Nonlinear Waves: Woods Hole GFD Program 2009

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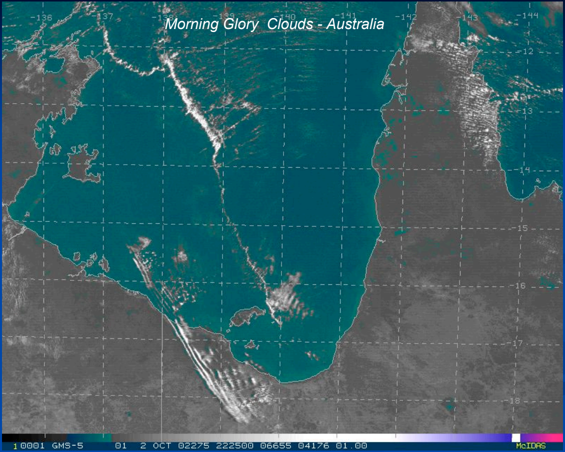
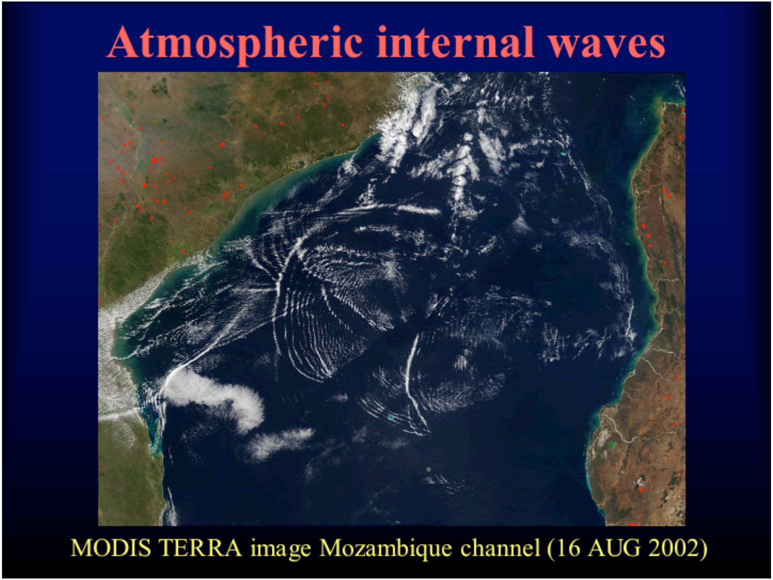
Morning Glory Waves

An atmospheric internal gravity wave train in Northern Australia

Satellite view of a Morning Glory wave



Atmospheric solitary waves near Mozambique



Undular Bore on the Dordogne river

5.1 Modulated periodic wave trains

The aim of the Whitham modulation theory is to provide an asymptotic theory to describe **slowly varying periodic waves**, essentially a nonlinear WKB theory. The equations describing the slow evolution of the parameters of nonlinear periodic waves (such as the amplitude, wavelength, frequency, etc.) are called the **modulation (or Whitham) equations**. It transpires that the Whitham equations have a remarkably rich mathematical structure, and at the same time are a powerful analytic tool for the description of nonlinear waves in a wide variety of physical contexts.

One of the most important aspects of the Whitham theory is the analytic description of the formation and evolution of **dispersive shock waves, or undular bores**. These are coherent nonlinear wave structures which resolve a wave-breaking singularity when it is dominated by dispersion rather than the dissipation.

There are also a number of important connections between the Whitham theory, the inverse scattering transform (IST) and the general theory of integrable hydrodynamic type systems.

5.2 KdV Cnoidal waves

We shall describe the Whitham modulation theory for the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0\,. \tag{1}$$

The one-phase periodic travelling wave solution (**cnoidal wave**) of the KdV equation (1) is

$$u(x,t) = r_2 - r_1 - r_3 + 2(r_3 - r_2)\mathrm{cn}^2(\sqrt{r_3 - r_1}\,\theta; m)\,, \tag{2}$$

where $\mathrm{cn}(y;m)$ is the Jacobi elliptic cosine function. Here $r_1 \le r_2 \le r_3$ are the three parameters, and the phase variable θ and the modulus m, $(0 < m < 1)$, are given by

$$\theta = x - Vt\,, \quad V = -2(r_1 + r_2 + r_3)\,, \tag{3}$$

$$m = \frac{r_3 - r_2}{r_3 - r_1}\,, \quad \text{and} \quad L = \oint d\theta = \frac{2K(m)}{\sqrt{r_3 - r_1}}\,, \tag{4}$$

where $K(m)$ is the complete elliptic integral of the first kind, and L is the "wavelength" along the x-axis. As $m \to 1$, $\mathrm{cn}(y;m) \to \mathrm{sech}(y)$ and (2) becomes a solitary wave, while as $m \to 0$ it reduces to a sinusoidal wave.

5.3 KdV Cnoidal waves

It is advantageous to use these parameters $r_{1,2,3}$ instead of the more "physical" parameters such as the amplitude, speed, wavelength etc., as they arise from the basic ordinary differential equation for the KdV travelling wave solution (2). Thus is we substitute (2) into the KdV equation (1) we get

$$u_\theta^2 = -2u^3 + Vu^2 + Cu + D\,, \tag{5}$$

where C, D are constants. This is transformed to

$$w_\theta^2 = -4P(w), \tag{6}$$

$$\text{where} \quad w = \frac{u}{2} - \frac{V}{4}\,, \quad \text{and} \quad P(w) = \prod_{i=1}^{3}(w - r_i)\,. \tag{7}$$

That is, the cnoidal wave (2) is parameterized by the zeros r_1, r_2, r_3 of the cubic polynomial $P(w)$.

In a modulated periodic wave, the parameters r_1, r_2, r_3 are slowly varying functions of x, t, described by the **Whitham modulation equations**. These can be obtained either by a multi-scale asymptotic expansion, or more conveniently by **averaging conservation laws**.

5.4 Averaged conservation laws

Introduce averaging over the period of the cnoidal wave (2) by

$$\langle \mathfrak{F} \rangle = \frac{1}{L} \oint \mathfrak{F} d\theta = \frac{1}{L} \int_{r_2}^{r_3} \frac{\mathfrak{F} d\mu}{\sqrt{-P(\mu)}} \,. \tag{8}$$

In particular,

$$\langle u \rangle = 2(r_3 - r_1)\frac{E(m)}{K(m)} + r_1 - r_2 - r_3 \,, \tag{9}$$

$$\langle u^2 \rangle = \frac{2}{3}[V(r_3 - r_1)\frac{E(m)}{K(m)} + 2Vr_1 + 2(r_1^2 - r_2 r_3)] + \frac{V^2}{4} \,, \tag{10}$$

where $E(m)$ is the complete elliptic integral of the second kind. Next, consider a set of three conservation laws for the KdV equation,

$$P_t + Q_x = 0 \,, \quad j = 1, 2, 3 \,, \tag{11}$$

Then apply the averaging (8) to the system (11) to obtain

$$\langle P_j \rangle_t + \langle Q_j \rangle_x = 0, \quad j = 1, 2, 3 \,. \tag{12}$$

This system (12) describes the slow evolution of the parameters r_j for the cnoidal wave (2).

5.5 Averaged conservation laws

For the KdV equation (1) two conservation laws are

$$\begin{aligned} u_t + (3u^2 + u_{xx})_x &= 0\,, & (13) \\ (u^2)_t + (4u^3 + 2uu_{xx} - u_x^2)_x &= 0\,. & (14) \end{aligned}$$

These are just the first two conservation laws, for respectively "mass" and "momentum", in an infinite set of conservation laws. The next would be that for "energy". But here only two are needed, because after averaging, we can replace the third equation by the law for the **conservation of waves**

$$k_t + \omega_x = 0\,, \quad \text{where} \quad k = \frac{2\pi}{L}\,, \quad \omega = kV\,. \tag{15}$$

This must be consistent with the modulation system (12), and can be introduced instead of any of three averaged conservation laws (12). Indeed any three independent conservation laws can be used, and will lead to equivalent modulation systems. Here, the Whitham modulation equations for the KdV equation are obtained by averaging (13, 14) and combining with (15).

5.6 Whitham modulation equations

In general the Whitham modulation equations have the structure

$$\mathbf{b}_t + \mathbf{A}(\mathbf{b})\mathbf{b}_x = 0\,. \tag{16}$$

Here $\mathbf{b} = (r_1, r_2, r_3)^t$, and the coefficient matrix $\mathbf{A}(\mathbf{b}) = \mathbf{P}^{-1}\mathbf{Q}$ where the matrices $\mathbf{P}, \mathbf{Q}$ have the entries $P_{ij} = \langle P_i \rangle_{r_j}$ and $Q_{ij} = \langle Q_i \rangle_{r_j}$ for $i, j = 1, 2, 3$. The eigenvalues of the coefficient matrix $\mathbf{A}$ are called the characteristic velocities. If all the eigenvalues $v_j(\mathbf{b})$ of $\mathbf{A}(\mathbf{b})$ are real-valued, then the system is **nonlinear hyperbolic** and the underlying travelling wave is **modulationally stable**. Otherwise the travelling wave is modulationally unstable.

For this KdV case all the eigenvalues are real and so the cnoidal wave is modulationally stable. It can be shown that

$$v_j = -2\sum r_j + \frac{2L}{\partial L/\partial r_j}\,, \quad j = 1, 2, 3, \tag{17}$$

5.7 Whitham modulation equations

The parameters r_j have been chosen because they are the **Riemann invariants** of the system (16) for the present case of the KdV equation. Thus this system has the **diagonal form**

$$r_{jt} + v_j r_{jx} = 0\,, \quad j = 1, 2, 3, \tag{18}$$

where we recall that $v_j(r_1, r_2, r_3)$ are the characteristic velocities (17)

$$\begin{aligned} v_1 &= -2\sum r_j + 4(r_3 - r_1)(1-m)K/E\,, \\ v_2 &= -2\sum r_j - 4(r_3 - r_2)(1-m)K/(E - (1-m)K)\,, \\ v_3 &= -2\sum r_j + 4(r_3 - r_2)K/(E - K)\,. \end{aligned} \tag{19}$$

where $K(m), E(m)$ are the elliptic integral so the first and second kind. This system is integrable, and the complete solution can be found.

5.8 Limiting cases of Whitham modulation equations

Sinusoidal waves limit: $m \to 0$: A solution of (18) is $r_2 = r_3, m = 0$, $v_1 = -6r_1, v_2 = v_3 = 6r_1 - 12r_3$ so that the system collapses to

$$\begin{aligned} & r_{1t} - 6r_1 r_{1x} = 0\,, \quad r_{3t} + (6r_1 - 12r_3) r_{3x} = 0\,. \\ \text{or} \quad & d_t + 6dd_x = 0\,, \quad k_t + \omega_x = 0\,. \end{aligned} \tag{20}$$

Here $-r_1 = d$ is the mean level, and $r_3 - r_1 = k^2/4$ is the wavenumber, and the dispersion relation is $\omega = 6dk - k^3$. An expansion for small m is needed to recover the wave action equation.

Solitary wave limit: $m \to 1$: A solution of (18) is $r_1 = r_2, m = 1$, $v_1 = v_2 = -4r_1 - 2r_3, v_3 = -6r_3$, so that the system collapses to

$$\begin{aligned} & r_{1t} + (-4r_1 - 2r_3) r_{1x} = 0\,, \quad r_{3t} - 6r_3 r_{3x} = 0\,. \\ \text{or} \quad & d_t + 6dd_x = 0\,, \quad a_t + Va_x = 0\,. \end{aligned} \tag{21}$$

Now $-r_3 = d$ is the background level and $2(r_3 - r_1) = a$ is the solitary wave amplitude, and $-4r_1 - 2r_3 = 6d + 2a = V$ is its speed.

5.9 Shocks

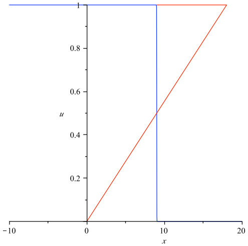
Next we consider the similarity solution of the modulation system (18) which describes an undular bore developing from an initial discontinuity

$$u(x,0) = \Delta \quad \text{for} \quad x < 0\,, \quad \text{and} \quad u(x,0) \quad \text{for} \quad x > 0\,, \tag{22}$$

where $\Delta > 0$ is a constant. It is now useful to consider the solution when the dispersive term in the KdV equation (1) is omitted, so that it becomes the **Hopf equation**

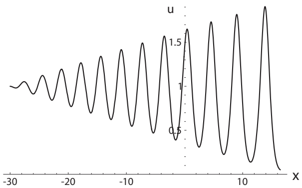
$$u_t + 6uu_x = 0\,. \tag{23}$$

This is readily solve by characteristics and the solution is multivalued, $u = \Delta$ for $-\infty < x < 6\Delta t$ and $u = 0$ for $0 < x < \infty$. Hence a shock is needed, whose speed is 3Δ, shown for $\Delta = 1, t = 3$.



5.10 Undular bore

The aim here is to replace the shock with a modulated wave train, generating a **dispersive shock wave** or **undular bore** .



Behind the undular bore $u = \Delta$ or in terms of the Riemann invariants $r_1 = -\Delta, r_2 = r_3$, and ahead $u = 0, r_1 = r_2, r_3 = 0$. Because of the absence of a length scale in this problem, the corresponding solution of the Whitham modulation system must depend on the self-similar variable $\tau = x/t$ alone, which reduces the system (18) to

$$(v_j - \tau)\frac{dr_j}{d\tau} = 0\,, \quad i = 1, 2, 3\,. \tag{24}$$

Hence two Riemann invariants must be constant, namely $r_1 = -\Delta$, $r_3 = 0$ and then r_2 varies in the range $-\Delta < r_2 < 0$, given by $v_2 = \tau$.

5.11 Undular bore

Thus, using the expressions (2, 3, 4) for the cnoidal wave, we finally get the solution for the undular bore, expressed in terms of the modulus m,

$$\frac{x}{\Delta t} = 2(1+m) - \frac{4m(1-m)K(m)}{E(m) - (1-m)K(m)}\,, \tag{25}$$

$$\frac{u}{\Delta} = 1 - m + 2m\,\mathrm{cn}^2(\Delta^{1/2}(x - Vt); m)\,, \quad \frac{V}{\Delta} = -2(1+m)\,. \tag{26}$$

The leading and trailing edges of the undular bore are determined from (25) by putting $m = 1$ and $m = 0$, so that it exists in the zone

$$-6 < \frac{x}{\Delta t} < 4\,. \tag{27}$$

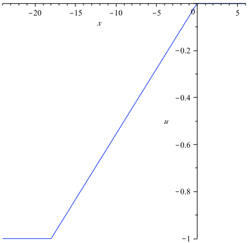
Thus this solution is an **unsteady undular bore**, and spreads out with time. A steady undular bore requires some friction. The leading solitary wave amplitude is 2Δ, exactly twice the height of the initial jump. Also the wavenumber is constant. For each wave in the wave train, $m \to 1$ as $t \to \infty$, so each wave tends to a solitary wave.

5.12 Rarefraction wave

When $\Delta < 0$, the resolution of the initial discontinuity (22) is the rarefraction wave

$$\begin{aligned} u &= 0\,, \quad \text{for} \quad x > 0\,, \\ u &= \frac{x}{6t}\,, \quad \text{for} \quad 6\Delta t < x < 0\,, \\ u &= \Delta\,, \quad \text{for} \quad x < 6\Delta t\,. \end{aligned} \tag{28}$$

This is a solution of the full KdV equation (1), but needs smoothing at the corners with a weak modulated periodic wave.



5.13 Further developments

1 : The Whitham can be applied to any nonlinear wave equation which has a (known) periodic travelling wave solution. These include the NLS equations, Boussinesq equations, Su-Gardner equations.

2: For a broad class of integrable nonlinear wave equations, a simple universal method has been developed by Kamchatnov (2000), enabling the construction of periodic solutions and the Whitham modulation equations directly in terms of Riemann invariants.

3: The "undular bore" solution can be extended to the long-time evolution from arbitrary localized initial conditions, described by Gurevich and Pitaevskii (1974) (and many subsequent works), and by Lax and Levermore (1983).

4: There have been applications in many physical areas, including surface and internal undular bores, collisionless shocks in rarefied plasmas (e. g. Earth's magnetosphere bow shock), nonlinear diffraction patterns in laser optics, and in Bose-Einstein condensates.

Lecture 5: References

Linear and Nonlinear Waves, **Whitham, G.B.** 1974, Wiley, New York.

Nonstationary structure of a collisionless shock wave **Gurevich, A.V. and Pitaevskii, L.P.** 1974 *Sov. Phys. JETP* **38**, 291-297.

The small dispersion limit of the Korteweg de Vries equation I, II, III. **Lax, P.D. and Levermore, C.D.** 1983 *Comm. Pure Appl. Math.* , **36**, 253-290, 571-593, 809-829.

Nonlinear Periodic Waves and Their Modulations—An Introductory Course **Kamchatnov, A.M.** 2000, *World Scientific, Singapore*.

Solitary Waves in Fluids, ed. **Grimshaw R.** 2007 *Advances in Fluid Mechanics, Vol 47, WIT press, UK*, 208pp

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