Nonlinear Ekman Dynamics for an Advective Flow

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1 Introduction

The Ekman layer was first described by Walfrid Ekman in his 1902 doctoral dissertation [2]. It is now a general term for a horizontal frictional boundary layer in a rotating frame of reference. These boundary layers are ubiquitous in geophysical fluid dynamics, at the bottom of the atmosphere, in the ocean surface layer, and (we think) above the seabed. The critical feature of these boundary layers is that the combination of rotation and friction induces flow across lines of constant pressure, causing convergence in some regions and divergence in others. Where there is convergence, fluid is forced vertically out of the boundary layer, a phenomenon known as *Ekman pumping*. This small vertical velocity from the surface Ekman layer of the ocean is thought to drive much of the ocean circulation.

Ekman derived the solution for linear flows, neglecting the effects of momentum advection. Because the Ekman layer is so central to oceanic circulation, the details of its physics are of great interest to the fluid dynamics community, and several efforts have have been made to understand its higher order behavior. In 1964, Benton, Lipps, and Tuann [1] examined the nonlinear modifications to the Ekman layer for a flow with locally uniform shear far from the boundary, calculating the corrections numerically to five orders of Rossby number. Inspired by this example, Eliassen (1971) [3] showed that nonlinear effects tended to suppress the pumping of fluid out of the boundary layer in the center of a cyclonic vortex. This result was of considerable interest, as it has been observed on many occasions that the center of cyclones in the atmosphere, like the eyes of hurricanes, tend to be relatively cloud–free. If the Ekman layer was inducing downwelling—sucking fluid out of the far field instead of pumping it in—this might explain the clear–eyed cyclones. In 2000, Hart [4] calculated analytically the nonlinear corrections to the Ekman pumping up to five orders of Rossby number. However, he restrict himself to unidirectional flows, that is flows that do not vary in the along–flow direction and where one of the velocity components is zero. He was able to show that Eliasson's result was in fact an artifact of the assumption of locally uniform shear adopted from Benton et al. As a result, it was unlikely that Ekman dynamics explained the paucity of clouds in cyclone centers.

All of these authors restrict their discussion to a limited class of flows, usually unidirectional flows. This simplifies the computation significantly, but it excludes important physics: the effects of curvature and the advection of material properties such as vorticity. However, we expect the lowest order nonlinear correction to the Ekman pumping velocity to be proportional to the advection of vorticity. To redress the oversight of previous studies, I will here calculate the weakly nonlinear form of the Ekman layer, and discuss the effect that including vorticity advection has on the structure of the boundary layer and the vertical velocity induced. I confine myself to discussing a case with no–slip boundary conditions, analogous to the ocean bottom. Though this is the less physically relevant case for ocean circulation, it is mathematically simpler and so a good starting point. I'll begin by reviewing the linear Ekman layer in Section 2, followed by the expansion in Rossby number in Section 3, including the weakly nonlinear solution for the Ekman pumping. Section 4 will discuss an illustrative example to help build intuition. Section 5 contains an alternative derivation of the weakly nonlinear solution for the illustrative example. I will conclude in Section 6.

2 The Linear Ekman Layer

We begin with the dimensionless Navier–Stokes equations for a homogeneous, steady flow:

$$\epsilon\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)-v = -\frac{\partial p}{\partial x}+\frac{E}{2}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) \quad (1)$$

$$\epsilon\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)+u = -\frac{\partial p}{\partial y}+\frac{E}{2}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) \quad (2)$$

$$\epsilon\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z}+\frac{E}{2}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right) \quad (3)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0 \quad (4)$$

In these equations, our dimensionless parameters are the Rossby number, $\epsilon=\frac{U_0}{2\Omega L}$, and the Ekman number, $E=\frac{\nu}{\Omega L^2}$, where Ω is the rotation rate of the frame, ν is the frictional parameter, L is a length scale, and U_0 is a velocity scale taken from the flow far from the boundary. In the inviscid interior flow far from the boundary, both E and ϵ can be considered small, so to lowest order, the above system of equations becomes:

$$-v = -\frac{\partial p}{\partial x}$$

$$+u = -\frac{\partial p}{\partial y}$$

$$0 = -\frac{\partial p}{\partial z}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0$$

These are the standard equations of *geostrophic balance*, where rotational and pressure effects balance each other in the horizontal and there is no vertical motion. We will assume that the flow far from the boundary always satisfies these equations, and we will denote the geostrophically balanced far field velocities $U(x,y)$ and $V(x,y)$. The continuity equation implies that $w_z=0$. If we imagine that somewhere there is a horizontal boundary that fluid cannot penetrate, we know that $w=0$ everywhere.

Though the interior is inviscid, there must be a region near the boundary in which the frictional terms are of the same order as the rotational terms. Therefore, we introduce a new stretched coordinate ζ, defined so that the region where friction is important—the boundary layer—is the region where ζ is $O(1)$:

$$z=\sqrt{E}\zeta \quad (5)$$

Assume $z=\zeta=0$ on the bottom boundary. Therefore:

$$\frac{\partial}{\partial z}=\frac{1}{\sqrt{E}}\frac{\partial}{\partial\zeta} \quad ; \quad \frac{\partial^2}{\partial z^2}=\frac{1}{E}\frac{\partial^2}{\partial\zeta^2}$$

We similarly rescale our vertical velocity so that:

$$\frac{\partial w}{\partial z}=\frac{\partial W}{\partial\zeta}$$

Our equations of motion (**??**) – (**??**) then become:

$$\epsilon\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+W\frac{\partial u}{\partial\zeta}\right)-v = -\frac{\partial p}{\partial x}+\frac{1}{2}\left(E\frac{\partial^2 u}{\partial x^2}+E\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial\zeta^2}\right)$$

$$\epsilon\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+W\frac{\partial v}{\partial\zeta}\right)+u = -\frac{\partial p}{\partial y}+\frac{1}{2}\left(E\frac{\partial^2 v}{\partial x^2}+E\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial\zeta^2}\right)$$

$$\begin{aligned}
\epsilon E\left(u\frac{\partial W}{\partial x}+v\frac{\partial W}{\partial y}+W\frac{\partial W}{\partial \zeta}\right) &= -\frac{\partial p}{\partial \zeta}+\frac{E}{2}\left(E\frac{\partial^2 W}{\partial x^2}+E\frac{\partial^2 W}{\partial y^2}+\frac{\partial^2 W}{\partial \zeta^2}\right)\\
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial W}{\partial \zeta} &= 0
\end{aligned}$$

The scale over which frictional effects are important has now been included in our governing equations, so we can neglect all terms that are still of order E. This gives:

$$\epsilon\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+W\frac{\partial u}{\partial \zeta}\right)-v = -\frac{\partial p}{\partial x}+\frac{1}{2}\frac{\partial^2 u}{\partial \zeta^2} \quad (6)$$

$$\epsilon\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+W\frac{\partial v}{\partial \zeta}\right)+u = -\frac{\partial p}{\partial y}+\frac{1}{2}\frac{\partial^2 v}{\partial \zeta^2} \quad (7)$$

$$0 = -\frac{\partial p}{\partial \zeta} \quad (8)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial W}{\partial \zeta} = 0 \quad (9)$$

To solve these equations, we assume that the Rossby number ϵ is small and expand all of our varying quantities in it:

$$\begin{aligned}
u &= u_0+\epsilon u_1+\dots\\
p &= p_0+\epsilon p_1+\dots\\
&\vdots
\end{aligned}$$

To $O(1)$, we get:

$$-v_0 = -\frac{\partial p_0}{\partial x}+\frac{1}{2}\frac{\partial^2 u_0}{\partial \zeta^2} \quad (10)$$

$$u_0 = -\frac{\partial p_0}{\partial y}+\frac{1}{2}\frac{\partial^2 v_0}{\partial \zeta^2} \quad (11)$$

$$0 = -\frac{\partial p_0}{\partial \zeta} \quad (12)$$

$$\frac{\partial u_0}{\partial x}+\frac{\partial v_0}{\partial y}+\frac{\partial W_0}{\partial \zeta} = 0 \quad (13)$$

These equations describe the linear Ekman layer problem. To solve it, we recall first that:

$$\begin{aligned}
U &= -\frac{\partial p_0}{\partial y}\\
V &= \frac{\partial p_0}{\partial x}
\end{aligned}$$

We can then define an auxiliary variable, Λ_0 such that:

$$\Lambda_0 = (u_0-U)+i(v_0-V)$$

The x– and y–momentum equations (**??**) — (**??**) then can be compactly expressed:

$$\frac{\partial^2 \Lambda_0}{\partial \zeta^2}-2i\Lambda_0 = 0$$

This second–order equation has two solutions, but we are only interested in the solution that is bounded as $\zeta \to \infty$, that is as we move away from the boundary:

$$\Lambda_0 = -\lambda e^{-\zeta(1+i)} \quad (14)$$

Applying the no slip boundary condition and converting back into real velocities, we find $\lambda = U + iV$, and:

$$u_0 = U + e^{-\zeta}\left(-U\cos\zeta - V\sin\zeta\right) \tag{15}$$
$$v_0 = V + e^{-\zeta}\left(U\sin\zeta - V\cos\zeta\right) \tag{16}$$

We can find W_0 using the continuity equation (**??**):

$$\begin{aligned} W_0 &= \frac{1}{2}\left(\frac{\partial U}{\partial y} \underbrace{-\frac{\partial V}{\partial y} - \frac{\partial U}{\partial x}}_{=0} - \frac{\partial V}{\partial x}\right) e^{-\zeta}\cos\zeta \\ &\quad + \frac{1}{2}\left(\frac{\partial U}{\partial y} + \underbrace{\frac{\partial V}{\partial y} + \frac{\partial U}{\partial x}}_{=0} - \frac{\partial V}{\partial x}\right) e^{-\zeta}\sin\zeta + C(x,y) \\ W_0 &= -\frac{1}{2}\omega e^{-\zeta}\left(\cos\zeta + \sin\zeta\right) + \frac{1}{2}\omega \end{aligned} \tag{17}$$

We find the constant of integration by applying a no normal flow boundary condition at the bottom boundary. In this, $\omega = \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}$ is the relative vorticity of the far field flow. We see that even infinitely far from the boundary, friction has induced a vertical velocity $W_0 = \frac{1}{2}\omega$ that is proportional to ω, the vorticity of the flow.

We have now completed the linear Ekman layer problem. We can see why it is worth pressing on to a weakly nonlinear solution by taking a look at the vorticity equation, found by taking the curl of the momentum equations (**??**) —(**??**):

$$\epsilon\left(u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y}) + w\frac{\partial \omega}{\partial z}\right) - \frac{\partial w}{\partial z} = \frac{E}{2}\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}) + \frac{\partial^2 \omega}{\partial z^2}\right)$$

Far from the boundary, where the effects of friction given by the right hand side of the equation are small, the vertical velocity gradient is balanced by the advection of vorticity. This strongly suggests that advective effects will play an important role in the weakly nonlinear solution. Let's calculate it.

3 Rossby Number Expansion

To find the weakly nonlinear correction, we examine our governing equations (**??**) to (**??**) to $O(\epsilon)$:

$$\left(u_0\frac{\partial u_0}{\partial x} + v_0\frac{\partial u_0}{\partial y} + W_0\frac{\partial u_0}{\partial \zeta}\right) - v_1 = -\frac{\partial p_1}{\partial x} + \frac{1}{2}\frac{\partial^2 u_1}{\partial \zeta^2} \tag{18}$$
$$\left(u_0\frac{\partial v_0}{\partial x} + v_0\frac{\partial v_0}{\partial y} + W_0\frac{\partial v_0}{\partial \zeta}\right) + u_1 = -\frac{\partial p_1}{\partial y} + \frac{1}{2}\frac{\partial^2 v_1}{\partial \zeta^2} \tag{19}$$
$$0 = -\frac{\partial p_1}{\partial \zeta} \tag{20}$$
$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial W_1}{\partial \zeta} = 0 \tag{21}$$

As in the linear problem, we can define an auxiliary variable $\Lambda_1 = u_1 + iv_1$, and express the $x-$ and $y-$ momentum equations in terms of it:

$$\begin{aligned} \frac{\partial^2 \Lambda_1}{\partial \zeta^2} - 2i\Lambda_1 &= 2\Big[\frac{\partial p_1}{\partial x} + i\frac{\partial p_1}{\partial y} + u_0\frac{\partial u_0}{\partial x} + v_0\frac{\partial u_0}{\partial y} + W_0\frac{\partial u_0}{\partial \zeta} + \\ &\quad iu_0\frac{\partial v_0}{\partial x} + iv_0\frac{\partial v_0}{\partial y} + iW_0\frac{\partial v_0}{\partial \zeta}\Big] \end{aligned} \tag{22}$$

Advection by the first–order velocity field becomes the forcing we apply to the order ϵ velocities. Note that we include the pressure p_1 with the forcing terms because it does not induce any cross-isobar flow, and so will not induce any change in the Ekman pumping or boundary layer structure. We assume that they cancel with the advection terms that do not decay to zero far from the boundary. After much long and tedious algebra, we find that these forcing terms can be expressed:

$$\frac{\partial^2 \Lambda_1}{\partial \zeta^2} - 2i\Lambda_1 = ae^{-\zeta(1-i)} + be^{-\zeta(1+i)} + ce^{-2\zeta} \tag{23}$$

where

$$a = i\lambda\omega - 2U\frac{\partial \lambda}{\partial x} - 2V\frac{\partial \lambda}{\partial y} \tag{24}$$

$$b = \lambda\omega - 2U\frac{\partial \lambda}{\partial x} - 2V\frac{\partial \lambda}{\partial y} \tag{25}$$

$$c = -\lambda\omega + (U - iV)\left(\frac{\partial \lambda}{\partial x} + i\frac{\partial \lambda}{\partial y}\right) \tag{26}$$

Solving this differential equation, and once again applying the no–slip boundary condition $\Lambda_0 = 0$ at $\zeta = 0$, we find a solution:

$$\Lambda_1 = \frac{i}{4}ae^{-\zeta(1-i)} - \underbrace{\frac{b}{2(1+i)}\zeta e^{-\zeta(1+i)}} + \frac{c}{4-2i}e^{-2\zeta} - \left(\frac{i}{4}a + \frac{c}{4-2i}\right)e^{-\zeta(1+i)} \tag{27}$$

Note that the term with an underbrace is a secular term that arises because the forcing resonates with the homogeneous solution. Since it grows with ζ, if you go sufficiently far from the boundary the expansion in ϵ will become disordered and so invalid. However, we can use the approximation $e^x \approx 1 + x$ to combine it with the homogeneous solution, and interpret it as a modification to the structure of the boundary layer. While this may seem presumptuous, it yields the same boundary layer thickness as approaching this problem with a multiple–scale expansion, so it is probably true. The multiple–scale approach is shown in Section 5. With this approximation, we find for the total velocity field of the fluid to order ϵ:

$$\begin{aligned} \Lambda &= (u - U) + i(v - V) = \Lambda_0 + \epsilon\Lambda_1 + O(\epsilon^2) \\ &= -\left(\lambda + \epsilon\left(\frac{i}{4}a + \frac{c}{4-2i}\right)\right)\underbrace{\exp\left[-\zeta\left(1 + i + \epsilon\frac{b}{2\lambda(1+i)}\right)\right]} \\ &\qquad + \epsilon\frac{i}{4}ae^{-\zeta(1-i)} + \epsilon\frac{c}{4-2i}e^{-2\zeta} \end{aligned} \tag{28}$$

This is admittedly difficult to interpret. However, we can see that the thickness of the boundary layer is modified by the exponential term indicated with an underbrace. We can also use Λ and the continuity equation to calculate the Ekman pumping out of the boundary layer for an arbitrary geostrophic flow. Continuity (**??**) tell us:

$$\begin{aligned} W &= -\int\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right]d\zeta \\ &= -\frac{\partial}{\partial x}\left(\Re\left[\int \Lambda d\zeta\right]\right) - \frac{\partial}{\partial y}\left(\Im\left[\int \Lambda d\zeta\right]\right) \end{aligned}$$

We are interested in the value of W as we move far away from the boundary—the amount of fluid that is actually pumped out of the boundary layer and into the interior fluid. We can see from equation (**??**) that we will find a solution of the form:

$$\Lambda = -\left(\lambda + \epsilon(\theta + \mu)\right)e^{-\zeta(1+i+\epsilon\beta)} + \epsilon\theta e^{-\zeta(1-i)} + \epsilon\mu e^{-2\zeta} \tag{29}$$

In this, $\theta = ia/4$, $\beta = b/(2\lambda(1+i))$, and $\mu = c/(4-2i)$. This integrates very easily to give:

$$\int \Lambda d\zeta = \frac{\lambda + \epsilon(\theta+\mu)}{1+i+\epsilon\beta} e^{-\zeta(1+i+\epsilon\beta)} - \epsilon\frac{\theta(1+i)}{2} e^{-\zeta(1-i)} - \epsilon\frac{\mu}{2}e^{-2\zeta} + C(x,y) \tag{30}$$

We find the constant of integration $C(x,y)$ by applying the boundary condition $W(\zeta = 0) = 0$. Since all of the terms in Λ decay with increasing ζ, $W(\zeta \to \infty) = C(x,y)$. Evaluating this by hand is not anyone's idea of fun, so I use a computer algebra program, Maple 11, to find:

$$\begin{aligned} W_\infty \;=\; & \frac{1}{2}\omega - \epsilon\frac{7}{40}\left(\left(\frac{\partial U}{\partial y}\right)^2 + U\frac{\partial^2 U}{\partial y^2}\right) + \epsilon\frac{1}{10}\Big(U\frac{\partial^2 V}{\partial y^2} - \frac{\partial U}{\partial x}\frac{\partial V}{\partial y} - V\frac{\partial^2 U}{\partial x^2}\Big) + \\ & +\epsilon\frac{13}{40}\Big(V\frac{\partial^2 U}{\partial y^2} - U\frac{\partial^2 U}{\partial x\partial y} + V\frac{\partial^2 V}{\partial x\partial y} + \frac{\partial U}{\partial x}\frac{\partial V}{\partial x} + \frac{\partial V}{\partial x}\frac{\partial V}{\partial y} - \frac{\partial U}{\partial x}\frac{\partial U}{\partial y} + -\frac{\partial U}{\partial y}\frac{\partial V}{\partial y}\Big) \\ & -\epsilon\frac{9}{20}\frac{\partial V}{\partial x}\frac{\partial U}{\partial y} + \epsilon\frac{17}{40}\Big(U\frac{\partial^2 V}{\partial x^2} - \left(\frac{\partial V}{\partial x}\right)^2 - V\frac{\partial^2 V}{\partial x^2}\Big) \\ & -\epsilon\frac{7}{10}\Big(\left(\frac{\partial U}{\partial x}\right)^2 + U\frac{\partial^2 U}{\partial x^2} + \left(\frac{\partial V}{\partial y}\right)^2 + V\frac{\partial^2 V}{\partial Y^2}\Big) \\ & -\epsilon\frac{11}{40}\Big(U\frac{\partial^2 V}{\partial x\partial y} + V\frac{\partial^2 U}{\partial x\partial y}\Big) \end{aligned} \tag{31}$$

This is the weakly non-linear solution for the Ekman pumping induced by a no–slip boundary in a rotating frame. As with many of the things I have presented in this essay, it is rather difficult to interpret. We notice that that there is no direct dependence on vorticity advection, though all the terms of the vorticity advection enter the above expression. Therefore, we will now turn to a simplified example to build intuition about the effects of nonlinearity on the Ekman layer.

4 An Illustrative Example

We can gain great insight into the effect of advection on the boundary layer by exploring the simplest test flow in which advection is present:

$$\begin{aligned} U &= U(y) \\ V &= \text{constant} \end{aligned}$$

This is the unidirectional flow examined by Hart, Pedlosky, and others, modified by a constant cross–stream velocity. The thickness of the boundary layer is controlled by the decaying exponential in Λ. It is the real part of the exponential designated by an underbrace in equation (**??**). It is:

$$\Lambda \propto \exp\left[-\zeta + \epsilon\zeta\frac{1}{4}\omega\left(1 - \frac{2V^2}{U^2+V^2}\right)\right] \tag{32}$$

In this flow, the relative vorticity $\omega = -\frac{\partial U}{\partial y}$. If the cross–flow velocity V goes to zero, this reduces to the solution for a unidirectional flow derived by Hart [4] and by Pedlosky [5] in the invited lectures of this year's summer school. That is, we find that the thickness of the Ekman layer is modified proportionally to the vorticity of the flow: where the vorticity is positive the Ekman layer is thicker, and where the vorticity is negative the Ekman layer is thinner. If, on the other hand, the cross–flow velocity V is very large, the thickness of the Ekman layer is modified in the opposite direction, thicker where the vorticity is negative and thinner where the vorticity is positive.

For this flow field, we find a solution of the form given in (**??**), with:

$$\beta \;=\; -\frac{1}{4\lambda}\left((i-1)U\frac{\partial U}{\partial y} + (i-3)V\frac{\partial U}{\partial y}\right) \tag{33}$$

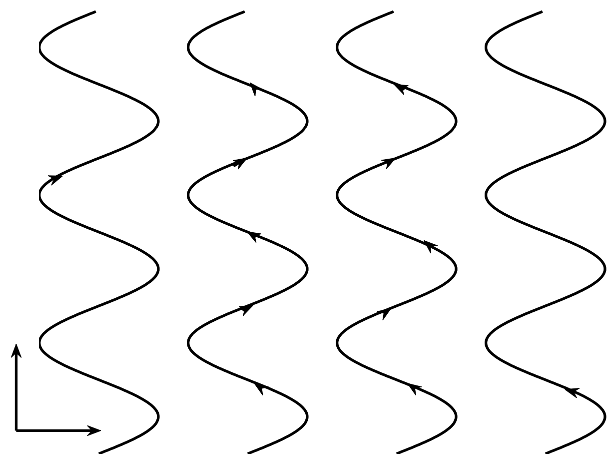

Figure 1: Streamlines for the Illustrative Example. Plan View.

$$\theta = \frac{1}{4}\left(U\frac{\partial U}{\partial y} - iV\frac{\partial U}{\partial y}\right) \tag{34}$$

$$\mu = \frac{3i+1}{10}\left(U\frac{\partial U}{\partial y} + V\frac{\partial U}{\partial y}\right) \tag{35}$$

Integrating, and taking the limit as $\zeta \to \infty$, we find

$$W_\infty = -\frac{1}{2}\frac{\partial U}{\partial y} - \epsilon\frac{7}{40}\left(\left(\frac{\partial U}{\partial y}\right)^2 + U\frac{\partial^2 U}{\partial y^2}\right) + \epsilon\frac{13}{40}\underbrace{V\frac{\partial^2 U}{\partial y^2}} \tag{36}$$

The final term, indicated with an underbrace, looks like the advection of vorticity. However, it is unclear if that is the appropriate physical interpretation because the general solution given in equation (**??**) is not directly proportional to vorticity advection.

To better understand the effects we've derived, let's examine a specific flow:

$$U = \cos ky$$
$$V = \text{constant}$$

The streamlines of this flow are shown in Figure **??**.

This flow makes an ideal test case because it is not computationally demanding, but has the physical characteristics we're interested in investigating: regions of positive and negative vorticity and fluid advected between them. The boundary layer thickness is given by:

$$\Lambda \propto \exp\left[-\zeta + \epsilon\zeta\frac{1}{4}k\sin ky\left(1 - \frac{2V^2}{\cos^2 ky + V^2}\right)\right]$$

This is shown in Figure **??** for three cases: no advection, weak advection, and strong advection. The Ekman pumping is:

$$W_\infty = \frac{k}{2}\sin ky - \epsilon\frac{7}{40}k^2 - \epsilon\frac{13}{40}Vk^2\cos ky$$

This solution is shown in Figure **??** for a variety of cross-flow velocities, V. Increasing the cross-flow velocity shifts the phase of the Ekman pumping from being in phase with the vorticity of the flow to being π out of phase with it. Note that the nonlinear correction to the Ekman pumping acts asymmetrically. It is somewhat intensified in regions of positive vorticity and weakened in regions negative vorticity because the vertical velocity induced by vorticity reinforce the Ekman pumping in regions of positive vorticity and suppress

Ekman pumping in regions of negative vorticity. The weakly nonlinear correction is also proportional to the curvature of the sinusoidal flow—as k increases, that is as the length scale of the oscillations decrease, the fluctuations in vorticity become more pronounced and the Ekman puming stronger.

5 Multiple Scale Expansion

Another way to derive a weakly nonlinear solution to the Ekman layer is by separating the vertical scales of the boundary layer, the far–field flow, and the transition region between them. As in the Rossby number expansion, we start with the boundary layer coordinate $\zeta = z/\sqrt{E}$, as defined in equation (**??**). We also define a coordinate that is of order one in the transition region between the boundary layer and the interior flow:

$$\eta = \epsilon\zeta = \frac{\epsilon}{\sqrt{E}} z$$

We assume that the scale of ζ and η are so widely separated that we can treat them as independent variables. Therefore:

$$\frac{\partial}{\partial z} = \frac{1}{\sqrt{E}}\frac{\partial}{\partial \zeta} + \frac{\epsilon}{\sqrt{E}}\frac{\partial}{\partial \eta} \qquad ; \qquad \frac{\partial^2}{\partial z^2} = \frac{1}{E}\frac{\partial^2}{\partial \zeta^2} + 2\frac{\epsilon}{E}\frac{\partial^2}{\partial \zeta \partial \eta} + \frac{\epsilon^2}{E}\frac{\partial^2}{\partial \eta^2}$$

Our equations of motion (**??**) – (**??**) then become:

$$\begin{aligned}
\epsilon\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + W\frac{\partial u}{\partial \zeta} + \epsilon W\frac{\partial u}{\partial \eta}\right) - v &= -\frac{\partial p}{\partial x} + \frac{1}{2}\left(E\frac{\partial^2 u}{\partial x^2} + E\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial \zeta^2} + 2\epsilon\frac{\partial^2 u}{\partial \zeta\partial \eta} + \epsilon^2\frac{\partial^2 u}{\partial \eta^2}\right) && (37)\\
\epsilon\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + W\frac{\partial v}{\partial \zeta} + \epsilon W\frac{\partial v}{\partial \eta}\right) + u &= -\frac{\partial p}{\partial y} + \frac{1}{2}\left(E\frac{\partial^2 v}{\partial x^2} + E\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial \zeta^2} + 2\epsilon\frac{\partial^2 v}{\partial \zeta\partial \eta} + \epsilon^2\frac{\partial^2 v}{\partial \eta^2}\right) && (38)\\
\epsilon E\left(u\frac{\partial W}{\partial x} + v\frac{\partial W}{\partial y} + W\frac{\partial W}{\partial \zeta} + \epsilon W\frac{\partial W}{\partial \eta}\right) &= -\frac{\partial p}{\partial \zeta} + \frac{E}{2}\left(E\frac{\partial^2 W}{\partial x^2} + E\frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial \zeta^2} + 2\epsilon\frac{\partial^2 W}{\partial \zeta\partial \eta} + \epsilon^2\frac{\partial^2 W}{\partial \eta^2}\right)\\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial W}{\partial \zeta} + \epsilon\frac{\partial W}{\partial \eta} &= 0
\end{aligned}$$

To order 1, we recover the same linear Ekman problem we discussed previously. It's solution is found in equation (**??**). However, instead of saying that the coefficient λ varies only in x and y, we allow it to vary in the transitional vertical coordinate η, as well. This coordinate is so dilated with respect to the boundary layer that variations in η seem constant within the boundary layer. If we let the coefficient $\lambda = A + iB$, the no slip boundary condition gives us $A(\zeta = 0) = -U$ and $B(\zeta = 0) = -V$. The Ekman pumping becomes:

$$\begin{aligned}
W_0 &= C(x,y,\eta) + \frac{1}{2}\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} + \frac{\partial B}{\partial y} + \frac{\partial A}{\partial x}\right)e^{-\zeta}\cos\zeta\\
&\quad + \frac{1}{2}\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} - \frac{\partial B}{\partial y} - \frac{\partial A}{\partial x}\right)e^{-\zeta}\sin\zeta && (39)
\end{aligned}$$

Since A and B now are considered to vary vertically, we can not assume that $\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} = 0$. To determine A and B we must solve the next order problem, which as governing equations:

$$\begin{aligned}
\left(u_0\frac{\partial u_0}{\partial x} + v_0\frac{\partial u_0}{\partial y} + W_0\frac{\partial u_0}{\partial \zeta}\right) - v_1 &= -\frac{\partial p_1}{\partial x} + \frac{1}{2}\left(\frac{\partial^2 u_1}{\partial \zeta^2} + 2\frac{\partial^2 u_0}{\partial \zeta\partial \eta}\right)\\
\left(u_0\frac{\partial v_0}{\partial x} + v_0\frac{\partial v_0}{\partial y} + W_0\frac{\partial v_0}{\partial \zeta}\right) + u_1 &= -\frac{\partial p_1}{\partial y} + \frac{1}{2}\left(\frac{\partial^2 v_1}{\partial \zeta^2} + 2\frac{\partial^2 v_0}{\partial \zeta\partial \eta}\right)\\
0 &= -\frac{\partial p_1}{\partial \zeta} - \frac{\partial p_0}{\partial \eta}\\
\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial W_1}{\partial \zeta} + \frac{\partial W_0}{\partial \eta} &= 0
\end{aligned}$$

We can again combine the $x-$ and y–momentum equations to find an inhomogeneous differential equation for $\Lambda_1 = u_1 + iv_1$, analogous to equation (**??**):

$$\begin{aligned}\frac{\partial^2 \Lambda_1}{\partial \zeta^2} - 2i\Lambda_1 &= 2\Big[\frac{\partial p_1}{\partial x} + i\frac{\partial p_1}{\partial y} + u_0\frac{\partial u_0}{\partial x} + v_0\frac{\partial u_0}{\partial y} + W_0\frac{\partial u_0}{\partial \zeta} - \frac{\partial^2 u_0}{\partial \zeta \partial \eta} + \\ &\quad + iu_0\frac{\partial v_0}{\partial x} + iv_0\frac{\partial v_0}{\partial y} + iW_0\frac{\partial v_0}{\partial \zeta} - \frac{\partial^2 v_0}{\partial \zeta \partial \eta}\Big]\end{aligned} \tag{40}$$

As in the single–scale expansion, we find that some of the forcing terms are of the same form as the homogeneous solution, $\propto \exp[-\zeta(1+i)]$. These forcing terms are dangerous, because they yield solutions proportional to $\zeta \exp[-\zeta(1+i)]$, giving terms in the final velocity like $(1+\epsilon\zeta)\exp[\zeta(1+i)]$. When $\eta = \epsilon\zeta$ is of order one, our expansion becomes disordered, that is terms of order ϵ are not smaller than those of order 1. In order to avoid this, we insist that all forcing terms of this form, called secular terms, sum to zero. This is the condition for the existence of a well–behaved expansion in Rossby number, so it is called the solvability condition. It is:

$$\frac{\partial \lambda}{\partial \eta} + \frac{1}{1+i}\left(U\frac{\partial \lambda}{\partial x} + V\frac{\partial \lambda}{\partial y}\right) - \lambda\left(C(x,y,\eta) + \frac{\omega}{2i(1+i)}\right) = 0 \tag{41}$$

This partial differential equation can be converted to a group of ordinary differential equations using the method of characteristics. We define a characteristic parameter s such that:

$$\frac{\partial x}{\partial s} = U \tag{42}$$

$$\frac{\partial y}{\partial s} = V \tag{43}$$

$$\frac{\partial \eta}{\partial s} = 1+i \tag{44}$$

Our solvability condition (**??**) now reduces to:

$$\frac{\mathrm{d}\lambda}{\mathrm{d}s} - \lambda\left((1+i)C(x,y,\eta) - i\frac{1}{2}\omega\right) = 0 \tag{45}$$

It is straightforward to integrate an equation of this form and find λ, but in order to do so we need an expression for $C(x,y,\eta)$. We find it using a vorticity equation formed from the $x-$ and y–momentum equations (**??**) and (**??**). Far from the boundary, where things are no longer changing over scales of ζ, the vorticity equation to order ϵ is:

$$u_0\frac{\partial \omega_0}{\partial x} + v_0\frac{\partial \omega_0}{\partial y} - \frac{\partial W_0}{\partial \eta} = 0$$

Far from the boundary, $u_0 = U$ and $v_0 = V$, so we integrate to find that:

$$W_0\Big|_{\zeta\to\infty} = C(x,y,\eta) = \left(U\frac{\partial \omega}{\partial x} + V\frac{\partial \omega}{\partial y}\right)\eta + C'(x,y) \tag{46}$$

The Ekman pumping to order ϵ should be proportional to the *vorticity advection* in the interior. We can use the no normal flow boundary condition at the bottom to find $C'(x,y) = -\frac{1}{2}\omega$. Unfortunately, this is as far as it is possible to go in the general case. In order to solve the equation for s (**??**), we need to know the specific geostrophic flow U and V. Let's consider the same flow that we discussed in Section 4, as an illuminating example: $U = \cos ky$ and $V = \text{constant}$. This flow is relatively simple to evaluate, but still has the crucial characteristic that it has varying vorticity and fluid being advected from regions of one vorticity to regions with a different local vorticity. Equations (**??**) and (**??**) give $y = Vx + y_0$ and $\eta = (1+i)s$.

Putting these into $C(x, y, \eta)$ from equation (**??**), we can solve equation (**??**) by direct integration in s. One finds eventually that:

$$\lambda = \lambda_0 \exp\Big[-(1+i)Uk\eta \sin ky - \frac{U}{V}(2i - \frac{1}{2})\cos ky + \\ + \frac{U}{V}(2i - \frac{1}{2})(\cos(ky - \frac{Vk\eta}{2})\cosh\frac{Vk\eta}{2} + +i\sin(ky - \frac{Vk\eta}{2})\sinh\frac{Vk\eta}{2}\Big]$$

This solution is only accurate to order ϵ, so we approximate all of the functions containing an $\eta = \epsilon\zeta$ to order ϵ, for example $\sinh x \approx x$. The details of the flow field calculated are less important that the induced variations in the thickness in the boundary layer. To find that, we apply our boundary conditions at $\zeta = \eta = 0$ to find λ_0 and take the real part of the exponential above, giving:

$$\Lambda \propto \exp\left[-\zeta + \epsilon\zeta\frac{1}{4}\omega\left(1 - \frac{2V^2}{U^2+V^2}\right)\right] \tag{47}$$

This is exactly the same boundary layer thickness that we calculated using the single–scale expansion, shown in equation (**??**). The fact that one can calculate it in two different ways substantially reinforces one's confidence that it might be correct.

Unfortunately, the two–scale method proved to be both less general and more complicated than a simple single–scale expansion. However, it was useful for confirming our previous efforts. Moreover, it provided us with the prediction that the Ekman pumping should be proportional to the vorticity advection in the interior flow. Our simplified single–scale expansion suggests that this may be the case, though we would have to calculate the fully general Ekman pumping to see if that were actually true.

6 Conclusions

In this study, we found the weakly nonlinear correction to the Ekman boundary layer of an arbitrary horizontal flow over a plate. Previous studies had concentrated on unidirectional flows of the form $U = U(y)$, $V = 0$. Guided by the intuition that the Ekman pumping out of the boundary layer should be related to the advection of vorticity, a quantitiy that goes to zero in the case of unidirectional flow, we performed an expansion in Rossby number. We found that both the structure of the boundary layer and the Ekman pumping were strongly effected to order ϵ by advection. We derived a general expression for the weakly nonlinear Ekman pumping for an arbitrary far–field flow, and studied a simple example in detail. In our example, both the Ekman pumping and the modification to the thickness of the boundary layer were shifted in phase by the introduction of vorticity.

There are a number of interesting applications for continued research in this area. Repeating these calculations for a stress boundary condition, analgous to the surface of the ocean, should be straight–forward mathematically. Once that is done, there are a number of high–resolution data sets already collected that might allow us to see if the non–linear corrections derived here are important in oceanographic contexts. In areas of very high wind stress, such as storms, or in areas where there is a strong and narrow current like the Gulf Stream, these corrections may prove to be significant.

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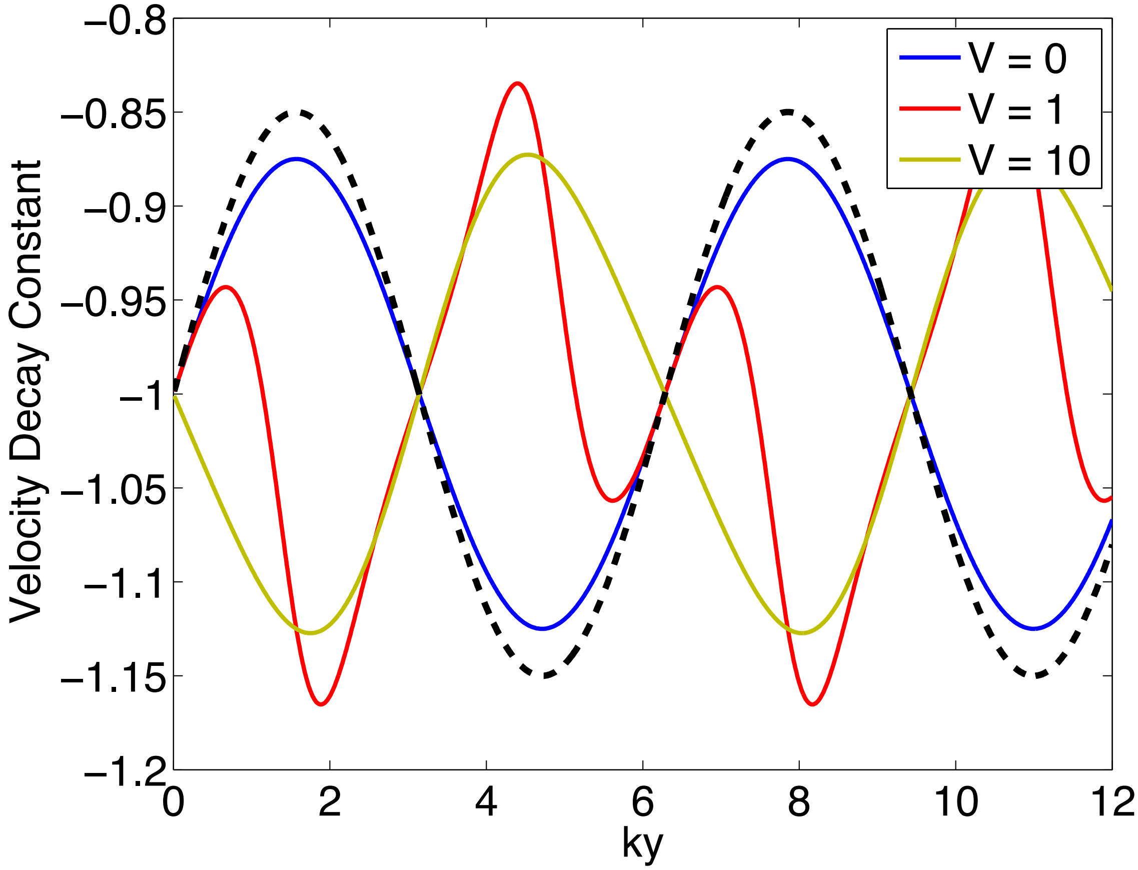

Figure 2: Modification of the Boundary Layer Thickness for varying advection strength. The base flow in this figure is $U = \cos ky$, $V =$ constant. The arrow shows the transition of the traces as V increases from 0 to 100 on a near–logarithmic scale. Notice how for low cross–stream velocities, shown in blue, the modification boundary layer thickness is proportional to the vorticity, shown in the dashed line. For high cross–stream velocities, the modification of the boundary layer thickness is opposite to the vorticity.

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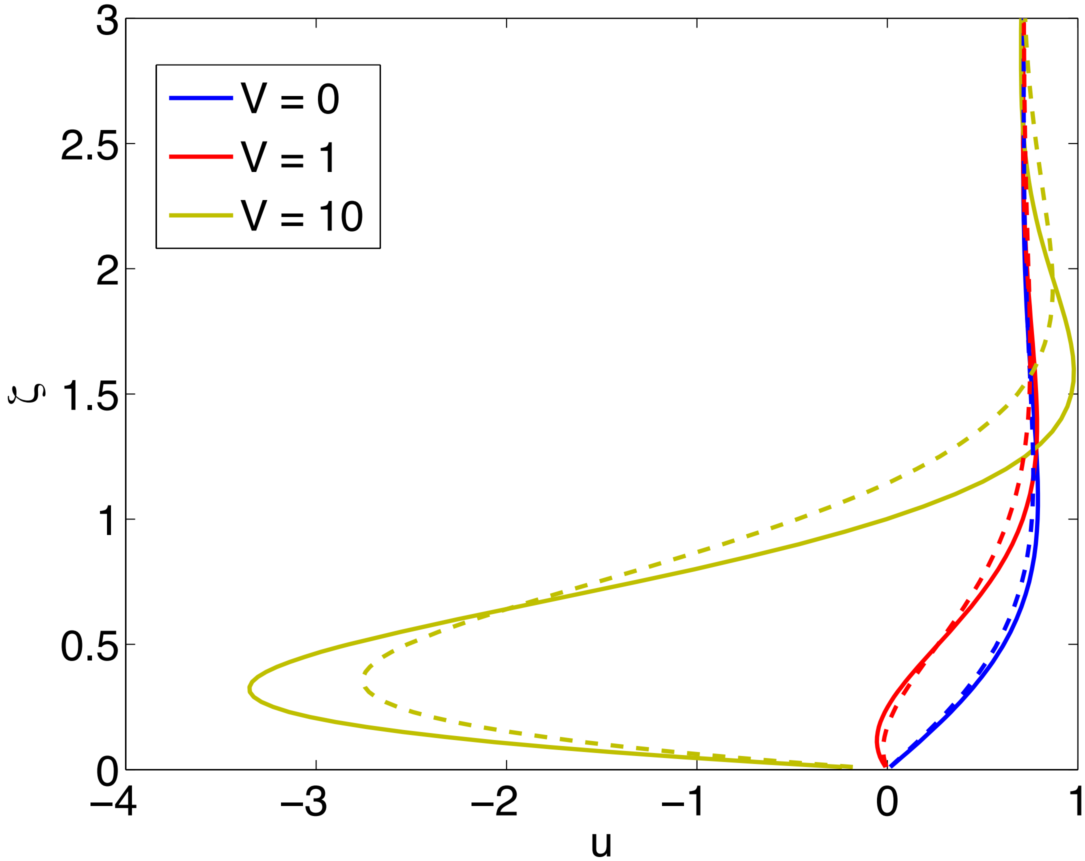

Figure 3: x–velocity in the boundary layer for varying strengths of advection for $ky = \frac{\pi}{4}$. Dashed lines indicate the linear Ekman solution, and solid lines indicate weakly nonlinear solution. All of the velocities have been rescaled by the maximum far–field velocity. Note that the boundary layer may be either thickened or thinned, depending on the strength of the advection.

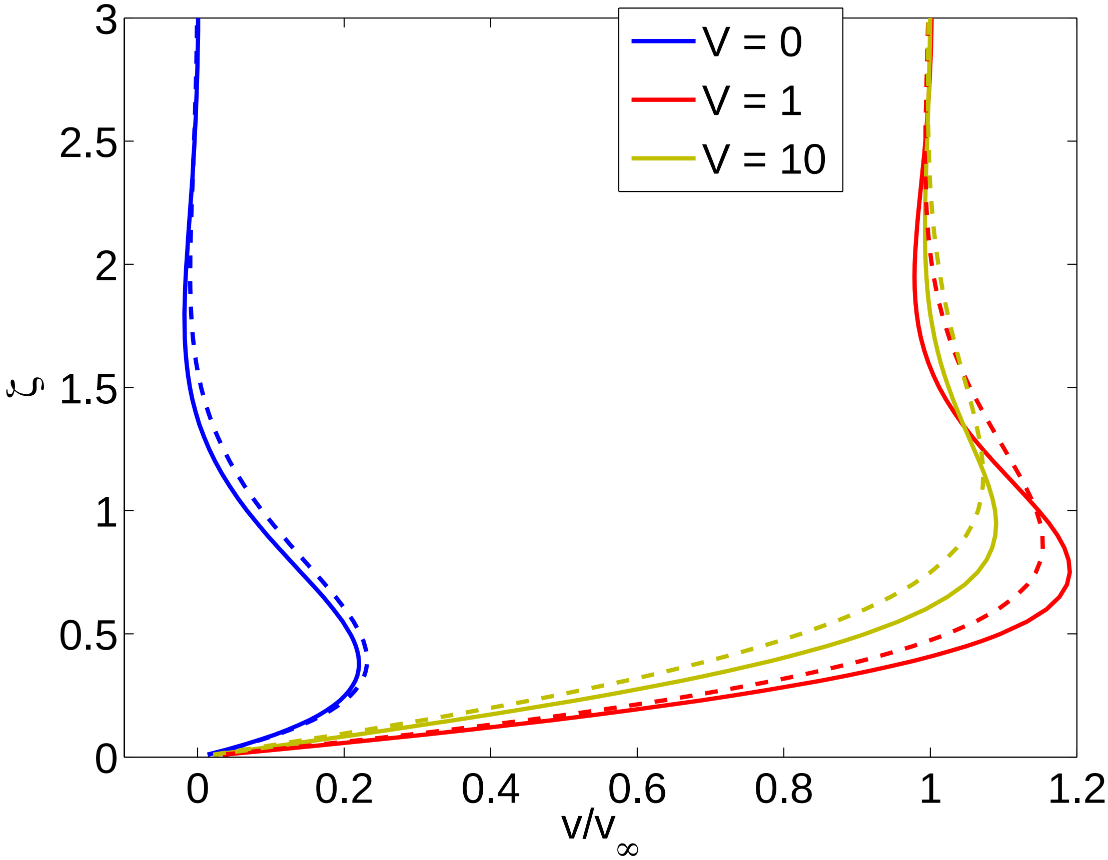

Figure 4: Advection velocity in the boundary layer for varying strengths of advection for $ky = \frac{\pi}{4}$. As in Figure **??**, dashed lines give the linear solution and solid lines give the weakly non-linear solution. For the strong advection case of $V = 10$, the velocities have been rescaled by V.

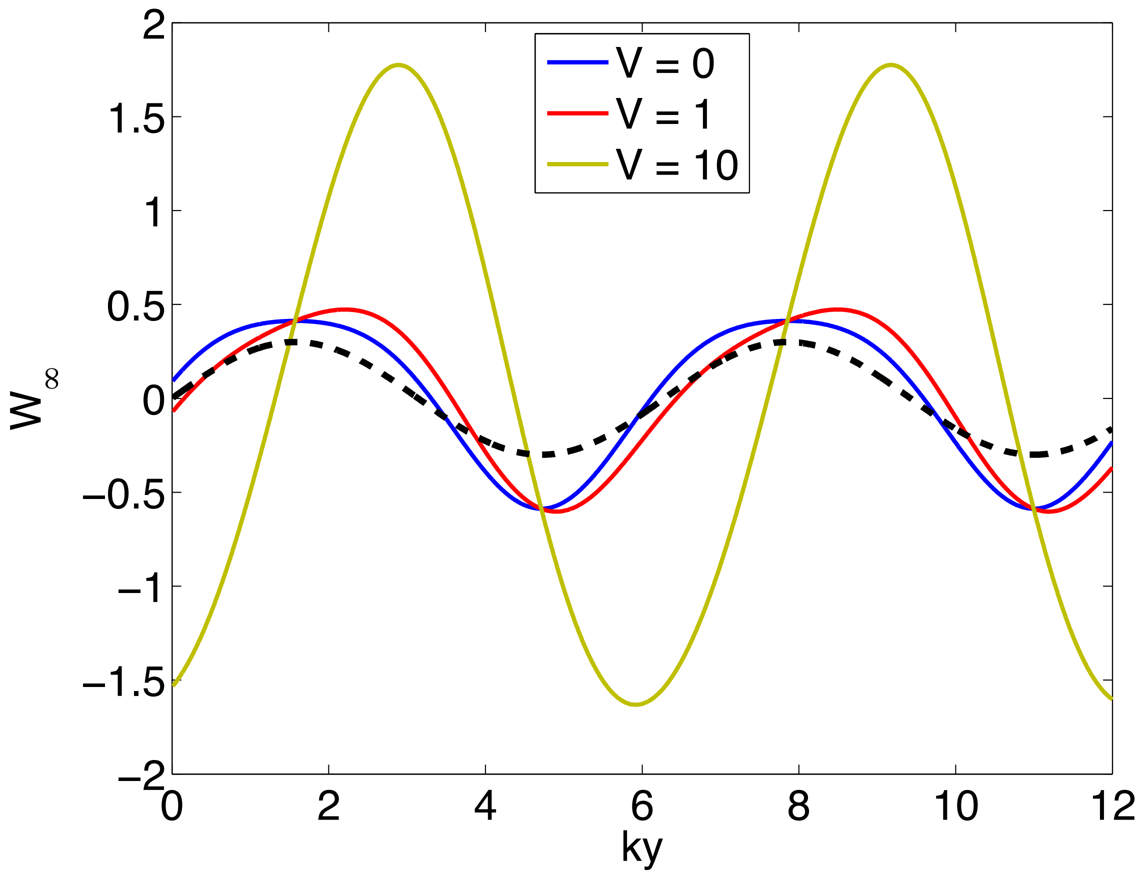

Figure 5: Ekman pumping. The dashed black line gives the vorticity of the far–field flow. The introduction of vorticity advection shifts the phase of the Ekman pumping by $\pi/2$. Note also that there is a weak asymmetry in the Ekman pumping between areas of positive and negative vorticity.