Transport in cellular flows from the viewpoint of stochastic differential equations

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July 18, 2006

1 Introduction

The behaviour of passive scalar tracers moving in a prescribed velocity field can be described using two equivalent formulations: (i) the passive scalar equation which in two dimensions can be written as

$$\partial_t \theta + \nabla^\perp H \cdot \nabla \theta = \varepsilon \Delta \theta, \tag{1}$$

(ii) probabilistic description in terms of stochastic differential equations

$$dX_t = -\partial_y H(X_t, Y_t) + \sqrt{2\varepsilon}\, dW_t^{(x)}; \tag{2a}$$

$$dY_t = \partial_x H(X_t, Y_t) + \sqrt{2\varepsilon}\, dW_t^{(y)}. \tag{2b}$$

Here $H(x, y)$ is a periodic stream function defined in such a way that the velocity field is given by $\mathbf{u} = (-\partial_y H, \partial_x H)$. In what follows the velocity field will always be deterministic and time-independent.

The subject of study in this report is the asymptotic behaviour of the process (2) in the limit $\varepsilon \to 0$. In fact, this question can be analyzed using the passive scalar equation (1). Here the homogenization technique from the theory of partial differential equations is applied which gives a description of the behaviour of solutions of (1) on large scales, see [1]. Separating slow and fast variables and performing a multiscale analysis one obtains the effective diffusion equation for the evolution of θ on large scales

$$\partial_t \theta = \nabla \cdot D_\varepsilon^* \nabla \theta. \tag{3}$$

The effective diffusivity D_ε^* is a constant matrix given by the following expression

$$D_\varepsilon^*(\mathbf{e}) = \langle (\varepsilon \mathbb{I} + \mathbf{\Psi})(\nabla \chi + \mathbf{e}) \cdot \mathbf{e} \rangle, \tag{4}$$

where χ is the solution of the so called cell problem

$$\nabla \cdot [(\varepsilon \mathbb{I} + \mathbf{\Psi})(\nabla \chi + \mathbf{e})] = 0. \tag{5}$$

Note that equation (5) has to be solved in the domain of periodicty of the streamfunction $H(x, y)$. One class of flows for which the cell problem (5) is amenable to analysis is a special

case of the well known ABC flow, namely the case $A = 1$, $C = 0$. The streamfunction is of the form

$$H(x, y) = \frac{1 + B}{2} \cos x \cos y + \frac{1 - B}{2} \sin x \sin y. \tag{6}$$

The case $B = 1$ has been treated in [2] using boundary layer techniques where it was shown that $D^*_\varepsilon = O(\sqrt{\varepsilon})$. More precisely, the boundary layer analysis is performed in a boundary layer of width $O(\sqrt{\varepsilon})$ which forms at the boundaries of the impermeable cells of the underlying velocity field $\nabla^\perp H(x, y)$. Heuristically, since the amount of the tracer transported along the boundary layer is proportional to its area, the effective diffusivity has to be proportional to $\sqrt{\varepsilon}$. An analytical solution of the cell problem in the boundary layer approximation was given in [3] using Wiener–Hopf technique. In the case $B < 1$ the effective diffusivity is anisotropic and as has been shown in [4] the effective diffusivity across the streamlines is of order $O(\varepsilon)$ while along the stream lines of the flow it is of order $O(1/\varepsilon)$.

The large-scale picture obtained by the homogenization approach as presented above does not give any information on the microscale structure of the diffusion processes happening in the flow. However, small scales do play a major role in the form of boundary layers which is typical for diffusion processes at high Péclet numbers.

In principle, stochastic differential equations (2) allow us to describe the diffusion of particles in much more detail. But how much information can we obtain on the limit $\varepsilon \to 0$? A standard approximating technique in this framework is the so called Wentzell–Freidlin method. It describes the behaviour of randomly perturbed Hamiltonian systems on large time scales at high Péclet numbers by means of continuous diffusion processes on graphs, as is explained in Section 2. A priori, this technique works only only for Hamiltonians $H(x, y)$ such that $H(x, y) \to +\infty$ when $|(x, y)| \to \infty$ and does not allow for existence of heteroclinic orbits.

If we try to apply this technique to the case of unbounded cellular flow with $H(x, y)$ given by (6) we arrive at a paradox. Namely, the transition from one cell to another will happen instantaneously no matter how large the spatial separation between the cells. Furthermore, the characteristic time scale of diffusion will be of order $O(1/\varepsilon)$ which contradicts the results obtained in the homogenization framework where the characteristic time scale is of order $O(1/\sqrt{\varepsilon})$.

However, for bounded domains this technique works for sufficiently small ε such that $1/\sqrt{\varepsilon}$ is much larger that the domain size (therefore it is not in contradiction to the homogenization method). This gives us an indication that the Wentzell-Freidlin method remains locally valid. In fact, it fails on unbounded domains because of its global structure which is determined by the method of "gluing" together single cells.

Actually, by changing the "gluing" prescription between the processes obtained in single cells we can make the Wentzell–Freidlin approach to be consistent with the results given by the homogenization. The main ingredient here is the conservation of probability: the probability current going out of a cell has to be matched with the probability current across the cell boundary (dominated by the boundary layer effects) into the neighbouring cells.

The structure of the present report is as follows: Section 2 is entirely devoted to the Wentzell–Freidlin technique. In Section 2.1 we explain the averaging principle for a single cell and its deterministic background. Asymptotics $\varepsilon \to 0$ of solutions to (2) on bounded domains is described in Section 2.2. In Section 2.3 we discuss some simple models for

diffusion in unbounded cellular flows. Furthermore, using results of numerical simulations we argue that the approximating process on unbounded domain will be discontinuous. In Section 3 we outline the procedure for obtaining this process and state the result. Possible applications are discussed in Section 4.

2 Random perturbation of Hamiltonian systems

For small values of ε (i.e. for high Péclet numbers) equations (2) describe small random perturbation of the determinstic system

$$\dot{x} = -\partial_y H(x(t), y(t)), \tag{7a}$$
$$\dot{y} = \partial_x H(x(t), y(t)) \tag{7b}$$

which represents the limiting case $\varepsilon = 0$. Heuristically one would expect that the behaviour of solutions to (2) in the limit $\varepsilon \to 0$ is to a large extent determined by the deterministic solutions, i.e. solutions of (7). The simplest ansatz of this type consists in making a perturbative expansion of solutions to (2) in powers of $\sqrt{\varepsilon}$ around the deterministic solutions[1], see [11, 12]. However, this approximation is of little use when we want to study long-time behaviour. Indeed, it works well only for a finite period of time (which also remains true when we include higher-order terms) and does not take into account the separation into fast and slow variables. The latter point is of special interest to us because the underlying structure of the deterministic case (to be described in Section 2.1.1) is at the basis of the approximating method discussed in Section 2.1.

2.1 Slow-scale motion inside a cell

2.1.1 Case of vanishing viscosity

In terms of (1) the deterministic case corresponds to the passive scalar equation with vanishing viscosity $\varepsilon = 0$

$$\partial_t \theta + \nabla^\perp H \cdot \nabla \theta = 0 \tag{8}$$

and an initial condition θ_0. The charateristics of this equation are given by (7) so that the solutions of the Cauchy problem for equation (8) can be constructed in terms of the one-parametric flow (by taking the inverse Lagrangian mapping) generated by the velocity field $\nabla^\perp H(x, y)$, see e.g. [5].

If we choose the streamfunction (6), then equations (7) can be solved explicitly, see [6]. For the particular case $B = 1$ the solution is given in terms of Jacobi elliptic functions (see [7])

$$x(t, h) = \arcsin\left(\sqrt{1-h^2}\,\mathrm{sn}(t, \sqrt{1-h^2})\right), \tag{9a}$$
$$y(t, h) = \arcsin\left(\sqrt{1-h^2}\,\mathrm{cd}(t, \sqrt{1-h^2})\right), \tag{9b}$$

[1] In the case of the Hamiltonian (6) with $B = 1$ the explicit solution of (7) given by (9) allows us to compute analytically the first order term in the expansion. It turns out to be a stochastic integral the integrand being a complicated expression involving Jacobi elliptic, hyperbolic and logarithmic functions.

where $x(0) = 0$ and $H(x, y) = h$ is kept fixed along a trajectory. Obviously, a particle initially located in one particlular cell will stay inside this cell. The period of motion along a level line with $H(x, y) = h$ is given by a complete elliptic integral $4K(\sqrt{1-h^2})$. For the special case $h = 0$ the equation of motion along the separatrix can be solved in terms of elementary functions

$$x(t, 0) = 2 \arctan e^t - \frac{\pi}{2}, \tag{10}$$

where the initial conditions are specified as $x(0, 0) = 0$ and $y(0, 0) = \pi/2$. Note that in the deterministic case it takes an infinite time to reach the equilibrium point $(\pi/2, \pi/2)$.

It is crucial for the following analysis that we can interpret (7) as a Hamiltonian system with one degree of freedom by identifying $(x, y) \to (p, q)$. The streamfunction $H(x, y) \to H(p, q)$ is then identified with the Hamiltonian of the system. As usually, an action variable $I(h)$ can be introduced as

$$I(h) = \frac{2}{\pi} \int_h^1 K\left(\sqrt{1-h'^2}\right) dh', \tag{11}$$

which is equal to the area inclosed inside the orbit $H(p, q) = h$ divided by 2π, see [8].

2.1.2 Effective Fokker-Planck equation

We will first study the behaviour of solutions to (2) with H given by (6) with $B = 1$ such that the particle is staying inside one cell. The typical technique for analyzing the evolution of slow variables of such a system subject to small random perturbations is the technique of averaging out the fast variables. Then an effective evolution equation for slow variables inside a cell is obtained in a way similar to the analysis of small perturbations in classical mechanics [8]. In the framework of stochastic differential equations this ansatz was introduced by Wentzell and Freidlin, see [12] and references therein. For analogous consideration in the framework of passive scalar equation see [17].

In our case the averaging principle can be briefly summarized as follows: Inside the cell the deterministic system (7) can be described in terms of motion on invariant tori. We parametrize these tori by the values of $H(x, y) = h$. In the perturbed system (at least for small perturbations) particles will still rotate rapidly along the tori, however they will slowly drift across the tori. To describe this slow drift we calculate $dH(X_t, Y_t)$. Using Itô's formula we obtain

$$dH(X_t, Y_t) = \sqrt{2\varepsilon}\, \nabla H(X_t, Y_t) \cdot d\mathbf{W}_t + \varepsilon\, \Delta H(X_t, Y_t) dt, \tag{12}$$

where $d\mathbf{W}_t = (dW_t^{(x)}, dW_t^{(y)})$ is the two-dimensional Brownian motion. Of course, we cannot evaluate the terms $\nabla H(X_t, Y_t) \cdot d\mathbf{W}_t$ and $\Delta H(X_t, Y_t)$ explicitly without solving equations (2). However, using the integral form of (12)

$$H(X_t, Y_t) = H(X_0, Y_0) + \sqrt{2\varepsilon} \int_0^t \nabla H(X_s, Y_s) \cdot d\mathbf{W}_s + \varepsilon \int_0^t \Delta H(X_s, Y_s) ds, \tag{13}$$

we see that because of the smallness of the perturbation (i) the second integral is approximately equal[2] to the integral $\int_0^t \langle \Delta H \rangle_{H=h}(X_s, Y_s) ds$, where

$$\langle \Delta H \rangle_{H=h} = \left(\oint \frac{dl}{|\nabla H|} \right)^{-1} \oint \frac{\Delta H}{|\nabla H|} \, dl \tag{14}$$

and the integrals are taken over the level set $\{(x, y) : \ H(x, y) = h\}$. Moreover, (ii) the first (stochastic) integral in (13) can be represented as

$$\int_0^t \nabla H(X_s, Y_s) \cdot d\mathbf{W}_s = W \left(\int_0^t |\nabla H(X_s, Y_s)|^2 ds \right), \tag{15}$$

where $W(\cdot)$ is a one-dimensional Wiener process. With the same argument as before $\int_0^t |\nabla H(X_s, Y_s)|^2 ds$ can be approximated by the ergodic average of $|\nabla H|^2$

$$\langle |\nabla H|^2 \rangle_{H=h} = \left(\oint \frac{dl}{|\nabla H|} \right)^{-1} \oint |\nabla H| \, dl \tag{16}$$

The obtained effective diffusion process is most conveniently formulated in terms of an effective Fokker-Planck equation with time rescaled as $t \to \varepsilon t$

$$\partial_t p = \partial_h^2 \left(A(h) p \right) - \partial_h \left(B(h) p \right) \tag{17}$$

The coefficients $A(h)$ and $B(h)$ given by

$$A(h) = \langle |\nabla H|^2 \rangle_{H=h}, \quad B(h) = \langle \Delta H \rangle_{H=h}. \tag{18}$$

In the case of H given by (6) with $B = 1$ these coefficients can be calculated explicitly using formulas (9)

$$A(h) = 2 \frac{E(\sqrt{1-h^2})}{K(\sqrt{1-h^2})} - 2h^2, \qquad B(h) = -2h. \tag{19}$$

Here $K(\cdot)$ and $E(\cdot)$ are complete elliptic integrals of the first and second kind, see [7, 15]. Note that instead of the Hamiltonian we could have used the action variable (11). Indeed, the formulation in terms of the action turns out to be very convenient for generalization of the averaging principle to higher dimensions.

2.2 Construction of an approximating Feller process (Wentzell-Freidlin technique)

In this subsection we study asymptotic behaviour of solutions of (2) constrained to a bounded domain (with periodic boundary conditions) approximating them by a process with essentially one-dimensional state space. We have seen that (17) specifies completely the behaviour of the system up to the first exit time out of a cell. Once having left the cell after some transitional time (during which it will stay in some neighbourhood of the boundary of the original cell) the particle will again slowly diffuse, either in the original cell or in another neighbouring cell.

[2] This is due to the averaging principle.

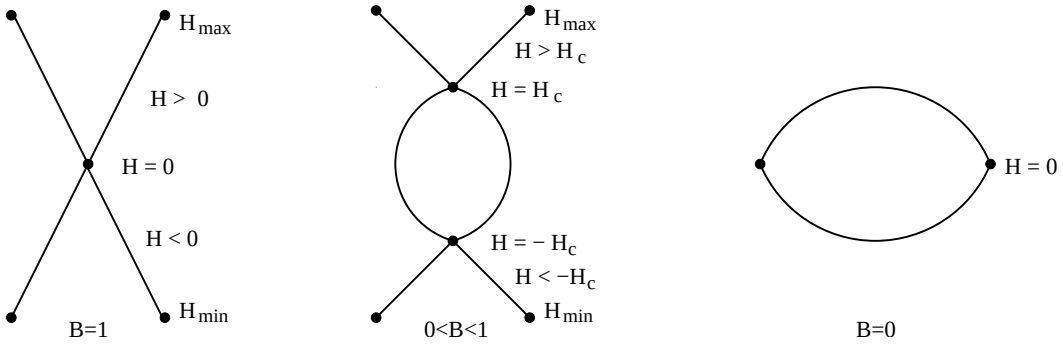

Figure 1: Reeb graphs of the flow (6) for the cases $B = 1$, $0 < B < 1$ and $B = 0$ on the domain of width $2\pi \times 2\pi$.

To describe the transition phase of the particle from one cell to another we need to (i) give a geometrical description of the way in which the cells are connected to each other, (ii) analyze in detail the behaviour of the diffusion process during this transition.

The geometrical description is given using the topological notion of a Reeb graph of $H(x, y)$ [9]. For a general Hamiltonian $H(x, y)$ it is defined in the following way: Let (x_c, y_c) be a critical point, i.e. a point such that $\nabla H(x_c, y_c) = 0$. Then each connected component of the level set $H^{-1}(H(x_c))$ is identified with a vertex. The points on the edges which connect the vertices are identified with connected components of the noncritical level sets $H^{-1}(H(x, y))$. Figure 1 shows the Reeb graphs of $H(x, y)$ given by (6) on the domain $[0, 2\pi] \times [0, 2\pi]$ for the cases $B = 0$, $0 < B < 1$, $B = 1$. Note that in the case $B = 1$ the edges of the Reeb graph correspond to the interior of the cells.

The geometrical description above suggests that the appropriate state space of the process approximating the solutions of (2) is the Reeb graph Γ of the Hamiltonian $H(x, y)$ with edges denoted by e_i and vertices denoted by O_k. In side each edge e_i the approximating process is governed by the evolution equation of the type (17)

$$\partial_t p_i = \partial_h^2 \left(A_i(h) p_i\right) - \partial_h \left(B_i(h) p_i\right) \tag{20}$$

To determine the process completely we have to specify the boundary conditions at the ends of each edge, "gluing" the edges together in a consistent way. This "gluing" procedure determines the behaviour of the process during the passage through the vertices of the graph. Physically it describes transitions of a particle from one cell to another.

Specifying boundary conditions for a stochastic process is in general a quite delicate point. One possibility to specify the boundary conditions is to require the approximating process to have "nice" mathematical properties.[3] In [13] all possible continuous Markov processes with Feller property[4] on graphs were described such that the diffusion inside an edge is governed by a second order elliptic operator (which can possibly depend on the edge). In our case the ellpitic operator is given by the right hand side of (20). Furthermore, at each edge the sum of the incoming probability currents has to vanish. This leads to the

[3] As we shall see later, this requirement is not always consistent with the behaviour of solutions of (2).

[4] This means that in course of time continuous distributions of probability remain continuous.

following boundary conditions at a vertex O_k

$$\sum_{i:\ e_i \sim O_k} J_i(O_k) = 0, \quad p_i(O_k) = p_j(O_k), \quad i,j : e_i \sim O_k,\ e_j \sim O_k \tag{21}$$

where $e_i, e_i \sim O_k$ denote the edges incident to the vertex O_k.

Thus, the approximating technique (in the following referred to as Wentzell–Freidlin technique) consists in (i) approximating the solutions of (2) inside of the cell by (17), (ii) gluing together the cells by matching continuously the probability distributions at the boundaries of the cells.

In the case of the Hamiltonian specified by (6) on bounded domain we expect that the process on the graph approximates well solutions of (2). Furthermore, similarly to [12] a particle spends a zero time (on the time scale εt) at the vertex. From this follows that once the particle reaches the cell (edge) boundary, it can hop instantaneously to any other cell. Furthermore, the behaviour of the process after it reaches the vertex does not depend on its prehistory, i.e. on the edge it came from. Thus, the transition probability from one edge to another is not dependent on their spatial separation. Note that for $\varepsilon \to 0$ this does not result in any contradiction, because of the very long relevant time $O(1/\varepsilon) \to \infty$.

Altogether the Wentzell–Freidlin technique yields the following picture for advection of passive scalar in a bounded domain: Let us chose an initial coondition which corresponds to the passive scalar being concentrated in the center of a cell at $t = 0$. Then, as time goes on, the passive scalar will slowly (i.e. on time scale $O(1/\varepsilon)$) diffuse untill it reaches the cell boundary. As soon as it reaches the boundary it will very quickly (instantaneously on time scale $O(1/\varepsilon)$) spread across the domain along the network of separatrices. After this the passive scalar will penetrate the cells on slow time scale $O(1/\varepsilon)$ untill the stationary distribution is established.

However, the above implies immediately that the Wentzell-freidlin technique cannot be applied in the unbounded case. Indeed, it would state that a transition from one edge to another happens instantaneously, no matter how large the spatial distance. It can also be readily seen from the structure of the Reeb graph corresponding to the unbounded case. Indeed, in this case the Reeb graph consists of one vertex with infinitely many incoming vertices.

2.3 Diffusion in cellular flows: unbounded case

As we have seen previously, the knowledge of the behaviour of solutions to (2) in the vicinity of the cell boundaries is of crucial importance. To describe this behaviour one has to take into account events such as a particle crossing of a separatrice and going to another cell. Qualitatively, we can discribe them as follows: We artificially separate the cells by channels (which play the role of the boundary layers along the separatrices) to take into account the transport along the boundaries. Let $\delta(\varepsilon)$ be the width of the channel. The motion of a particle consists of two types of events: (i) slow motion across streamlines inside the cells with typical time spent inside a cell $t_{\text{cell}} = O(1/\varepsilon)$ and mean square displacement $O(\varepsilon^0)$; (ii) fast transport in the channels with velocity of order $O(\varepsilon^0)$ and time spent inside the

channel $O(\delta^2/\varepsilon)$. The effective diffusion is given by

$$\kappa_{\text{eff}} = O\left(\frac{\langle X^2\rangle_{\text{channel}}}{t_{\text{cell}}}\right) = O\left(\delta^2\right) \tag{22}$$

Setting $\delta(\varepsilon) = \varepsilon^{\frac{1}{4}}$ we obtain $\kappa_{\text{eff}} = O(\sqrt{\varepsilon})$. However, the width of the channel is something that we have to insert by hand into this model. Furthermore, for solution of the cell problem the width of the boundary layer is usually assumed to be $\sqrt{\varepsilon}$. Nevertheless, this simple model is usefull because it gives an additional intuition about the nature of diffusion in the cellular flow. Indeed, numerical simulations of (2) show long flights along the cell boundaries interupted by trapping of the particle inside the cells, see Fig. 2.

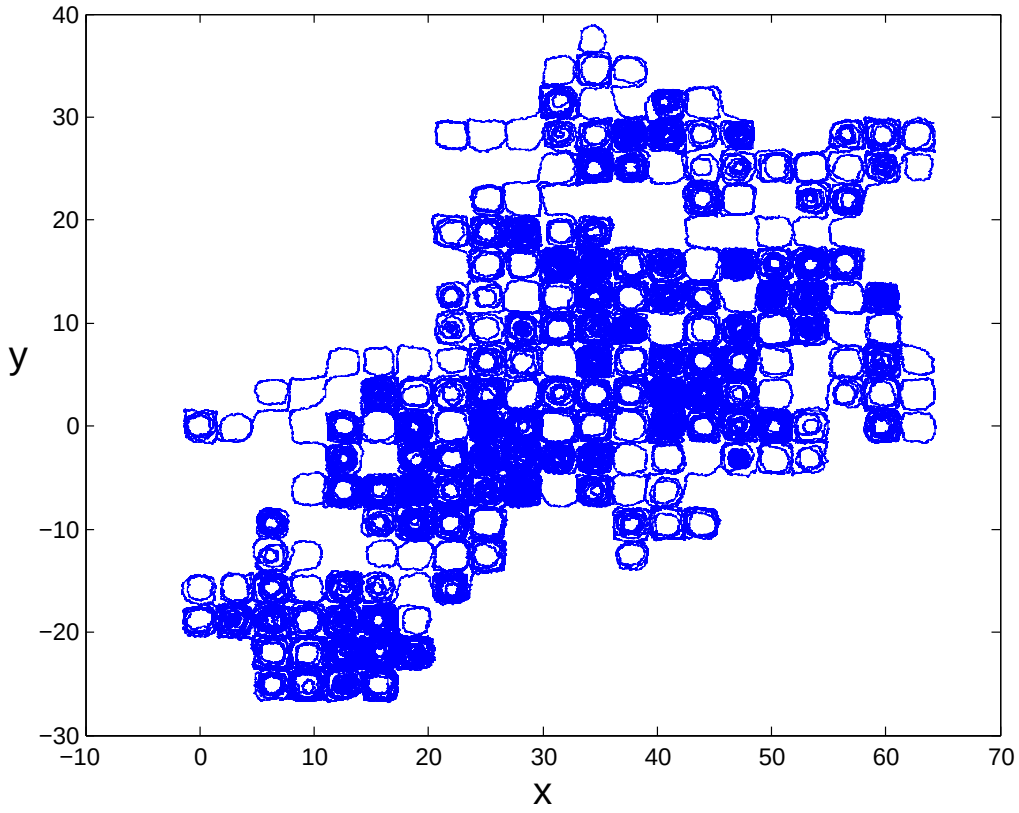

Figure 2: Random motion of a Brownian particle the cellular flow $B = 1$.

During the excursions the stochastique motion can be described by a very simple one-dimensional model

$$d\Theta_t = |\cos\Theta_t| + \sqrt{2\varepsilon}dW_t, \tag{23}$$

where $\theta \in [0, 4\pi]$. The backward Kolmogorov equation is given by

$$\partial_t p + |\cos\theta|\partial_\theta p = \varepsilon\partial_\theta^2 p. \tag{24}$$

Solutions[5] are most easily found by writing the drift $|\cos\Theta|$ as the derivative of the following continuous potential

$$V(\theta) = \begin{cases} -4k - \sin\theta & \text{for } x \in [-\frac{\pi}{2} + 2k\pi, \frac{\pi}{2} + 2k\pi]; \\ -2(2k+1) + \sin\theta & \text{for } x \in [-\frac{\pi}{2} + (2k+1)\pi, \frac{\pi}{2} + (2k+1)\pi], \end{cases} \tag{25}$$

such that $|\cos\theta| = -\partial_\theta V(\theta)$. Note that this potential contains two parts: a periodic part $V_0(\theta)$ with period π and a tilting force $F = \frac{2}{\pi}$. Obviously, the function $V_0(\theta) = V(\theta) + \theta F$ satisfies $V_0(\theta + \pi) = V_0(\theta)$. Stationary distributions are easily found which allows to determine the mean velocity and the effective diffusion

$$< v >= \lim_{t\to\infty} \frac{\mathbb{E}[\Theta_t]}{t} \qquad D = \lim_{t\to\infty} \frac{\mathbb{E}[\Theta_t^2] - (\mathbb{E}[\Theta_t])^2}{2t} \tag{26}$$

It turns out that the mean velocity does not vanish and the effective diffusion is proportional to $1/\epsilon$, see [22]. The behaviour of the diffusion process in the neighbourhood of the cell boundaries has been studied in [14] using boundary layer asymptotics. It turns out that the time a particle spends in a cell is of order $O(\varepsilon)$.

Note that events (i) and (ii) have two different time scales: (i) is on time scale $O(1/\sqrt{\varepsilon})$ while (ii) is on time scale $O(1/\varepsilon)$. A similar situation has been discussed in [18] for effective diffusion along a pipe with semiinfinite pipes branching off the main pipe. The fast motion happens inside the main pipe while trapping occurs inside the side branches.

Since the stochastic motion of a particle inside of the unbounded cellular flow is a mixture of random walk on the lattice of cells (jump process) and slow diffusion inside the cells (continuous process) we cannot expect the Wentzell–Freidlin technique to hold on unbounded domains. Indeed, as Fig. 3 shows, in numerical simulations of diffusion of passive tracer on the cellular flow (6) we find high gradients of the passive scalar accross the cell boundaries.

3 Approximating process with jumps

In this section we propose a generalization of Wentzell-Freidlin method to the case of unbounded domains. The main idea is to give up the mathematical condition of continuity of the process at the vertices of the Reeb graph which in the previous paragraph has been shown to inconsistent and allow for processes which can have jumps at vertices. The continuity condition has to be replaced by another condition which takes into account processes happening at the separatrices analyzing them more carefully than it has been done in Section 2.2.

We begin by outlining the procedure which can be used for obtaining the approximating process on the domain $\mathbb{R}^2$. We label each cell by a two-dimensional integer vector $(n_1, n_2) \in \mathbb{Z}^2$. The effective evolution equation inside each cell is given by

$$\partial_{\varepsilon t}\, p_{(n_1,n_2)} = \partial_h^2 \left(A(h) p_{(n_1,n_2)}\right) - \partial_h \left(B(h) p_{(n_1,n_2)}\right) \tag{27}$$

Consider now, analogously to [20] a water-pipe network $\Omega_N^\varepsilon = \{(x,y) \in \Omega : \ |H(x,y)| \le N\sqrt{\varepsilon}\}$ around the separatrices. The corresponding water-pipe approximation is

$$\epsilon \Delta \theta_N^\varepsilon - \nabla^\perp H \cdot \nabla \theta_N^\varepsilon = 0, \qquad (x,y) \in \Omega_N^\varepsilon, \tag{28}$$

[5] Stochastic equations with periodic drift are intensively studied in [21].

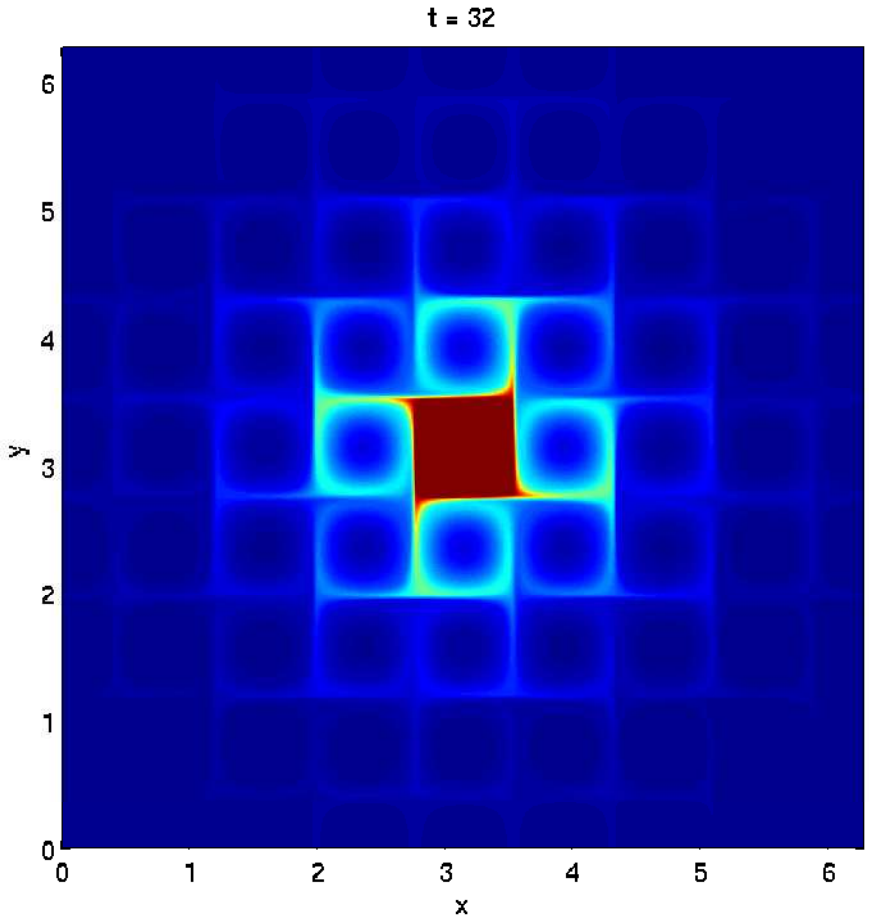

Figure 3: Diffusion in the cellular flow given by (6) with $B = 1$ for $\varepsilon = 0.0009765625$ at $t = 32$.

however, instead of zero gradient von Neumann boundary conditions at the level set

$$\mathcal{L}_{(n_1,n_2)}(N\sqrt{\varepsilon}) = \{(x,y) \in \Omega : \ |H(x,y)| = N\sqrt{\varepsilon}, \ (x,y) \text{ lies in the cell } (n_1,n_2)\} \tag{29}$$

used in [20], we introduce the following conditions

$$\frac{\partial \theta_N^\varepsilon}{\partial \mathbf{n}} = (D\theta)_{(n_1,n_2)}, \quad (x,y) \in \mathcal{L}_{(n_1,n_2)}(N\sqrt{\varepsilon}). \tag{30}$$

The two descriptions, in the interior of the cells and at the cell boundary, have to be glued together. It is here that the conservation of probability enters. We have to match probablity current leaving the edge with the probablity current entering the water-pipe network. This gives the equation

$$\oint_{\mathcal{L}_k(N\sqrt{\varepsilon})} (D\theta)_k dl = \varepsilon \left[B(h) p_{(n_1,n_2)}(h) - \partial_h \left(A(h) p_{(n_1,n_2)} \right) \right]_{h=N\sqrt{\varepsilon}}. \tag{31}$$

The expression on the right hand side is just the probability current entering the boundary of the cell.

Now we turn to description of the approximating process. First of all, in order to describe the transport from one cell to another in the asymptotics $\varepsilon \to 0$ we find it convenient to consider the network of lines connecting the neighbouring cells which is in fact a network dual to the water-pipe network. In our case of $H(x, y)$ given by (6) with $B = 1$ it is just the two-dimensional lattice $\mathbb{Z}^2$. At each vertex of this network we specify a function $f_{(n_1,n_2)}(t)$. This function serves as a boundary condition for the effective Fokker–Planck equation (27) (with the original time) in the cell (n_1, n_2). For a particle the probability of leaving this cell through the boundary adjacent to one of the neighbouring cell, e.g. the cell $(n_1, n_2 + 1)$ is proportional to $f_{(n_1,n_2+1)} - f_{(n_1,n_2)}$. Therefore the discontinuous part of the process is governed by the Laplace lattice operator on the lattice $\mathbb{Z}^2$

$$\Delta_{\mathbb{Z}^2} f_{(n_1,n_2)}(t) = f_{(n_1+1,n_2)}(t) + f_{(n_1-1,n_2)}(t) + f_{(n_1,n_2+1)}(t) + f_{(n_1,n_2-1)}(t) - 4f_{(n_1,n_2)}(t) \quad (32)$$

The evolution equation for $f_{(n_1,n_2)}(t)$ is then given by

$$\frac{d}{dt} f_{(n_1,n_2)} + \frac{d}{dt} \int p_{(n_1,n_2)}(h)dh = \Delta_{\mathbb{Z}^2} f_{(n_1,n_2)} \quad (33)$$

In this way we obtain a coupled system of equations which yield the complete description of the approximating process.[6]

4 Discusion

The generalization of the Wentzell–Freidlin technique proposed in the previous paragraph is easily generalized to the cases of nonperiodic cellular flows. In fact, it suffices to replace the Laplace operator on $\mathbb{Z}^2$ by the Laplace operator on the network (graph) dual to the network of separatrices of the original flow. Of course, the spectral properties of the graph Laplacian then depend strongly on the topological structure of the dual network. Therefore the cellular structure of the flow can nontrivially influence the solutions of (33).

One application of the Wentzell–Freidlin technique is connected to the study of reaction-diffusion equations. However, as has been stated in [10], in its usual formulation it is not applicable without any restrictions. The generalized form of this technique proposed in the report seems to be suitable to a wider range of applications, including nonperiodic cellular flows.

5 Acknowledgements

I would like to thank E. Vanden Eijnden, G. Papanicolaou, W. Young and J.-L. Thiffeault for helping me in my work on this project. Many thanks to G. Veronis and C. Doering for softball coaching. And of course, I would like to thank all fellows for the wonderful summer we spent together.

[6] In fact, analogous constructions were discussed in [19, 18].

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