode**, and henceforth omit the index "n". Then, as time increases, the hitherto neglected **nonlinear terms** begin to have an effect, and cause wave steepening. However, this is opposed by the terms representing **linear wave dispersion**, also neglected in the linear long wave theory. A balance between these two effects emerges as time increases and the outcome is the **Korteweg-de Vries (KdV)** equation for the wave amplitude.

4.4 Asymptotic expansion

The formal derivation of the evolution equation requires the introduction of **two small parameters,** α **and** ϵ, respectively characterising the wave amplitude and dispersion. The **KdV balance requires** $\alpha = \epsilon^2$, with a corresponding **time scale of** ϵ^{-3}. The asymptotic analysis required is well understood, so we give only the outcome here. Introduce the scaled variables

$$T = \epsilon\alpha t \,, \quad X = \epsilon(x - ct) \,, \tag{6}$$

and then put

$$\zeta = \alpha A(X, T)\phi(z) + \alpha^2\zeta_2 + \ldots \,, \tag{7}$$

Since the modal equations are homogeneous, we are free to impose a normalization condition on $\phi(z)$. A commonly used condition is that $\phi(z_m) = 1$ where $|\phi(z)|$ achieves a maximum value at $z = z_m$. In this case the amplitude αA is uniquely defined as the amplitude of ζ (to $0(\alpha)$) at the depth z_m.

Then, at the next order, we obtain a linear inhomogeneous equation for ζ_2, whose homogeneous part is just the modal equation. Hence the solution requires a compatibility condition, which yields the **Korteweg-de Vries (KdV) equation**,

$$A_T + \mu A A_X + \delta A_{XXX} = 0\,, \tag{8}$$

where the coefficients μ and δ are given by

$$I\mu = 3\int_{-h}^{0} \rho_0 (c - u_0)^2 \phi_z^3\, dz\,, \tag{9}$$

$$I\delta = \int_{-h}^{0} \rho_0 (c - u_0)^2 \phi^2\, dz\,, \tag{10}$$

where $$I = 2\int_{-h}^{0} \rho_0 (c - u_0) \phi_z^2\, dz\,. \tag{11}$$

4.6 Korteweg-de Vries equation

The KdV equation (8) is **integrable**, It is to be solved with the initial condition $A(X, T = 0) = A_0(X)$ where $A_0(X)$ is determined from the linear long wave theory. The outcome is a set of **rank-ordered solitary waves**, and dispersive radiation. The solitary waves have **polarity** determined by the sign of $\mu\delta$.

$$I\mu = 3\int_{-h}^{0} \rho_0(c - u_0)^2 \phi_z^3 \, dz \, , \quad I\delta = \int_{-h}^{0} \rho_0(c - u_0)^2 \phi^2 \, dz \, ,$$

$$I = 2\int_{-h}^{0} \rho_0(c - u_0)\phi_z^2 \, dz \, .$$

For waves propagating to the right, $c > u_M = \max u_0(z)$, so that $I > 0$ and $\delta > 0$. For the surface mode, $\mu > 0$. The internal modes are normalized so that $\phi(z_m) = 1$ where z_m is an extremal point. Then it is readily shown that for a **near-surface pycnocline, μ is negative for the first internal mode**. However, in general μ can take either sign.

4.7 Extended Korteweg-de Vries equation

Proceeding to the next highest order yields a higher-order KdV equation,

$$A_T + \mu A A_X + \delta A_{XXX} +$$

$$\alpha\{\delta_1 A_{XXXXX} + \mu_1 A^2 A_X + \sigma_1 A A_{XXX} + \sigma_2 A_X A_{XX}\} = 0\,. \qquad (12)$$

Explicit expressions for the coefficients are available. A particularly important special case of the higher-order KdV equation (12) arises when the nonlinear coefficient μ (8) in the KdV equation is close to zero. In this situation, the cubic nonlinear term in the higher-order KdV equation is the most important higher-order term. The KdV equation (8) may then be replaced by the **extended KdV** (Gardner) equation,

$$A_T + \mu A A_X + \alpha\mu_1 A^2 A_X + \delta A_{XXX} = 0\,. \qquad (13)$$

For $\mu \approx 0$, a rescaling is needed and the optimal choice is to assume that μ is $0(\epsilon)$, and then replace A with A/ϵ. In effect the amplitude parameter is ϵ in place of ϵ^2. Henceforth we assume that this rescaling has been done. Like the KdV equation, (13) is **integrable**, and has solitary wave solutions. There are two independent forms of the eKdV equation (13), depending on the sign of $\delta\mu_1$.

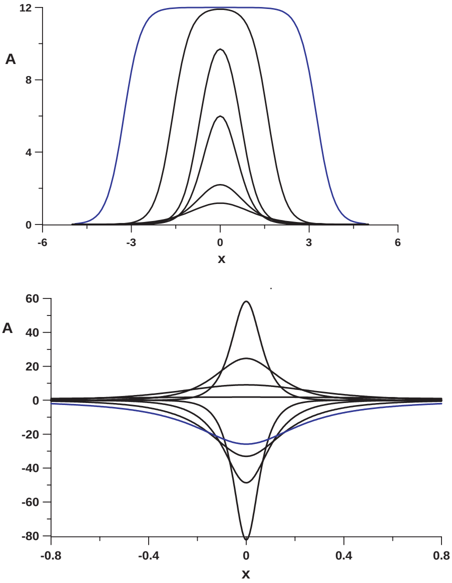
4.8 Solitary waves of the eKdV equation

$$A = \frac{H}{1 + B \cosh K(X - VT)}\,, \tag{14}$$

$$\text{where } V = \frac{\mu H}{6} = \delta K^2\,,\ \ B^2 = 1 + \frac{6\delta\mu_1 K^2}{\mu^2}\,, \tag{15}$$

with a **single parameter** B. For $\delta\mu_1 < 0$, $0 < B < 1$, and the family ranges from small-amplitude waves of KdV-type ("sech^2"-profile) ($B \to 1$) to a limiting flat-topped wave of amplitude $-\mu/\mu_1$ ($B \to 0$) (**"table-top" wave**). For $\delta\mu_1 > 0$ there are two branches; one has $1 < B < \infty$ and ranges from small-amplitude KdV-type waves ($B \to 1$), to large waves with a "sech"-profile ($B \to \infty$). The other branch, $-\infty < B < 1$, has the opposite polarity and ranges from large waves with a "sech"-profile to a limiting algebraic wave of amplitude $-2\mu/\mu_1$. Waves with smaller amplitudes do not exist, and are replaced by **breathers** .

4.9 Solitary waves of the eKdV equation



4.10 Variable coefficients

The derivation sketched so far is for a waveguide with constant properties in the horizontal direction. But, in the ocean there is **varying depth, and variations in the basic state hydrology and background currents**. These effects can be formally incorporated into the theory by supposing that the basic state is a function of the slow variable $\chi = \epsilon^3 x$. That is, $h = h(\chi)$, $u_0 = u_0(\chi, z)$ with a corresponding vertical velocity field $\epsilon^3 w_0(z, \chi)$, a density field $\rho_0(z, \chi)$ a corresponding pressure field $p_0(\chi, z)$ and a free surface displacement $\eta_0(\chi)$. With this scaling, the slow background variability enters the asymptotic analysis at the same order as the weakly nonlinear and weakly dispersive effects. An asymptotic analysis analogous to that described above then produces a **variable coefficient extended KdV** equation. The modal system is again defined by (4, 5), but now $c = c(\chi)$ and $\phi = \phi(z, \chi)$, where the χ-dependence is parametric. We find that $Q = C^2 I$ where I is defined by (11) Note also that this expression for Q can also be simply determined by requiring that QA^2 should be the wave action flux in the linear long wave limit.

4.11 Variable-coefficient extended KdV equation

With all small parameters removed, this is

$$A_\tau + \alpha A A_\xi + \alpha_1 A^2 A_\xi + \lambda A_{\xi\xi\xi} = 0\,. \tag{16}$$

$$\tau = \int^x \frac{dx}{c}\,, \quad \xi = t - \tau\,, \tag{17}$$

where the original amplitude A has been replaced by $\sqrt{Q}\,A$. Q is the linear magnification factor, defined so that QA^2 **is the wave action flux**. The coefficients $\alpha(\tau), \alpha_1(\tau), \delta(\tau)$ and $Q(\tau)$ are given by

$$\alpha = \frac{\mu}{cQ^{1/2}}\,,\ \alpha_1 = \frac{\mu_1}{cQ}\,,\ \lambda = \frac{\delta}{c^3}\,,\ Q = c^2 I\,. \tag{18}$$

4.12 Slowly-varying solitary waves

The evKdV equation (16) possesses two relevant conservation laws,

$$\int_{-\infty}^{\infty} A\,dx = \text{constant}\,, \tag{19}$$

$$\int_{-\infty}^{\infty} A^2\,dx = \text{constant}\,, \tag{20}$$

representing conservation of "mass" and "momentum" respectively (more strictly an approximate representation of the physical mass and wave action flux).

The slowly-varying solitary wave is then given as before, but its parameters $B(\tau)$ etc. now vary slowly in a manner determined by **conservation of momentum** (20). Mass is conserved by the generation of a **trailing shelf**.

4.13 Slowly-varying solitary waves

$$G(B) = \text{constant}\left|\frac{\alpha_1^3}{\lambda\alpha^2}\right|^{1/2}, \tag{21}$$

$$\text{where } G(B) = |B^2-1|^{3/2}\int_{-\infty}^{\infty}\frac{du}{(1+B\cosh u)^2}.$$

The integral term in $G(B)$ can be explicitly evaluated, and so these expressions provide explicit formulas for the variation of $B(\tau)$ as the environmental parameters vary.

But since the conservation of momentum completely defines the slowly-varying solitary wave, total mass (19) is conserved by a **trailing shelf** (linear long wave) whose amplitude A_{shelf} at the rear of the solitary wave is

$$VA_{shelf} = -\frac{\partial M_{sol}}{\partial\tau}, \qquad M_{sol} = \int_{\infty}^{\infty} A_{sol}\, d\xi, \tag{22}$$

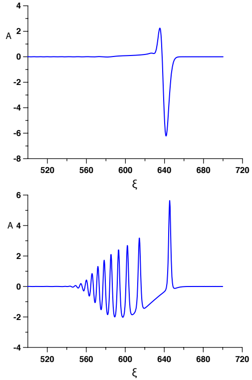
and where A_{sol} is the solitary wave solution.

4.14 Critical point $\alpha = 0, \alpha_1 < 0$

The adiabatic expressions (21, 22) show that the **critical points** where $\alpha = 0$ (or where $\alpha_1 = 0$) are sites where we may expect a dramatic change in the wave structure. First, as α passes through zero, assume that $\boldsymbol{\alpha_1 < 0, 0 < B < 1}$ at the critical point $\tau = 0$ where $\alpha = 0$. Then as $\alpha \to 0$, it follows from (21) that $B \to 0$ and the wave profile approaches the limiting "table-top" wave. But in this limit, $K \sim |\alpha|$, and so the amplitude approaches the limiting value $-\alpha/\alpha_1$. Thus the wave amplitude decreases to zero, the mass M_0 of the solitary wave grows as $|\alpha|^{-1}$ and the amplitude A_1 of the trailing shelf grows as $1/|\alpha|^4$. Essentially the trailing shelf passes through the critical point as a disturbance of the opposite polarity to that of the original solitary wave, which then being in an environment with the opposite sign of α, can generate **a train of solitary waves of the opposite polarity, riding on a pedestal of the same polarity as the original wave**.

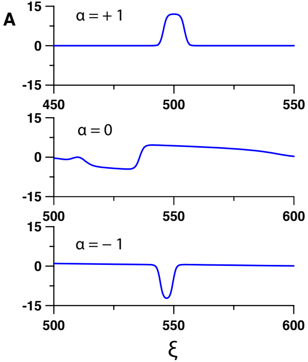
4.15 Critical point $\alpha = 0, \alpha_1 = 0$: KdV case

$\delta = 1, \alpha_1 = 0$ and α varies from -1 to 1 (that is the variable-coefficient KdV equation). The upper panel is when $\alpha = 0$ and the lower panel is when $\alpha = 1$. This is **conversion of a depression wave into a train of elevation waves riding on a negative pedestal**.



4.16 Critical point $\alpha = 0, \alpha_1 < 0$: eKdV case

$\delta = 1, \alpha_1 = -0.083$ and α varies from 1 to -1, that is, the variable coefficient eKdV equation, with a negative cubic nonlinear coefficient. This shows the conversion of an elevation "table-top" wave into a **depression** "table-top" wave, riding on a **positive** pedestal.

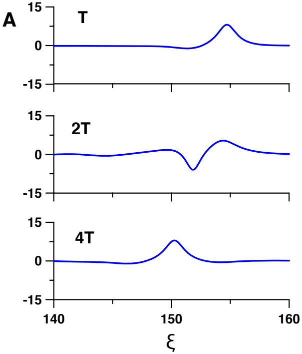


4.17 Critical point $\alpha = 0, \alpha_1 > 0$: eKdV case

Next, let us suppose that at the critical point where $\alpha = 0$, $\alpha_1 > 0$. In this case, $1 < |B| < \infty$ and there are the two sub-cases to consider, $B > 1$ or $B < -1$, when the the solitary wave has the same or opposite polarity to α. Then, as $\alpha \to 0, |B| \to \infty$ as $|B| \sim 1/|\alpha|$. It follows from (15) that then $K \sim 1$, $H \sim 1/|\alpha|, a \sim 1, M_0 \sim 1$. It follows that the wave adopts the "sech"-profile, but has *finite* amplitude, and so can **pass through the critical point $\alpha = 0$ without destruction**. But the wave **changes branches** from $B > 1$ to $B < -1$ as $|B| \to \infty$, or *vice versa*. An interesting situation then arises when the wave belongs to the branch with $-\infty < B < -1$ and the amplitude is reducing. If the limiting amplitude of $-2\alpha/\alpha_1$ is reached, then there can be no further reduction in amplitude for a solitary wave, and instead a **breather** will form.

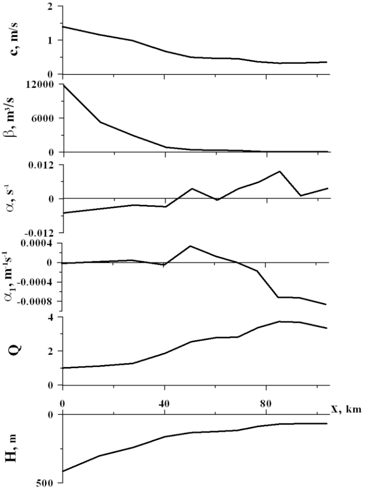
4.18 Critical point $\alpha = 0, \alpha_1 > 0$: eKdV case

$\delta = 1, \alpha_1 = 0.3$ and α varies from 1 to -1 for $-T < \tau < T$, that is, the variable coefficient eKdV equation, with a positive cubic nonlinear coefficient. This shows the adiabatic evolution of an elevation wave from $\tau = -T$ to $\tau = T$, where its amplitude is too small, and so the wave becomes a **breather**.

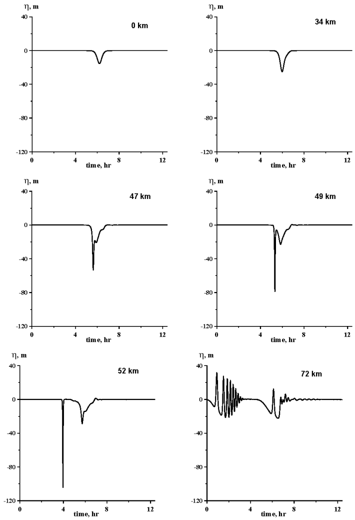


4.19 Wave propagation, deformation and disintegration

Simulation of veKdV (16) of the passage of an initial solitary wave of depression across the **North West Shelf of Australia**.

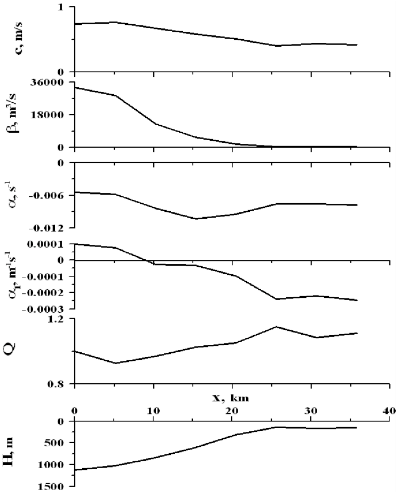


4.20 Wave propagation, deformation and disintegration: NWS, initial depression wave of 15m amplitude

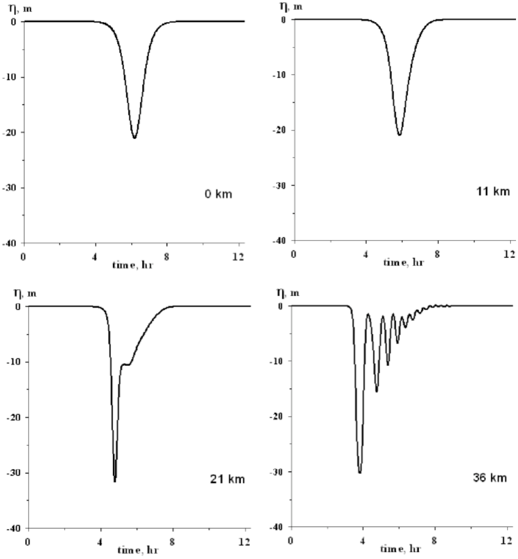


4.21 Wave propagation, deformation and disintegration

Simulation using (16) of the passage of an initial solitary wave of depression across the **Malin shelf off west coast of Scotland**.

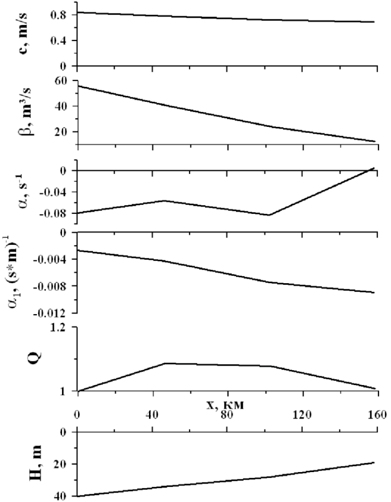


4.22 Wave propagation, deformation and disintegration: Malin Shelf, fission, initial amplitude of 21m

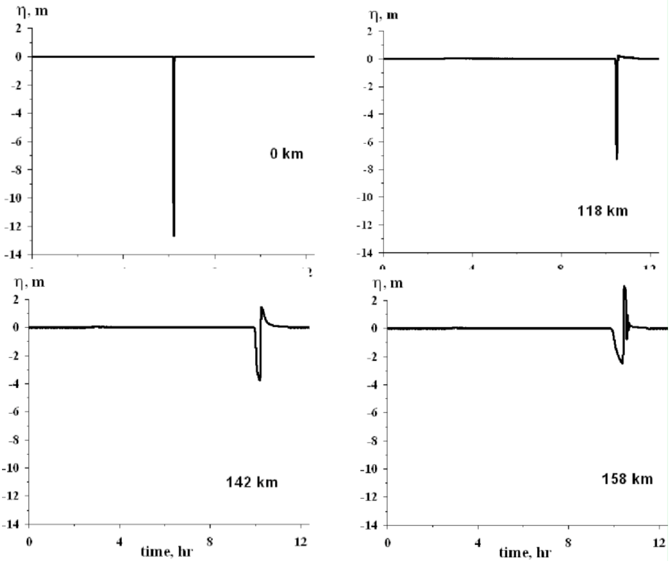


4.23 Wave propagation, deformation and disintegration

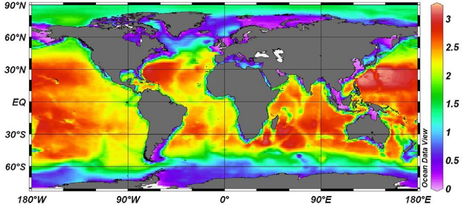
Simulation using (16) of the passage of an initial solitary wave of depression across the **Arctic shelf off north coast of Russia** .



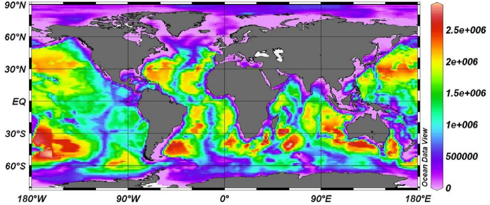
4.24 Wave propagation, deformation and disintegration: Arctic Shelf, adiabatic, initial amplitude of 13m



4.25 World map of eKdV coefficients

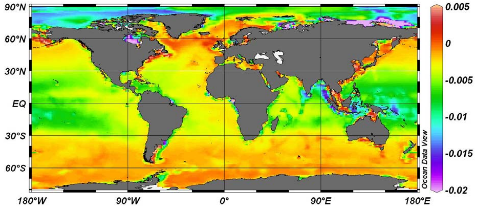


a) speed of propagation, c (m/s)

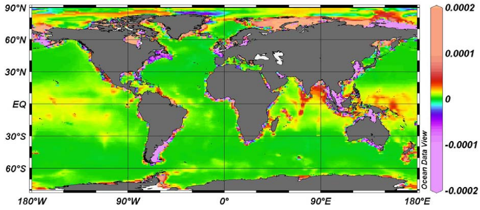


b) dispersion coefficient, δ (m^3/s)

4.26 World map of eKdV coefficients



c) coefficient of quadratic nonlinearity, μ (s^{-1})



d) coefficient of cubic nonlinearity, μ_1 ($m^{-1}s^{-1}$)

Lecture 4: References

Internal solitary waves ed. Grimshaw, R. 2001 *Environmental Stratified Flows*, ed. R. Grimshaw, Kluwer, Boston.

Simulation of the transformation of internal solitary waves on oceanic shelves Grimshaw, R., Pelinovsky, E., Talipova, T. and Kurkin, A. 2004, *J. Phys. Ocean.*, **34** , 2774-2779

Modeling internal solitary waves in the coastal ocean Grimshaw, R., Pelinovsky, E. and Talipova, T. 2007 *Surveys in Geophysics*, **28**, 273-298.