Nonlinear Waves: Woods Hole GFD Program 2009

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Lecture 9: Wave-Mean Flow Interaction, Part I

Water Waves: Painting by Katsushika Hokusai

Part 1: Water waves and currents

In the **linear approximation**, the surface elevation for sinusoidal unidirectional waves is

$$\zeta(t,x) = a\cos\theta\,, \quad \theta = kx - \omega t + \alpha\,, \tag{1}$$

for waves of amplitude a, wavenumber $k > 0$ and frequency ω. Here α is an arbitrary constant ensemble parameter. The dispersion relation is

$$\omega = Uk + \omega^*\,, \quad \omega^{*2} = gk\tanh kH\,. \tag{2}$$

Here U is a constant horizontal mean current, g is gravity and H is the constant mean water depth. The **total** frequency is decomposed into the **Doppler shift** Uk and the **intrinsic frequency** ω^*, which has two branches.

9.2 Slowly varying, small amplitude waves

Now suppose that the amplitude, wavenumber, frequency, mean current and mean depth vary slowly relative to the wave field. Then (1) is replaced by

$$\zeta(t,x) \sim a(x,t)\cos\theta + \nu_2 a^2 \cos 2\theta + O(a^3)\,, \tag{3}$$

$$\theta = \phi(x,t) + \alpha\,, \quad k = \phi_x \quad \omega = -\phi_t\,. \tag{4}$$

The ensemble parameter α is constant, and the coefficient ν_2 depends on ω^*, k, U, H. The equation for **conservation of waves** is, from (4),

$$k_t + \omega_x = 0\,. \tag{5}$$

The issue now is to determine how the amplitude, wavenumber, frequency, mean current and mean depth vary (slowly) in space and time. The mean current and depth $U(x,t), H(x,t)$ can be decomposed into background components $u(x,t), h(x)$ and a wave-induced $O(a^2)$ component.

9.3 Averaged Lagrangian

The modulation equations for the wave amplitude, etc. are found using **Whitham's averaged Lagrangian method**. The water wave system can be obtained from a Lagrangian, which is averaged to give

$$\bar{L} = \frac{1}{2\pi}\int_0^{2\pi} L d\alpha = \bar{L}^{(m)}(U, B, H, h) + \bar{L}^{(w)}(E^*, \omega^*, k, H)\,, \qquad (6)$$

$$E^* = \frac{ga^2}{2}\,,\ k = \phi_x\,,\ \omega = -\phi_t\,, \quad U = \psi_x\,,\ B = -\psi_t\,. \qquad (7)$$

$$\textbf{Mean} \ :\ \bar{L}^{(m)} = (B - \frac{U^2}{2})H - \frac{gH^2}{2} + gHh\,, \qquad (8)$$

$$\textbf{Wave} \ :\ \bar{L}^{(w)} = \frac{DE^*}{2} + \frac{D_2 k^2 E^{*2}}{2g} + \cdots\,, \qquad (9)$$

$$D = \frac{\omega^{*2}}{gkT} - 1\,, D_2 = -\frac{9T^4 - 10T^2 + 9}{8T^4}\,, T = \tanh kH. \qquad (10)$$

9.4 Modulation equations

The modulation equations are obtained from $\bar{L}$ by variations of E^*, ϕ, ψ, H, to yield the dispersion relation, the **wave action** equation, the mean flow and mean momentum equations. To these we add equation (5) for conservation of waves.

$$\bar{L}^{(w)}_{E^*} = \frac{D(\omega^*, k, H)}{2} + \frac{k^2 D_2 E^*}{g} + \cdots = 0\,, \tag{11}$$

$$A_t + F_x = 0\,, \quad A = \bar{L}^{(w)}_{\omega}\,, \quad F = -\bar{L}^{(w)}_{k}\,, \tag{12}$$

$$H_t + (HV)_x = 0\,, \quad V = U + \frac{kA}{H}\,, \tag{13}$$

$$(HV)_t + (HV^2)_x + (\frac{gH^2}{2})_x + S_x = gHh_x\,, \tag{14}$$

$$S = k(F - VA) + \bar{L}^{(w)} - H\bar{L}^{(w)}_H\,, \tag{15}$$

$$k_t + \omega_x = 0\,. \tag{16}$$

9.5 Wave action

These equations are fully nonlinear. A is the **wave action** density and F is the wave action flux. In the linearized approximation the dispersion relation (11) becomes

$$D(\omega^*, k, h) = 0\,, \quad \omega = \omega^* - ku\,, \quad \omega^{*2} = gk\tanh kh\,, \tag{17}$$

$$A = D_{\omega^*}\frac{E^*}{2} = \frac{E^*}{\omega^*}\,, \quad F = c_g A = (c_g^* + u)A\,, \tag{18}$$

where $c_g^* = \partial\omega^*/\partial k$ is the intrinsic group velocity. S is the **radiation stress**, and in the linearized approximation is

$$S = (kc_g^* + h\omega_h^*)A = (2kc_g^* - \frac{\omega^*}{2})A\,. \tag{19}$$

since for water waves, $h\omega_h^* = kc_g^* - \omega^*/2$. The equation for conservation of waves (16) becomes

$$k_t + c_g k_x = -ku_x - \omega_h^* h_x\,, \quad \omega_x + c_g\omega_x = -ku_t\,. \tag{20}$$

Note that for steady backgrounds the frequency is conserved.

9.6 Waves on a current

Consider a unidirectional steady current $u = u(x)$ with constant depth h. Then in the linearized approximation , $H, U \approx h, u$ and equation (16) becomes

$$k_t + \omega_x = 0\,, \quad \omega = uk + \omega^*\,, \quad {\omega^*}^2 = gk\tanh kh\,. \tag{21}$$

The steady solution is $\omega = \omega_0$ **(a constant)**, with $k(x)$ then being found from the dispersion relation (21). The wave amplitude is obtained from the wave action equation (12), which reduces to

$$A_t + (c_g A)_x = 0\,, \quad c_g = u + c_g^*\,, \quad A = \frac{E^*}{\omega^*} \tag{22}$$

The steady solution has **constant wave action flux** F_0,

$$2c_g A = c_g c^* a^2 = 2F_0\,, \quad c^* = \frac{\omega^*}{k}\,. \tag{23}$$

9.7 Waves on an advancing current

For simplicity, we now make the **deep-water** approximation $kh \to \infty$, so that $\omega^{*2} = gk, c_g^* = c^*/2$. To fix ideas suppose that $u(x=0) = 0$, and at $x = 0$, the intrinsic phase speed $c^* = c_0 > 0$. Then the solution of (21) is

$$c^*(x) = \frac{c_0}{2} \pm \{c_0 u(x) + \frac{c_0^2}{4}\}^{1/2} . \qquad (24)$$

Here we must initially at $x = 0$ choose the plus sign. Note that the group velocity is

$$c_g(x) = u(x) + \frac{c^*}{2} = u(x) + \frac{c_0}{4} \pm \frac{1}{2}\{c_0 u(x) + \frac{c_0^2}{4}\}^{1/2} . \qquad (25)$$

Thus for an **advancing current** $u(x) > 0, x > 0$, we must choose only the plus sign, and so $c^*(x), c_g(x)$ both increase as $u(x)$ increases, while then $k(x) = g/c^{*2}$ decreases. Since $c_g c^* a^2 = 2F_0$, the wave amplitude decreases.

9.8 Waves on an opposing current

The solutions (24, 25) are

$$c^*(x) = \frac{c_0}{2} \pm \{c_0 u(x) + \frac{c_0^2}{4}\}^{1/2} \, ,$$

$$c_g(x) = u(x) + \frac{c_0}{4} \pm \frac{1}{2}\{c_0 u(x) + \frac{c_0^2}{4}\}^{1/2} \, .$$

Hence for an **opposing current** $u(x) < 0, x > 0$, there is a stopping velocity at $x = x_c, u(x_c) = -c_0/4$, and the waves cannot penetrate past this point, since $c_g(x_c = 0)$. Instead the waves reflect, with the minus sign in (24, 25). Both $c^*(x), c_g(x)$ decrease as $|u(x)|$ increases, while $k(x)$ increases. Since $c_g c^* a^2 = 2F_0 = c_0^2 a_0^2$, the wave amplitude increases from the initial value a_0, and $a^2 \rightarrow \infty$ as $x \rightarrow x_c$. Of course, this result is outside the linear approximation, and in practice the waves will **break** at $x_b < x = x_c$. Here we use a **breaking criterion,** $ak(x = x_b) = 0.44$; note that x_b depends on a_0, c_0.

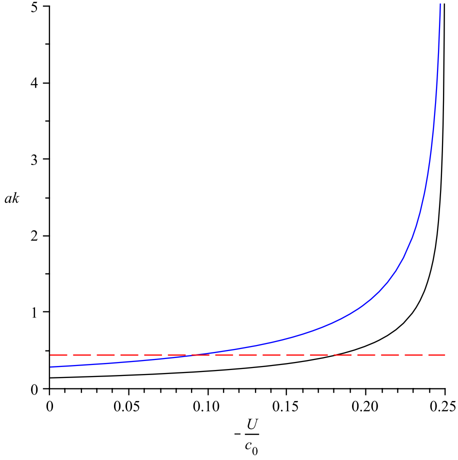
9.9 Waves on an opposing current: breaking waves

This rather simple theory has applications to the formation of **giant (rogue, freak) waves** in the ocean, for example on the Agulhas current. There also applications to the modulation of water waves by an underlying **internal solitary wave**, whose surface current is $u(x) = u_0 \operatorname{sech}^2(Kx)$ say. To explore these further, we take a wave packet solution of the wave action equation (22)

$$c_g A = c_g c^* a^2 = c_0^2 a_0^2 b^2(t - \tau)\,, \quad \tau = \int_0^x \frac{dx}{c_g}\,. \qquad (26)$$

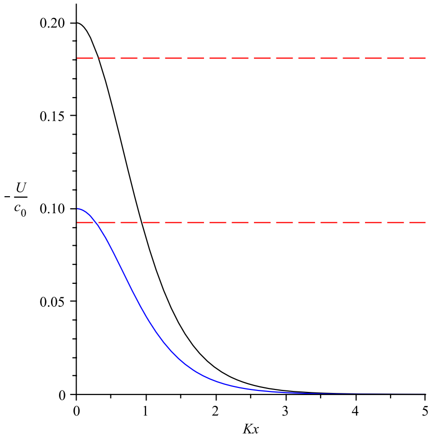
Here $a_0 b(t)$ is the wave amplitude at $x = 0$, and we assume that the shape function $b(t)$ is localized (e.g. Gaussian), varying from 0 to a maximum of 1 at $t = 0$. Then the waves break throughout the zone, $x_b < x < x_c$, over a time interval determine by the width of the packet.

9.10 Waves on an opposing current: breaking zones



Wave steepness ak versus u/c_0; $a_0 k_0 = 0.1, 0.2$ (black, blue); wave breaking criterion $ak = 0.44$ (red dash), yields breaking for $|u|/c_0 > 0.18, 0.092$.

9.11 Waves on an internal wave current



Breaking waves on the internal wave current $u = u_0 \text{sech}^2(Kx)$. for $u_0/c_0 = -0.2, -0.1$ (black, blue), where the red lines give the breaking zones for $a_0 k_0 = 0.1, 0.2$(upper, lower).

9.12 Waves on a current: nonlinear effects

In deep water, the wave-induced components of U, H are negligble and so the Lagrangian (6) becomes just (9) given now by

$$\bar{L}^{(w)} = (\frac{\omega^{*2}}{gk} - 1)\frac{E^*}{2} - \frac{k^2 E^{*2}}{2g} + \cdots \,, \tag{27}$$

where now $\omega^* = \omega - ku(x)$. The nonlinear dispersion relation (11) becomes, from $\bar{L}^{(w)}_{E^*} = 0$,

$$\omega^{*2} = gk + 2k^3 E^* + \cdots \,, \tag{28}$$

Conservation of wave action (12) and conservation of waves (16) again yield, for a steady solution

$$F = -\bar{L}^{(w)}_k = F_0 \,, \quad \omega_0 = \omega^* + u(x)k \,, \tag{29}$$

where F_0, ω_0 are constants. When combined with (28) these yield two coupled equations for k, E^* in terms of $u(x)$.

9.13 Waves on an opposing current: nonlinear effects

Now the dispersion relation (28) depends on the amplitude, $\omega^* = \omega^*(k, E^*)$ as well as the wavenumber. Conservation of wave action flux becomes

$$WA = F_0\,, W = -\frac{\bar{L}_k^{(w)}}{\bar{L}_\omega^{(w)}} = u(x) + \frac{\omega^*}{2k} + k^2 A\,, \tag{30}$$

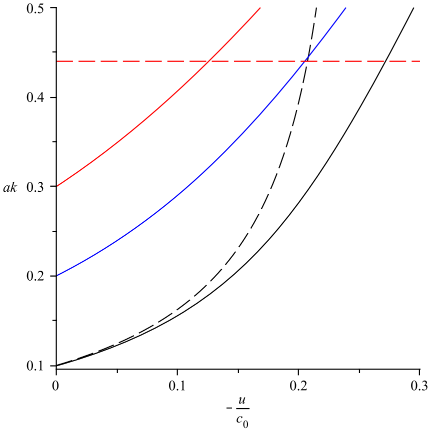
$$A = \bar{L}_\omega^{(w)} = \frac{E^*}{\omega^*}(1 + \frac{2k^2 E^*}{g})\,. \tag{31}$$

These are combined with (28) and (29),

$$\omega^{*2} = gk + 2k^3\omega^* A\,, \quad \omega_0 = \omega^* + u(x)k\,, \tag{32}$$

to yield two equations for k, A in terms of $u(x)$. Note that for an opposing current $u(x) < 0$ $(x > 0)$ there is now no stopping velocity, as $W \to 0, A \to \infty$ is not allowed.

9.14 Waves on an opposing current: nonlinear effects



Wave steepness ak versus u/c_0; $a_0 k_0 = 0.1, 0.2, 0.3$ (black, blue,red); wave breaking criterion $ak = 0.44$ (red dash) yields breaking for $|u|/c_0 > 0.27, 0.21, 0.13$. The dash line is the linear solution for $a_0 k_0 = 0.1$.

9.15 Waves on a beach: Modulation equations

We recall that the full modulation equations are

$$A_t + F_x = 0\,, \quad A = \overline{L}^{(w)}_{\omega}\,, \quad F = -\overline{L}^{(w)}_{k}\,, \tag{33}$$

$$H_t + (HV)_x = 0\,, \quad V = U + \frac{kA}{H}\,, \tag{34}$$

$$V_t + VV_x + g\bar{\zeta}_x + \frac{S_x}{H} = 0\,, \tag{35}$$

$$S = k(F - VA) + \overline{L}^{(w)} - H\overline{L}^{(w)}_{H}\,, \quad H = \bar{\zeta} + h(x)\,, \tag{36}$$

$$k_t + \omega_x = 0\,, \quad \overline{L}^{(w)}_{E^*} = 0\,, \tag{37}$$

$$\text{where} \quad \overline{L}^{(w)}(\omega^*, k, H, E^*) = \frac{DE^*}{2} + \frac{D_2 k^2 E^{*2}}{2g} + \cdots\,, \tag{38}$$

$$\text{and} \quad D(\omega^*, k, H) = \frac{\omega^{*2}}{gk\tanh kH} - 1\,, \quad \omega^* = \omega - Uk\,. \tag{39}$$

The mean momentum equation (35) has been rewritten.

9.16 Waves on a beach: wave set-up

Suppose that $h = h(x) \to 0$ as $x \to 0$, and that there is no background current. Then the steady solution $(\partial/\partial t = 0)$ of these modulation equations yields the dispersion relation (37, 39), constant frequency $\omega = \omega_0$, and constant wave action flux and zero mass transport,

$$-\bar{L}_k^{(w)} = F_0 \,, \quad V = U + \frac{kA}{H} = 0 \,, \quad \omega^* = \omega_0 - Uk \,. \tag{40}$$

Thus there is a **mean Eulerian flow $U = -kA/H$, opposing the Stokes drift due to the waves**. The mean momentum equation (35) then yields **the wave set-up $\bar{\zeta}$**,

$$gH\bar{\zeta}_x + S_x = 0 \,, \quad S = kF_0 + \bar{L}^{(w)} - H\bar{L}_H^{(w)} \,. \tag{41}$$

From (40), S as known in terms of H, and so

$$g\bar{\zeta} = -\int^H \frac{S_H}{H}\, dH \,. \tag{42}$$

9.17 Waves on a beach: wave set-up in linear theory

To illustrate, first make the small amplitude approximation. Then $\omega^* \approx \omega_0$, so that the dispersion relation becomes $\omega_0^2 = gk \tanh kh$ and yields $k = k(h)$. The constant wave action flux condition reduces to

$$c_g a^2 = c_{g0} a_0^2 \,, \tag{43}$$

where the subscript "0" indicates the values at the depth $h = h_0$ offshore. The expression (42) becomes

$$\bar{\zeta} = -\frac{ka^2}{2\sinh 2kh} \,, \tag{44}$$

where $\bar{\zeta}_0 = 0$. This is always negative, and so is a **set-down**. In shallow water as $kh \to 0$, $c_g \approx (gh)^{1/2}$, and

$$\frac{k}{k_0} \approx (\frac{h_0}{h})^{1/2} \,, \quad \frac{a}{a_0} \approx (\frac{h_0}{h})^{1/4} \,, \quad \bar{\zeta} \approx -\frac{a^2}{4h} \, (\frac{a_0^2 h_0^{1/2}}{4h^{3/2}}) \,. \tag{45}$$

Since this small-amplitude theory predicts infinite amplitudes as $h \to 0$, we must consider nonlinear effects. One option is to impose an empirical wave-beaking condition $a/h = 0.44$, which defines the depth $h = h_b$, beyond which there is a **surf zone**. Here, we shall examine nonlinear effects in $h > h_b$ in the shallow water approximation $kH \to 0$. Then the Lagrangian (38) becomes

$$\bar{L}^{(w)} \approx \frac{DE^*}{2} - \frac{9E^{*2}}{16gk^2H^4}\,, \quad D \approx \frac{\omega^{*2}}{gHk^2}(1 + \frac{k^2H^2}{3}) - 1\,. \qquad (46)$$

It is apparent that this can only be valid when $ak << k^3H^3$, that is for a very small **Stokes number**. Using the linear shallow water expressions we require that $S_0 = a_0/k_0^2h_0^3 << (h/h_0)^{9/4}$, which must fail as $h \to 0$. Hence, we infer that in shallow water we need to use a new theory, valid for Stokes number of order unity, that is the **Korteweg-de Vries** model.

9.19 Waves on a beach: Korteweg-de Vries model

The Korteweg-de Vries (KdV) equation for **weakly nonlinear long water waves**, propagating on a constant undisturbed mean depth H, is given by

$$\zeta_t + c_0\zeta_x + \frac{3c_0}{2H}\zeta\zeta_x + \frac{c_0 H^2}{6}\zeta_{xxx} = 0\,,\ \ c_0 = (gH)^{1/2}\,. \qquad (47)$$

The KdV balance has linear dispersion, represented by $H^3\zeta_{xxx}$, balanced by nonlinearity, represented by $\zeta\zeta_x$. To leading order, the waves propagate unchanged in form with the **linear long wave speed** $c_0 = (gH)^{1/2}$. Nonlinearity leads to wave steepening, opposed by wave dispersion, resulting in the KdV balance and the well-known **solitary wave**

$$\zeta = a_s\,\mathrm{sech}^2\kappa(x - ct)\,, \quad \frac{c}{c_0} - 1 = \frac{a_s}{2H} = \frac{2\kappa^2 H^2}{3}\,. \qquad (48)$$

9.20 Waves on a beach: cnoidal waves

The periodic wave solution of the KdV equation (47) is

$$\zeta = 2a\{b(m) + \mathrm{cn}^2(\gamma\theta); m)\}\,, \quad \omega = -\theta_t\,, k = -\theta_x\,, \tag{49}$$

$$b = \frac{1-m}{m} - \frac{E(m)}{mK(m)}\,,\ \frac{a}{H} = \frac{2}{3}m\gamma^2(kH)^2\,,\ \gamma = \frac{K(m)}{\pi}\,, \tag{50}$$

$$\text{and} \quad c = \frac{\omega}{k} = c_0\{1 + \frac{a}{H}[\frac{2-m}{m} - \frac{3E(m)}{mK(m)}]\}\,, \tag{51}$$

Here $cn(x;m)$ is the elliptic function of modulus m where $0 < m < 1$, and $K(m), E(m)$ are elliptic integrals of the first and second kind. The amplitude is a and the mean value is 0. As $m \to 1$, this becomes a **solitary wave**, since then $b \to 0$ and $\mathrm{cn}^2(x) \to \mathrm{sech}^2(x)$. As $m \to 0$, $\gamma \to 1/2$, and it reduces to sinusoidal waves of small amplitude $a \sim m$. This cnoidal wave (49) contains two free parameters; we take these to be the amplitude a and the wavenumber k.

9.21 Waves on a beach: modulated cnoidal waves

We now use the cnoidal wave expression (49) to evaluate the averaged Lagrangian (6), incorporating a mean current U,

$$\bar{L}^{(w)} = (\frac{c^{*2}}{gH} - 1)G(m)\frac{E^*}{2} + \cdots , \quad E^* = \frac{ga^2}{2} , \tag{52}$$

$$\text{where} \quad G(m) = 8(< cn^4(\gamma\theta; m) > - b^2) , \tag{53}$$

$$\text{or} \quad G(m) = \frac{8(EK(4-2m) - 3E^2 - K^2(1-m))}{3K^2m^2} . \tag{54}$$

To leading order the phase speed $c^* = W = (gH)^{1/2}$, while the wave action density, wave action flux and radiation stress now become, to leading order,

$$A = \bar{L}^{(w)}_\omega = \frac{G(m)E^*}{\omega^*} , \quad F = -\bar{L}^{(w)}_k = (U + c^*)A , \tag{55}$$

$$S = \frac{3\omega^* A}{2} = \frac{3G(m)E^*}{2} . \tag{56}$$

9.22 Waves on a beach: steady case

As before, we now seek the steady solutions, that is $\partial/\partial t = 0$, so that again $\omega = \omega_0$ is the constant wave frequency, so that to leading order $kh^{1/2} = k_0 h_0^{1/2}$ is constant. Next $F = F_0$ is the constant wave action flux, implying that, to leading order in wave amplitude,

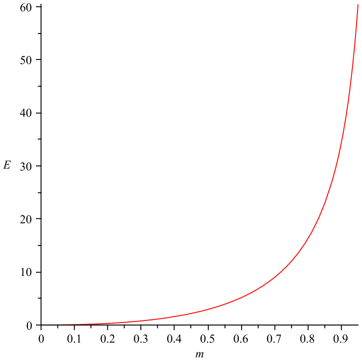
$$h^{1/2} G(m) a^2 = \text{constant}\,, \tag{57}$$

Then using the expression (50) we find that $a \propto mK^2 k^2 h^3$ and so finally we get that

$$\tilde{G}(m) = K^4 m^2 G(m) = \text{constant}\, h^{-9/2}\,. \tag{58}$$

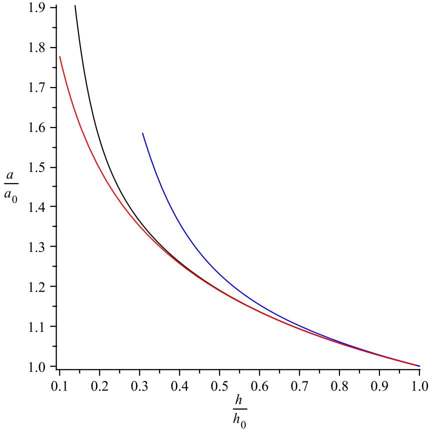
As $m \to 0$, $G \propto 1, \tilde{G} \propto m^2$, and so $m \propto h^{-9/4}$, $a \propto h^{-1/4}$ which is the linear Green's law result. But, as $m \to 1$, $G \propto K^{-1}$, $\tilde{G} \propto K^3$, $a \propto h^{-1}$.

9.23 Waves on a beach: cnoidal wave modulus



As h decreases, $E(m)$ increases and $m \to 1$. As the waves progress shorewards they become **solitary waves**, whose amplitude $a \propto h^{-1}$. But for small-amplitude sinusoidal waves $m \to 0$, $E(m) \propto m^2$ and $a \propto h^{-1/4}$.

9.24 Waves on a beach: cnoidal wave amplitude



The wave amplitude is determined from (57, 58). The plots are for an initial modulus $m_0 = 0.1, 0.5$ (black, blue), while the linear solution $\zeta \propto h^{-1/4}$ is the red curve.

9.25 Waves on a beach: cnoidal wave set-up

Wave set-up is found from (35, 56) and is given by

$$g\bar{\zeta} = -\frac{S_x}{h}, \quad S = \frac{3\omega A}{2} = \frac{3G(m)E^*}{2}. \tag{59}$$

But since the wave frequency $\omega = kc_0, c_0 = (gh)^{1/2}$ and the wave action flux $c_0 A$ are conserved (see (57)), we readily find that

$$\bar{\zeta} = -\frac{a^2 G(m)}{4h} = -\frac{a_0^2 h_0^{1/2} G(m_0)}{4h^{3/2}}, \tag{60}$$

This is just the linear law again, and is independent of how the wave amplitude varies. Note that for $a_0/h_0 << 1, m_0 \approx 0$, $G(m_0) \approx 1$.

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