Scattering past a cylinder with weak circulation

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1 Introduction

Wave phenomena arise in a wide variety of geophysical problems. Indeed, in this year's principal lectures a main focus was the modelling of waves in the ocean. It was in this context that ray tracing and the geometrical theory of diffraction were introduced.

An important distinguishing feature of waves in the atmosphere and the ocean is that they propagate through a fluid, and that fluid is often already in motion. Familiar examples include the propagation of acoustic waves in the atmosphere in the presence of winds, or gravity waves in the ocean in the presence of currents. Ray tracing has routinely been used to solve such problems, and there is a large amount of current research devoted to understanding these wave-mean interactions.

Diffraction is the apparent bending and spreading of waves when they meet an obstruction. It is a phenomenon not described by ordinary geometric optics. However, an extension to ray tracing called the geometrical theory of diffraction (GTD) can overcome this problem. On the whole GTD has been little applied to wave-mean problems, and the focus of this project is to understand how GTD can be used in the presence of a mean flow.

We consider a new twist on the canonical problem of scattering of a plane wave past a circular cylinder. Scattering past a cylinder is a classical problem with a long history. A good introduction to the ideas behind this work can be found in [5] and in particular the application of GTD to the circular cylinder can be found in [7]. Special functions abound in scattering problems, and [1] is an invaluable source for looking up their properties.

Our problem considers the addition of a weak circulation around the cylinder, which could be motivated by the problem of modelling weak currents around an island. We emphasise here that the circulation is weak as this simplifies matters considerably [3].

The geometry is shown in Figure 1. We let the radius of the circular cylinder be a and take coordinates centred on the cylinder. We take the plane wave to be incoming from $+\infty$ on the x-axis. The cylinder is taken to be impermeable.

2 The governing equations

Following [2], we set up our governing equations as those of 2-D compressible gas dynamics, which have as a special case the familiar shallow water equations. The continuity equation

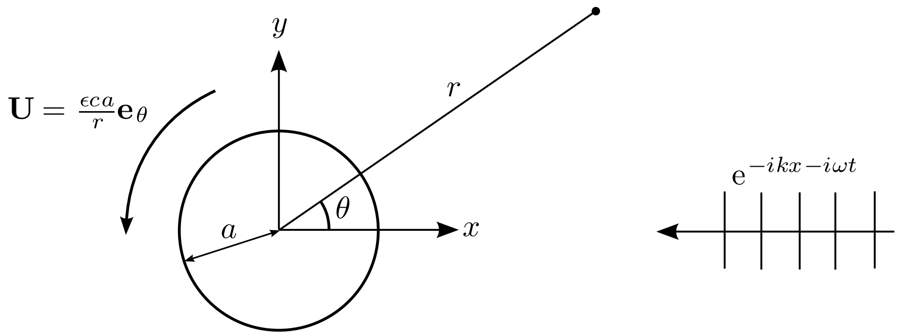

Figure 1: The geometry of the problem. An incoming plane wave is incident on a cylinder with circulation.

is

$$\frac{\mathrm{D}h}{\mathrm{D}t} + h\nabla\cdot\mathbf{u} = 0, \tag{1}$$

and the momentum equation is

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \frac{c_0^2}{\gamma-1}\nabla\left(h^{\gamma-1}\right) = 0. \tag{2}$$

Here $\mathbf{u}$ is the two dimensional velocity vector of the fluid, and h is the density of the gas, or the height of the free surface in the case of shallow water. For gas dynamics γ is the polytropic exponent and $c = c_0\sqrt{H^{\gamma-1}}$ is the undisturbed sound speed for a gas of uniform density H. For shallow water $\gamma = 2$, $c_0^2 = g$ the acceleration due to gravity, and the undisturbed gravity wave speed for a layer of uniform depth H is $c = \sqrt{gH}$. The corresponding equation of state is

$$p \equiv \frac{c_0^2}{\gamma}h^\gamma, \tag{3}$$

where the additive constant has been neglected. The momentum equation can then be written in momentum flux form as

$$\frac{\partial\left(h\mathbf{u}\right)}{\partial t} + \nabla\cdot\left(h\mathbf{u}\mathbf{u}\right) + \nabla p = 0. \tag{4}$$

We will assume that our flow is irrotational, $\nabla\times\mathbf{u} = 0$. This implies we can write $\mathbf{u}$ in terms of a velocity potential, $\mathbf{u} = \nabla\phi$. The momentum equation can then be integrated to give Bernoulli's equation

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + \frac{c_0^2h^{\gamma-1}}{\gamma-1} = G(t), \tag{5}$$

where $G(t)$ is an arbitrary function of time alone. Note that Bernoulli's equation determines h as a function of the velocity potential ϕ.

3 Small amplitude waves

3.1 Time averaged equations

When studying wave phenomena it is often useful to decompose fields into a time averaged mean part and a disturbance part, namely $\phi = \overline{\phi} + \phi'$, where $\overline{(\phi')} = 0$. Averaging (1), the averaged continuity equation is

$$\nabla \cdot \left(\overline{h}\overline{\mathbf{u}} + \overline{h'\mathbf{u}'} \right) = 0, \tag{6}$$

and averaging (5) the averaged Bernoulli equation is

$$c_0^2 \frac{\overline{h^{\gamma-1}}}{\gamma - 1} = \text{constant} - \frac{1}{2} \left(|\overline{\mathbf{u}}|^2 + \overline{|\mathbf{u}'|^2} \right). \tag{7}$$

We perform a standard perturbation analysis of the governing equations in terms of a small non-dimensional wave amplitude parameter η. We will assume the $O(1)$ flow has no disturbance part, and that the $O(\eta)$ flow has no mean part. For the mean flow we write

$$\overline{\phi} = \Phi + 0 + \eta^2 \overline{\phi_2} + \dots \,, \tag{8}$$

$$\overline{h} = H + 0 + \eta^2 \overline{h_2} + \dots \,, \tag{9}$$

$$\overline{\mathbf{u}} = \mathbf{U} + 0 + \eta^2 \overline{\mathbf{u}_2} + \dots \,. \tag{10}$$

where capital letters are used to denote the $O(1)$ flow. For the disturbance we write

$$\phi' = 0 + \eta \phi_1' + \eta^2 \phi_2' + \dots \,, \tag{11}$$

$$h' = 0 + \eta h_1' + \eta^2 h_2' + \dots \,, \tag{12}$$

$$\mathbf{u}' = 0 + \eta \mathbf{u}_1' + \eta^2 \mathbf{u}_2' + \dots \,. \tag{13}$$

3.2 Mean flow

The mean flow that we are applying is that of a line vortex:

$$\Phi = \epsilon c a \theta, \tag{14}$$

$$\mathbf{U} = \frac{\epsilon c a}{r} \mathbf{e}_\theta, \tag{15}$$

$$c_0^2 \frac{H^{\gamma-1}}{\gamma - 1} = \text{constant} - \frac{\epsilon^2 c^2 a^2}{2r^2}. \tag{16}$$

Here ϵ is a non-dimensional parameter determining the strength of the vortex. Throughout this work will neglect terms $O\left(\epsilon^2\right)$. Hence (16) becomes simply $H = \text{constant}$. We have non-dimensionalised on c the constant undisturbed wave speed, and a the radius of the cylinder. Note that the maximum mean flow occurs on the cylinder where $|\mathbf{U}| = \epsilon c$, so the non-dimensionalisation is such that ϵ is a ratio of mean flow speed to wave speed (a Froude/ Mach number). The circulation Γ associated with the line vortex is $\Gamma = 2\pi\epsilon c a$. Note that the chosen mean flow is incompressible, $\nabla \cdot \mathbf{U} = 0$.

3.3 Linear waves

The $O(\eta)$ continuity equation is found from (1) to be

$$\frac{\partial h_1'}{\partial t} + \mathbf{U} \cdot \nabla h_1' + H \nabla^2 \phi_1' = 0, \tag{17}$$

where we have used incompressibility of the mean flow, and that H is constant. The $O(\eta)$ Bernoulli equation is found from (5) to be

$$h_1' = \frac{H}{c^2} \left(-\frac{\partial \phi_1'}{\partial t} - \mathbf{U} \cdot \nabla \phi_1' \right). \tag{18}$$

Combining these equations we find

$$c^2 \nabla^2 \phi_1' - \frac{\partial^2 \phi_1'}{\partial t^2} - 2\mathbf{U} \cdot \left(\frac{\partial \nabla \phi_1'}{\partial t} \right) - \mathbf{U} \cdot \nabla \left(\mathbf{U} \cdot \nabla \phi_1' \right) = 0. \tag{19}$$

Neglecting $O(\epsilon^2)$ terms this becomes

$$c^2 \nabla^2 \phi_1' - \frac{\partial^2 \phi_1'}{\partial t^2} - 2\mathbf{U} \cdot \left(\frac{\partial \nabla \phi_1'}{\partial t} \right) = 0, \tag{20}$$

or in polar coordinates

$$c^2 \nabla^2 \phi_1' - \frac{\partial^2 \phi_1'}{\partial t^2} - 2\frac{\epsilon c a}{r^2} \frac{\partial^2 \phi_1'}{\partial \theta \partial t} = 0. \tag{21}$$

In the case of no mean flow, $\epsilon = 0$ and this reduces to the familiar wave equation.

We can define the local energy density E by

$$E = \frac{c^2 h_1'^2}{2H} + \frac{H |\mathbf{u}_1'|^2}{2}. \tag{22}$$

Using equations (17) and (18) the energy equation can be derived:

$$\frac{\partial E}{\partial t} + \nabla \cdot \left(E\mathbf{U} + c^2 h_1' \mathbf{u}_1' \right) = -H \mathbf{u}_1' \cdot \left(\frac{\nabla \mathbf{U} + \nabla \mathbf{U}^T}{2} \right) \cdot \mathbf{u}_1'. \tag{23}$$

3.4 Terms of second order in wave amplitude

On the whole we shall not be concerned with second order terms, but they are important when calculating the force on the cylinder. The quantity of interest is the time averaged pressure

$$\overline{p} = \frac{c_0^2}{\gamma} \overline{h^\gamma} = \frac{c_0^2}{\gamma} H^\gamma + \eta^2 \left(\frac{\gamma - 1}{2} c_0^2 H^{\gamma-2} \overline{h_1'^2} + c_0^2 H^{\gamma-1} \overline{h_2} \right) + O(\eta^3). \tag{24}$$

The time averaged Bernoulli equation (7) implies that to second order we have

$$\frac{\gamma - 2}{2} c_0^2 H^{\gamma-3} \overline{h_1'^2} + c_0^2 H^{\gamma-2} \overline{h_2} = \text{constant} - \mathbf{U} \cdot \overline{\mathbf{u}_2} - \frac{1}{2} \overline{|\mathbf{u}_1'|^2}. \tag{25}$$

Hence we can rewrite the averaged pressure as

$$\overline{p} = \text{constant} + \eta^2 \left(\frac{c^2}{2H}\overline{h_1'^2} - \frac{H}{2}\overline{|\mathbf{u}_1'|^2} - H\mathbf{U}\cdot\overline{\mathbf{u}_2} \right) + O(\eta^3). \tag{26}$$

Note that without a mean flow (26) gives the $O(\eta^2)$ pressure purely in terms of first order quantities. With a mean flow, $\overline{\mathbf{u}_2}$ (a second order term) must also be specified to calculate the pressure. To $O(\epsilon)$ we have from (18) that

$$h_1'^2 = \frac{H^2}{c^4}\left(\left(\frac{\partial \phi_1'}{\partial t}\right)^2 + 2\frac{\partial \phi_1'}{\partial t}\mathbf{U}\cdot\nabla\phi_1' \right). \tag{27}$$

Hence the time averaged pressure can be written in terms of the velocity potentials as

$$\overline{p} = \text{constant} + \eta^2 H \left(\frac{1}{2c^2}\overline{\left(\frac{\partial \phi_1'}{\partial t}\right)^2} - \frac{1}{2}\overline{|\nabla\phi_1'|^2} + \frac{1}{c^2}\overline{\frac{\partial \phi_1'}{\partial t}\mathbf{U}\cdot\nabla\phi_1'} - \mathbf{U}\cdot\nabla\overline{\phi_2} \right) + O(\eta^3). \tag{28}$$

The second order term we are interested in is $\overline{\phi_2}$, and since its term in the above expression is multiplied by $\mathbf{U}$ we may neglect $O(\epsilon)$ terms in its solution. From the time averaged continuity equation (6) we find neglecting $O(\epsilon)$ terms

$$H\nabla\cdot\overline{\mathbf{u}_2} = -\nabla\cdot\left(\overline{h_1'\mathbf{u}_1'}\right). \tag{29}$$

Now by time averaging the energy equation (23) we find that $\nabla\cdot\left(\overline{h_1'\mathbf{u}_1'}\right) = O(\epsilon)$. Hence, the leading order governing equation for $\overline{\phi_2}$ is simply Laplace's equation $\nabla^2\overline{\phi_2} = 0$.

4 Eigenfunction solution

We will seek time harmonic solutions to (20) of the form $\phi_1'(\mathbf{x},t) = \psi(\mathbf{x})\mathrm{e}^{-i\omega t}$, where ω is a chosen constant angular frequency, and the real part is assumed. (20) then reduces to

$$c^2\nabla^2\psi + \omega^2\psi + 2i\omega\mathbf{U}\cdot\nabla\psi = 0. \tag{30}$$

Let $k_\infty = \omega/c$, the constant wavenumber at infinity where the mean flow is absent. Then this can be written as

$$\nabla^2\psi + k_\infty^2\psi + 2ik_\infty\frac{\mathbf{U}}{c}\cdot\nabla\psi = 0, \tag{31}$$

or in polar coordinates as

$$\nabla^2\psi + k_\infty^2\psi + \frac{2i\epsilon k_\infty a}{r^2}\frac{\partial\psi}{\partial\theta} = 0. \tag{32}$$

which in the case of no mean flow is the familiar Helmholtz equation.

(32) can be solved by separation of variables. Let $\psi(\mathbf{x}) = R(r)\Theta(\theta)$. Then

$$r^2R'' + rR' + (k_\infty^2r^2 - \lambda^2)R = 0, \tag{33}$$

$$\Theta'' + 2i\epsilon k_\infty a\Theta' + \lambda^2\Theta = 0. \tag{34}$$

where λ is a constant. These have solutions of the form

$$R(r) = H_{\lambda}^{(1,2)}(k_\infty r), \tag{35}$$

$$\Theta(\theta) = \mathrm{e}^{i(\pm\lambda - \epsilon k_\infty a)\theta}. \tag{36}$$

where $H_\nu^{(1,2)}(z)$ are Hankel functions of the first and second kinds of order ν. Since we must have a single valued function of θ, we have that $\lambda = \pm(m + \epsilon k_\infty a)$, where $m \in \mathbb{Z}$. Also, since $H_{-\nu}^{(1,2)}(z) = \mathrm{e}^{-\nu\pi i} H_\nu^{(1,2)}(z)$, the eigenfunctions of (32) are thus just $H_{\tilde{m}}^{(1,2)}(k_\infty r)\mathrm{e}^{im\theta}$, where $\tilde{m} = m + \epsilon k_\infty a$. As we are solving a self adjoint problem these eigenfunctions are orthogonal and we can express the general solution in terms of these eigenfunctions as

$$\psi(r,\theta) = \sum_{m\in\mathbb{Z}} \left(A_m H_{\tilde{m}}^{(1)}(k_\infty r) + B_m H_{\tilde{m}}^{(2)}(k_\infty r) \right) \mathrm{e}^{im\theta} \tag{37}$$

for constants A_m, B_m to be determined.

4.1 Green's function for a point source

Consider a point source at $\mathbf{x}_0 = (r_0, \theta_0)$ in polar coordinates. The governing equation for the Green's function $G(\mathbf{x}, \mathbf{x_0})$ is

$$\nabla^2 G + k_\infty^2 G + \frac{2i\epsilon k_\infty a}{r^2} \frac{\partial G}{\partial \theta} = \delta(\mathbf{x} - \mathbf{x}_0). \tag{38}$$

Using the eigenfunction expansion (37) it can be shown that

$$G(\mathbf{x}, \mathbf{x_0}) = \frac{1}{8i} \sum_{m\in\mathbb{Z}} \frac{H_{\tilde{m}}^{(2)}(k_\infty r_{\min}) H_{\tilde{m}}'^{(1)}(k_\infty a) - H_{\tilde{m}}^{(1)}(k_\infty r_{\min}) H_{\tilde{m}}'^{(2)}(k_\infty a)}{H_{\tilde{m}}'^{(1)}(k_\infty a)} H_{\tilde{m}}^{(1)}(k_\infty r_{\max}) \mathrm{e}^{im(\theta-\theta_0)}, \tag{39}$$

where $r_{\max} = \max(r, r_0)$, $r_{\min} = \min(r, r_0)$. An alternative expression for $G(\mathbf{x}, \mathbf{x_0})$, easier to compute numerically, is

$$G(\mathbf{x}, \mathbf{x_0}) = \frac{1}{4} \sum_{m\in\mathbb{Z}} \frac{J_{\tilde{m}}(k_\infty r_{\min}) Y_{\tilde{m}}'(k_\infty a) - Y_{\tilde{m}}(k_\infty r_{\min}) J_{\tilde{m}}'(k_\infty a)}{H_{\tilde{m}}'^{(1)}(k_\infty a)} H_{\tilde{m}}^{(1)}(k_\infty r_{\max}) \mathrm{e}^{im(\theta-\theta_0)}, \tag{40}$$

where $J_\nu(z)$ and $Y_\nu(z)$ are Bessel functions of first and second order respectively.

Note that since the problem is self-adjoint, the Green's function satisfies a reciprocity relation $G(\mathbf{x}, \mathbf{x_0}) = G^*(\mathbf{x}_0, \mathbf{x})$, where * denotes complex conjugation. Since $G^*(\mathbf{x}, \mathbf{x_0})$ satisfies

$$\nabla^2 G^* + k_\infty^2 G^* - \frac{2i\epsilon k_\infty a}{r^2} \frac{\partial G^*}{\partial \theta} = \delta(\mathbf{x} - \mathbf{x}_0), \tag{41}$$

then the reciprocity relation can be simply stated as: the field at $\mathbf{x}$ due to a point source at $\mathbf{x}_0$ is the same as the field at $\mathbf{x}_0$ due to a point source at $\mathbf{x}$ with the direction of the vortex reversed.

4.2 Eigenfunction solution for an incoming plane wave

We want to find the field due to scattering of an incoming plane wave on the cylinder. The potential ψ_i for a plane wave incident from $+\infty$ on the x-axis is

$$\psi_i = \mathrm{e}^{-ik_\infty r\cos\theta - i\epsilon k_\infty a\theta}. \tag{42}$$

Note that the above expression satisfies (32) neglecting terms of $O(\epsilon^2)$. Note also that this expression has a branch, and so θ has to be defined so that $-\pi < \theta < \pi$. Unless $\epsilon k_\infty a$ is an integer, ψ_i will be discontinuous. ψ_i can be expanded in terms of Bessel functions as

$$\psi_i = \sum_{m\in\mathbb{Z}} J_{\tilde{m}}(kr)\mathrm{e}^{-i\tilde{m}\pi/2}\mathrm{e}^{im\theta} \tag{43}$$

$$= \frac{1}{2}\sum_{m\in\mathbb{Z}} \left(H^{(1)}_{\tilde{m}}(k_\infty r) + H^{(2)}_{\tilde{m}}(k_\infty r)\right)\mathrm{e}^{-i\tilde{m}\pi/2}\mathrm{e}^{im\theta}. \tag{44}$$

To solve the problem of scattering on the cylinder by the incident wave we propose a solution of the form $\psi = \psi_i + \psi_s$, where ψ_s is an outgoing scattered wave of the form

$$\psi_s = \sum_{m\in\mathbb{Z}} A_m H^{(1)}_{\tilde{m}}(k_\infty r)\mathrm{e}^{im\theta}. \tag{45}$$

Applying the boundary condition $\frac{\partial\psi}{\partial r} = 0$ on $r = a$ yields

$$\psi_s = -\frac{1}{2}\sum_{m\in\mathbb{Z}} \frac{H'^{(1)}_{\tilde{m}}(k_\infty a) + H'^{(2)}_{\tilde{m}}(k_\infty a)}{H'^{(1)}_{\tilde{m}}(k_\infty a)} H^{(1)}_{\tilde{m}}(k_\infty r)\mathrm{e}^{-i\tilde{m}\pi/2}\mathrm{e}^{im\theta}, \tag{46}$$

$$\psi = \frac{1}{2}\sum_{m\in\mathbb{Z}} \frac{H^{(2)}_{\tilde{m}}(k_\infty r)H'^{(1)}_{\tilde{m}}(k_\infty a) - H^{(1)}_{\tilde{m}}(k_\infty r)H'^{(2)}_{\tilde{m}}(k_\infty a)}{H'^{(1)}_{\tilde{m}}(k_\infty a)}\mathrm{e}^{-i\tilde{m}\pi/2}\mathrm{e}^{im\theta}. \tag{47}$$

The expression for ψ can be rewritten as

$$\psi = i\sum_{m\in\mathbb{Z}} \frac{J_{\tilde{m}}(k_\infty r)Y'_{\tilde{m}}(k_\infty a) - Y_{\tilde{m}}(k_\infty r)J'_{\tilde{m}}(k_\infty a)}{H'^{(1)}_{\tilde{m}}(k_\infty a)}\mathrm{e}^{-i\tilde{m}\pi/2}\mathrm{e}^{im\theta} \tag{48}$$

which is easier to compute numerically.

The eigenfunction solutions are useful for plotting for moderate $k_\infty a$. However for large $k_\infty a$ a large number of modes m must be taken to provide an accurate approximation. Ray tracing overcomes this restriction by providing an asymptotic theory for large $k_\infty a$.

5 Ray Tracing

Ray tracing can be used to provide an asymptotic solution to (21) for a slowly varying wavetrain embedded in a slowly varying background environment. Let

$$\phi'_1 \sim z(\mathbf{x})\mathrm{e}^{i\Theta(\mathbf{x},t)}, \tag{49}$$

where we will suppose the phase Θ is rapidly varying, and the wave amplitude z slowly varying. The local wavenumber $\mathbf{k}$ and local frequency ω are defined by

$$\mathbf{k} = \nabla\Theta, \qquad \omega = -\frac{\partial\Theta}{\partial t}. \tag{50}$$

The standard ray tracing equations are then given in terms of the dispersion relation

$$\omega = \Omega(\mathbf{x}, \mathbf{k}) = ck + \mathbf{U}\cdot\mathbf{k}, \tag{51}$$

where $k = |\mathbf{k}|$, as Hamilton's equations

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = +\frac{\partial\Omega}{\partial\mathbf{k}}, \qquad \frac{\mathrm{d}\mathbf{k}}{\mathrm{d}t} = -\frac{\partial\Omega}{\partial\mathbf{x}}. \tag{52}$$

The ray tracing equations imply that $\mathrm{d}\omega/\mathrm{d}t = 0$, i.e. that absolute frequency is conserved along a ray, and we will consider ω a global constant along all rays. The group velocity $\mathbf{c}_g$ is given by

$$\mathbf{c}_g = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = c\hat{\mathbf{k}} + \mathbf{U}, \tag{53}$$

where $\hat{\mathbf{k}} = \mathbf{k}/k$. Another important consequence of the ray tracing approximation is the conservation of wave action. Define the intrinsic frequency by $\tilde{\omega} = ck$, the frequency of the wave in a frame moving with the fluid. Then the wave action $A = E/\tilde{\omega}$, where E is energy density defined in (22), satisfies

$$\frac{\partial A}{\partial t} + \nabla\cdot(A\mathbf{c}_g) = 0. \tag{54}$$

Note also that the energy density satisfies equipartition in the ray tracing approximation, namely

$$\frac{c^2\overline{h_1'^2}}{2H} = \frac{H\overline{|\mathbf{u}_1'|^2}}{2}. \tag{55}$$

5.1 Consequences of the weak mean flow

For an irrotational mean flow there is a curious result which states that to order ϵ the ray paths are straight [3, 6]. However, there is still refraction of the wave due to the $O(\epsilon)$ variation in $\mathbf{k}$ given by

$$\mathbf{k} = \mathbf{k}_\infty - k_\infty\frac{\mathbf{U}}{c} = \mathbf{k}_\infty - \frac{\epsilon k_\infty a}{r}\mathbf{e}_\theta, \tag{56}$$

where $\mathbf{k}_\infty$ is the wavenumber vector at infinity for the ray in question, and $k_\infty = |\mathbf{k}_\infty|$ (Figure 2).

The phase progression along the ray is given by

$$\Theta = \int \mathbf{k}\cdot\mathrm{d}\mathbf{x} = \int (\mathbf{k}_\infty - \epsilon k_\infty a\nabla\theta)\cdot\mathrm{d}\mathbf{x} \tag{57}$$

$$= \text{constant} + (\mathbf{k}_\infty\cdot\mathbf{x} - \epsilon k_\infty a\theta) \tag{58}$$

Since $\mathbf{k}_\infty$ is in the direction of the ray, this can be written as

$$\Theta = \Theta_0 + k_\infty\left(s - s_0 - \epsilon a(\theta - \theta_0)\right) \tag{59}$$

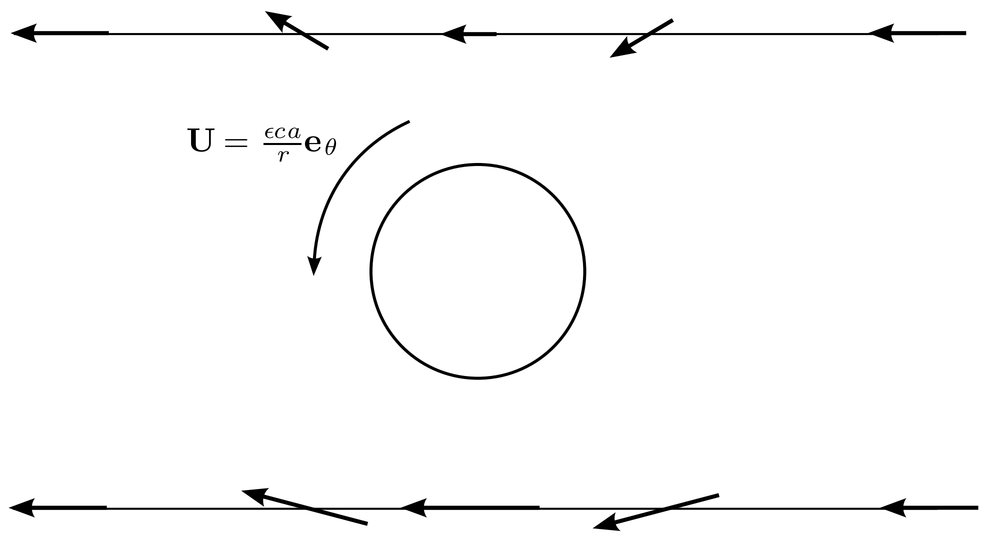

Figure 2: A cartoon of refraction of the incident wave. Two incident rays are shown, one above the cylinder and one below. The arrows along the rays indicate the direction and magnitude of the wavenumber vector at various points along the ray. The rays themselves are straight, but there is refraction from the changing wavenumber vector given by (56). Far away from the cylinder the wavenumber vector aligns with the ray direction. Note that the wavenumber becomes larger as the ray passes the cylinder for the bottom ray, but smaller for the top ray.

where $s - s_0$ is the distance travelled along the ray, and $\theta - \theta_0$ is the angular change along the ray.

The wave action is given by

$$A = \frac{E}{\tilde{\omega}} = \frac{Hkz^2}{c}. \tag{60}$$

(54) implies that $\nabla \cdot (A\mathbf{c}_g) = 0$, which to $O(\epsilon)$ implies simply

$$\nabla \cdot \left(z^2 \mathbf{k}_\infty\right) = 0. \tag{61}$$

Consider an infinitesimal ray tube R with ends E_1, E_2 orthogonal to the ray. Note that $\mathbf{k}_\infty$ is parallel to the sides of the ray tube and orthogonal to its ends. Then by applying the divergence theorem to (61)

$$0 = \int_R \nabla \cdot \left(z^2 \mathbf{k}_\infty\right)\, \mathrm{d}V = \int_{E_2} z^2 \mathbf{k}_\infty \cdot \mathbf{n}\, \mathrm{d}S - \int_{E_1} z^2 \mathbf{k}_\infty \cdot \mathbf{n}\, \mathrm{d}S. \tag{62}$$

Since $\mathbf{k}_\infty \cdot \mathbf{n}$ is a constant this leads to the simple result that $z^2 \mathrm{d}S$ is constant along a ray tube.

For a plane wave incident from $x = \infty$ it follows that the incident wave field is $\phi_i = \mathrm{e}^{-ik_\infty(x+\epsilon a\theta)}$, where we have prescribed that the incident wave has unit amplitude.

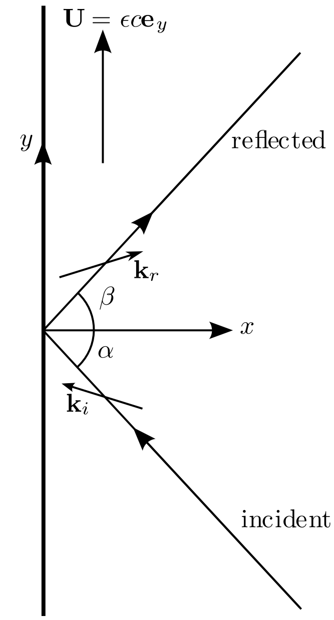

Figure 3: Close up of reflection at the cylinder, where it can be considered locally as a flat wall. Note that the wavenumber vectors (the small arrows) are not in the same direction as the rays as they hit the wall.

5.2 Reflected Wave

We now consider reflection of a ray on the cylinder. Locally we can consider the cylinder as a flat wall (Figure 3). Define new coordinates with x perpendicular to the wall and y parallel. Let the mean flow along the wall be $\mathbf{U} = (0, \epsilon c)$. Let the incident ray hit the wall at $(0, 0)$ with angle of incidence α and the reflected ray leave with angle of reflection β. Far from the wall the wavenumber vector of each ray is in the same direction as the ray. Moreover, since ω/c is constant everywhere, both incident and reflected wavenumber vectors must have same magnitude far from the wall. Hence we may write

$$\mathbf{k}^i_\infty = k_\infty \left(-\cos\alpha, \sin\alpha\right), \tag{63}$$

$$\mathbf{k}^r_\infty = k_\infty \left(\cos\beta, \sin\beta\right), \tag{64}$$

for the incident and reflected wavenumbers at infinity respectively. From (56) we see that the incident and reflected wavenumbers at the wall are given by

$$\mathbf{k}^i_0 = k_\infty \left(-\cos\alpha, \sin\alpha - \epsilon\right), \tag{65}$$

$$\mathbf{k}^r_0 = k_\infty \left(\cos\beta, \sin\beta - \epsilon\right). \tag{66}$$

Hence locally we have that

$$\psi = \psi^i + \psi^r = z_i e^{i\mathbf{k}^i_0 \cdot \mathbf{x} + \Theta^i_0} + z_r e^{i\mathbf{k}^r_0 \cdot \mathbf{x} + \Theta^r_0}. \tag{67}$$

Using the boundary condition $\frac{\partial \psi}{\partial x} = 0$ at $x = 0$ we find

$$-z_i \cos\alpha + z_r \cos\beta = 0, \tag{68}$$

$$-\Theta_0^i + \Theta_0^r = 0. \tag{69}$$

From this it follows that $z_i = z_r$ and $\alpha = \beta$. Hence the angle of incidence is equal to the angle of reflection, and the reflected wave has the same amplitude and phase as the incident wave as it leaves the wall.

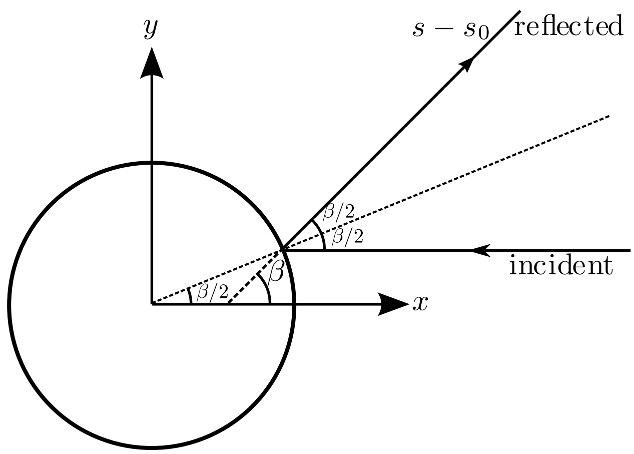

Figure 4: Reflection on the cylinder with an incident ray of angle of incidence $\beta/2$.

We now return to the global view (Figure 4). Consider a ray hitting the cylinder at an angle of incidence $\beta/2$. Then it hits the cylinder at $(x, y) = a(\cos\beta/2, \sin\beta/2)$. At that point the incident ray has phase

$$\Theta_0^i = -k_\infty(a\cos\beta/2 + \epsilon a\beta/2). \tag{70}$$

The phase progression along the reflected ray is given from (59) as

$$\Theta^r = \Theta_0^r + k_\infty\left(s - s_0 - \epsilon a(\theta - \beta/2)\right), \tag{71}$$

where s is the distance along the ray from the focus, and s_0 is the distance from the focus to the point at which the incident ray hits. The focus is the point inside the cylinder from which rays locally spread out from. Geometrically s_0 is found to be $s_0 = a/2\cos\beta/2$. Hence combining (69), (70), and (71) we find the phase progression along the reflected ray as

$$\Theta^r = k_\infty\left(s - \frac{3a}{2}\cos\beta/2 - \epsilon a\theta\right). \tag{72}$$

Rays spread out radially from the focus. $z^2 \mathrm{d}S$ = constant along a ray tube, and the incident wave has unit amplitude. Hence we have that the amplitude of the reflected ray is given by

$$z_r = \sqrt{\frac{s_0}{s}} = \sqrt{\frac{a\cos\beta/2}{2s}}. \tag{73}$$

Hence the reflected field takes the form

$$\psi_r = \sqrt{\frac{a\cos\beta/2}{2s}}\mathrm{e}^{ik_\infty(s-3a/2\cos\beta/2-\epsilon a\theta)}. \tag{74}$$

5.3 Diffracted Wave

To calculated the diffracted field we first go back to the problem of a point source rather than an incoming plane wave, as we will find diffraction coefficients by comparison with the Green's function of a point source. We apply the geometrical theory of diffraction (GTD) to the problem (Figure 5).

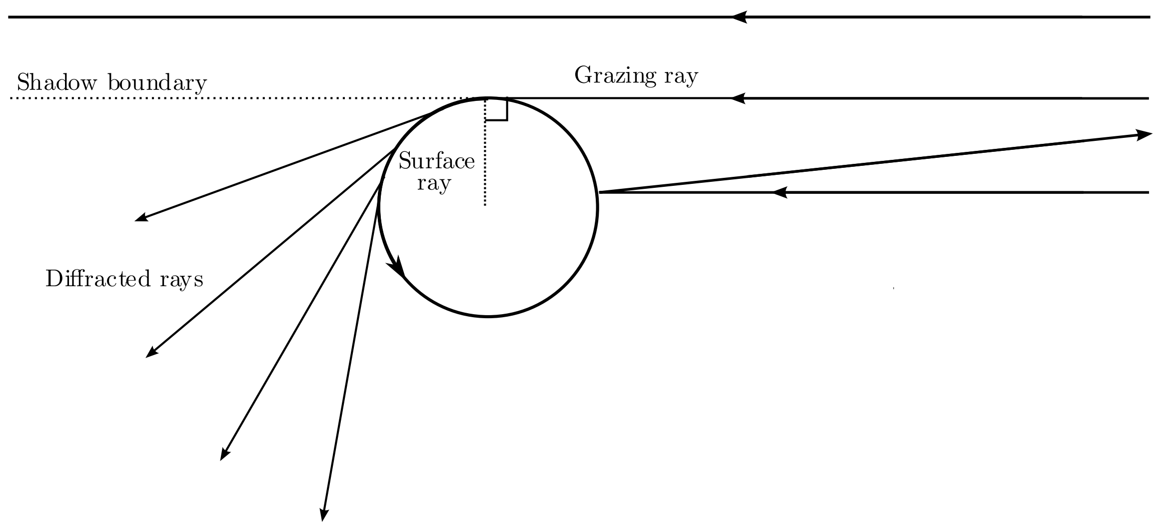

Figure 5: Cartoon of the geometrical theory of diffraction. The grazing ray hits normal to the cylinder and produces a surface ray. This surface ray travels around the cylinder constantly shedding diffracted rays normal to the cylinder.

5.3.1 Incident rays

Consider a point source located at the point $(r_0, 0)$ in Cartesian coordinates (Figure 6). Consider the two rays which leave this point and hit the cylinder at right angles. Let α be the angle between the point at which the rays hit the cylinder and the horizontal. Then the wavenumber at infinity for this ray is given by $\mathbf{k}_\infty = (-\sin\alpha, \pm\cos\alpha)$ where $+$ is the top ray and $-$ is the bottom ray. At the points at which the rays hit, $\mathbf{U} = \pm\epsilon c(-\sin\alpha, \pm\cos\alpha)$, so that the top ray hits going with the flow, and the bottom ray hits going against the flow. The wavenumber vector at the points the rays hit are then given by (56) as

$$\mathbf{k} = k_\infty(1\mp\epsilon)(-\sin\alpha, \pm\cos\alpha), \tag{75}$$

or in terms of the unit vector $\mathbf{e}_\theta$ as

$$\mathbf{k}^{\mathrm{top}} = k_\infty(1-\epsilon)\mathbf{e}_\theta, \qquad \mathbf{k}^{\mathrm{bot}} = -k_\infty(1+\epsilon)\mathbf{e}_\theta. \tag{76}$$

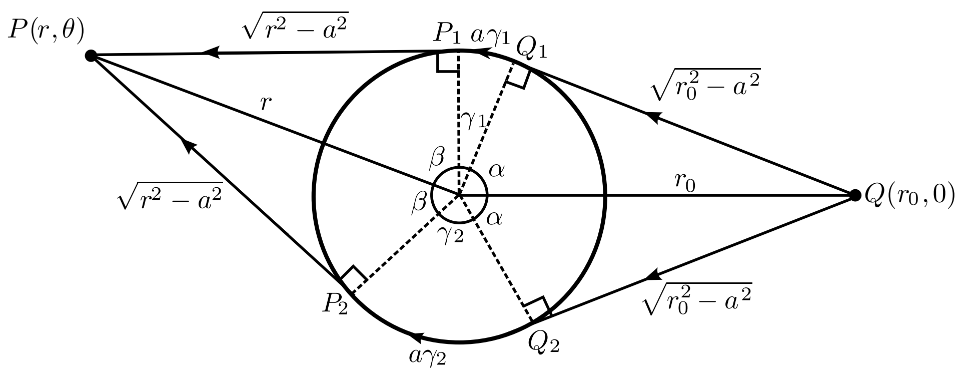

Figure 6: Geometrical theory of diffraction construction for a point source. There is a point source at Q and we are observing the field at a point P in the shadow region. Two rays paths are shown, one which involves an anticlockwise surface ray from Q_1 to P_1, and another which involves a clockwise surface ray from Q_2 to P_2.

Hence as these incident rays hit the cylinder their wavenumber vectors are tangent to the cylinder. The phase progression along these two rays is given by (59) as

$$\Theta(Q_1) = k_\infty\left(\sqrt{r_0^2 - a^2} - \epsilon a\alpha\right), \tag{77}$$

$$\Theta(Q_2) = k_\infty\left(\sqrt{r_0^2 - a^2} + \epsilon a\alpha\right). \tag{78}$$

The field due to a point source in free space with no mean flow has the form

$$\psi^i = \frac{i}{4}H_0^{(1)}(k_\infty r') \sim \frac{\mathrm{e}^{i\pi/4}\mathrm{e}^{ik_\infty r'}}{\sqrt{8\pi k_\infty r'}}, \tag{79}$$

where r' is the distance from the point source. Hence the amplitude of the rays as they hit the cylinder is given by

$$z^i(Q_1) = \frac{\mathrm{e}^{i\pi/4}}{\sqrt{8\pi k_\infty\sqrt{r_0^2 - a^2}}}. \tag{80}$$

5.3.2 Surface rays

In the geometrical theory of diffraction, a ray hitting the cylinder at right angles causes the production of a surface ray. This ray is contrained to go around the cylinder, and sheds diffracted rays tangent to the cylinder as it progresses around. On the surface ray $\mathbf{k} = \pm k\mathbf{e}_\theta$. The surface dispersion relation is then

$$\omega = \Omega_s(\mathbf{x},\mathbf{k}) = ck(1 \pm \epsilon) \tag{81}$$

which is a constant independent of position. Hence the magnitude of the wavenumber vector is a constant along the surface ray, and depends only on the direction of travel. For a surface ray travelling with the flow $\mathbf{k} = k_\infty(1-\epsilon)\mathbf{e}_\theta$, and against the flow $\mathbf{k} = -k_\infty(1+\epsilon)\mathbf{e}_\theta$. Thus the corresponding phase progression along the with flow surface ray is

$$\Theta(P_1) = \Theta(Q_1) + k_\infty(1-\epsilon)a\gamma_1, \tag{82}$$

where γ_1 is the angle travelled around the cylinder. Similarly for the against flow surface ray

$$\Theta(P_2) = \Theta(Q_2) + k_\infty(1+\epsilon)a\gamma_2, \tag{83}$$

where γ_2 is measured in the opposite direction.

Using the GTD we assume the amplitude of the surface ray is proportional to the amplitude of the incident ray that created it. Namely, that

$$z^s(Q_1) = d_1(Q_1)z^i(Q_1), \tag{84}$$

where $d_1(Q_1)$ is a diffraction coefficient depending only on the curvature of the surface. In the GTD it is proposed that the rate of decay of wave action A travelling along the ray is proportional to the wave action. Namely, that

$$\frac{\mathrm{d}A}{\mathrm{d}\sigma} = -2\alpha A, \tag{85}$$

where σ is arc length along the ray and α is a constant depending solely on the curvature of the surface. The wave action of the surface ray is proportional to the square of the amplitude and hence the surface ray decays exponentially in amplitude as

$$z^s(P_1) = \mathrm{e}^{-\alpha a\gamma} z^s(Q_1). \tag{86}$$

5.3.3 Diffracted rays

As the surface ray travels around the cylinder it sheds diffracted rays. A diffracted ray leaves tangent to the cylinder, so the phase progression along the diffracted rays is given by a similar equation to the incident rays, namely

$$\Theta(P) = \Theta(P_1) + k_\infty\left(\sqrt{r^2-a^2} - \epsilon a\beta\right), \tag{87}$$

$$\Theta(P) = \Theta(P_1) + k_\infty\left(\sqrt{r^2-a^2} + \epsilon a\beta\right), \tag{88}$$

where β is the appropriate angular progression after leaving the cylinder. We assume the amplitude of the diffracted ray is proportional to the amplitude of the surface ray which shed it. As diffracted rays leave the cylinder they spread out, and so the amplitude is inversely proportional to the square root of the distance from the cylinder. Hence

$$z^d(P) = \frac{d_2(P_1)}{(k_\infty\sqrt{r^2-a^2})^{1/2}} z^s(P_1). \tag{89}$$

where $d_2(P_1)$ is a further diffraction coefficient dependent only on curvature, and k_∞ is an appropriate non-dimensionalisation factor. By the reciprocity relation of the Green's function we must have that $d_1(Q_1) = d_2(P_1) = d$, a constant. However there is still a possibility that the diffraction coefficient d and decay coefficient α may depend on the direction of travel around the cylinder, whether going with or against flow. However, when considering an asymptotic evaluation of the Green's function later it will turn out that they do not.

5.3.4 The diffracted field

Diffracted rays are simply ordinary geometric optics rays. Combining the phase progression equations we find that for the top travelling rays the phase at P is

$$\Theta(P) = k_\infty \left(\sqrt{r_0^2 - a^2} + a\gamma_1 + \sqrt{r^2 - a^2} - \epsilon a\theta \right), \tag{90}$$

and for the bottom travelling rays the phase at P is

$$\Theta(P) = k_\infty \left(\sqrt{r_0^2 - a^2} + a\gamma_2 + \sqrt{r^2 - a^2} + \epsilon a \left(2\pi - \theta\right) \right), \tag{91}$$

where

$$\gamma_1 = \theta - \cos^{-1}\frac{a}{r_0} - \cos^{-1}\frac{a}{r}, \tag{92}$$

$$\gamma_2 = 2\pi - \theta - \cos^{-1}\frac{a}{r_0} - \cos^{-1}\frac{a}{r}. \tag{93}$$

Combining the amplitude equations we find

$$z(P) = \frac{e^{i\pi/4}\mathrm{e}^{-\alpha a\gamma} d^2}{(8\pi k_\infty^2 \sqrt{r_0^2 - a^2}\sqrt{r^2 - a^2})^{1/2}} \tag{94}$$

Hence

$$\begin{aligned}\phi_d(P) =& \frac{\mathrm{e}^{i\pi/4} d^2}{(8\pi k_\infty^2 \sqrt{r_0^2 - a^2}\sqrt{r^2 - a^2})^{1/2}} \mathrm{e}^{ik_\infty\left(\sqrt{r_0^2-a^2}+\sqrt{r^2-a^2}\right)-(ik_\infty-\alpha_j)a\left(\cos^{-1}\frac{a}{r_0}+\cos^{-1}\frac{a}{r}\right)} \\ &\times \left(\mathrm{e}^{(ik_\infty(1-\epsilon)-\alpha)a\theta} + \mathrm{e}^{(ik_\infty(1+\epsilon)-\alpha)a(2\pi-\theta)} \right). \end{aligned} \tag{95}$$

However, note also that there are also further rays due to multiple orbits of the cylinder by the surface ray. These rays just give additional factors of $2m\pi$ added to θ and $2\pi - \theta$ where $m \in \mathbb{N}$. These extra terms are easily summed as they form geometric series. Furthermore, when the ray hits it excites numerous surface rays with different α_j and d_j. This leads to the final expression

$$\begin{aligned}\phi_d(P) =& \sum_j \frac{\mathrm{e}^{i\pi/4} d_j^2}{(8\pi k_\infty^2 \sqrt{r_0^2 - a^2}\sqrt{r^2 - a^2})^{1/2}} \mathrm{e}^{ik_\infty\left(\sqrt{r_0^2-a^2}+\sqrt{r^2-a^2}\right)-(ik_\infty-\alpha_j)a\left(\cos^{-1}\frac{a}{r_0}+\cos^{-1}\frac{a}{r}\right)} \\ &\times \left(\frac{\mathrm{e}^{(ik_\infty(1-\epsilon)-\alpha_j)a\theta}}{1 - \mathrm{e}^{2\pi a(ik_\infty(1-\epsilon)-\alpha_j)}} + \frac{\mathrm{e}^{(ik_\infty(1+\epsilon)-\alpha_j)a(2\pi-\theta)}}{1 - \mathrm{e}^{2\pi a(ik_\infty(1+\epsilon)-\alpha_j)}} \right). \end{aligned} \tag{96}$$

Unfortunately, to obtain the coefficients α_j and d_j we must look back to the eigenfunction solution.

5.3.5 Asymptotics of the eigenfunction solution

We return to the Green's function solution for a point source (39). Write this as

$$G(\mathbf{x},\mathbf{x}_0) = \sum_{\nu\in\mathbb{Z}} F_{\nu+\epsilon k_\infty a} \mathrm{e}^{i\nu\theta} \tag{97}$$

where

$$F_\nu = \frac{1}{8i} \frac{H_\nu^{(2)}(k_\infty r_{\min}) H_\nu'^{(1)}(k_\infty a) - H_\nu^{(1)}(k_\infty r_{\min}) H_\nu'^{(2)}(k_\infty a)}{H_\nu'^{(1)}(k_\infty a)} H_\nu^{(1)}(k_\infty r_{\max}). \tag{98}$$

Let $F(\nu)$ be the function which gives analytic continuation of F_ν to all complex ν. Then by performing a Watson transform we may write

$$G(\mathbf{x},\mathbf{x}_0) = -\frac{i}{2} \oint_\Gamma \frac{\mathrm{e}^{i\nu(\theta-\pi)}}{\sin\nu\pi} F(\nu+\epsilon k_\infty a)\, \mathrm{d}\nu, \tag{99}$$

where Γ is a contour around the real axis. Exploiting the fact $F(\nu) = F(-\nu)$ this integral can then be rewritten as

$$G(\mathbf{x},\mathbf{x}_0) = \frac{i}{2} \int_{-\infty+i\delta}^{\infty+i\delta} \frac{\mathrm{e}^{i\nu(\theta-\pi)}}{\sin\nu\pi} F(\nu+\epsilon k_\infty a) + \frac{\mathrm{e}^{-i\nu(\theta-\pi)}}{\sin\nu\pi} F(\nu-\epsilon k_\infty a)\, \mathrm{d}\nu. \tag{100}$$

where $\delta > 0$, with the contour being just above the real axis. Consider the integral $I^\pm$ defined by

$$I^\pm = \int_{-\infty+i\delta}^{\infty+i\delta} \frac{\mathrm{e}^{\pm i\nu(\theta-\pi)}}{\sin\nu\pi} F(\nu\pm\epsilon k_\infty a)\, \mathrm{d}\nu. \tag{101}$$

We close the contour in the upper half plane. Note that this can only be done in the diffracted region. The only contribution to the integral comes from residues in the upper half plane. We get residue contributions wherever $H_{\tilde\nu}'^{(1)}(k_\infty a)$ has zeros, where $\tilde\nu = \nu \pm \epsilon k_\infty a$. For large $\tilde\nu$, $k_\infty a$ the zeros of $H_{\tilde\nu}(k_\infty a)$ are given by

$$\tilde\nu_j \sim k_\infty a - \left(\frac{k_\infty a}{2}\right)^{1/3} \mathrm{e}^{i\pi/3} q_j', \tag{102}$$

where q_j' are the roots of the derivative of the Airy function, $\mathrm{Ai}'(q_j') = 0$. Hence

$$\nu_j^\pm = \tilde\nu_j \mp \epsilon k_\infty a \sim k_\infty a(1\mp\epsilon) - \left(\frac{k_\infty a}{2}\right)^{1/3} \mathrm{e}^{i\pi/3} q_j'. \tag{103}$$

The poles are in the upper half plane, and are all simple so we find

$$I^\pm \sim -\frac{\pi}{4} \sum_j \frac{\mathrm{e}^{\pm i\nu_j^\pm(\theta-\pi)}}{\sin\nu_j^\pm\pi} H_{\tilde\nu_j}^{(1)}(k_\infty r_0) H_{\tilde\nu_j}^{(1)}(k_\infty r) \frac{H_{\tilde\nu_j}'^{(2)}(k_\infty a)}{\frac{\partial}{\partial\tilde\nu_j} H_{\tilde\nu_j}'^{(1)}(k_\infty a)}. \tag{104}$$

The factors in front can be rewritten as

$$\frac{\mathrm{e}^{+i\nu_j^+(\theta-\pi)}}{\sin\nu_j^+\pi} = \frac{2i\mathrm{e}^{i\nu_j^+\theta}}{\mathrm{e}^{2\pi i\nu_j^+} - 1}, \quad \frac{\mathrm{e}^{-i\nu_j^-(\theta-\pi)}}{\sin\nu_j^-\pi} = \frac{2i\mathrm{e}^{i\nu_j^-(2\pi-\theta)}}{\mathrm{e}^{2\pi i\nu_j^-} - 1}. \tag{105}$$

Since $\tilde{\nu} \sim k_\infty a$, for large $\tilde{\nu}$, $k_\infty r$ the Hankel function has asymptotic form

$$H^{(1)}_{\tilde{\nu}_j}(k_\infty r) \sim \sqrt{\frac{2}{\pi k_\infty \sqrt{r^2 - a^2}}} \mathrm{e}^{ik_\infty \sqrt{r^2-a^2} - i\tilde{\nu}_j \cos^{-1} \frac{a}{r} - i\pi/4}, \tag{106}$$

and $H^{(1)}_{\tilde{\nu}_j}(k_\infty r_0)$ will have a similar asymptotic expression. It can be shown that

$$\frac{H'^{(2)}_{\tilde{\nu}_j}(k_\infty a)}{\frac{\partial}{\partial \tilde{\nu}_j} H'^{(1)}_{\tilde{\nu}_j}(k_\infty a)} = \frac{\mathrm{e}^{5i\pi/6}}{2\pi(-q'_j)\left(\mathrm{Ai}(q'_j)\right)^2} \left(\frac{k_\infty a}{2}\right)^{1/3}. \tag{107}$$

Combining all these expressions, and comparing with GTD solution (96) eventually yields the diffraction coefficients as

$$\alpha_j = \frac{\mathrm{e}^{5i\pi/6}}{a} \left(\frac{k_\infty a}{2}\right)^{1/3} q'_j, \tag{108}$$

$$d_j = \frac{-\mathrm{e}^{i\pi/24}}{(2\pi)^{1/4}(-q'_j)^{1/2}|\,\mathrm{Ai}(q'_j)|} \left(\frac{k_\infty a}{2}\right)^{1/6}. \tag{109}$$

Note that q'_j is real and negative, and $\mathrm{Ai}(q'_j)$ is real.

5.4 The diffracted field of an incident plane wave

Now the diffraction coefficients have been found, the case of an incident plane wave can be solved by a similar construction. The corresponding solution is

$$\begin{aligned}\phi_d = &\sum_j \frac{d_j^2}{(k_\infty \sqrt{r^2 - a^2})^{1/2}} \mathrm{e}^{ik_\infty \sqrt{r^2-a^2} - (ik_\infty - \alpha_j)a\left(\pi/2 + \cos^{-1} \frac{a}{r}\right)} \\ &\times \left(\frac{\mathrm{e}^{(ik_\infty(1-\epsilon)-\alpha_j)a\theta}}{1 - \mathrm{e}^{2\pi a(ik_\infty(1-\epsilon)-\alpha_j)}} + \frac{\mathrm{e}^{(ik_\infty(1+\epsilon)-\alpha_j)a(2\pi-\theta)}}{1 - \mathrm{e}^{2\pi a(ik_\infty(1+\epsilon)-\alpha_j)}}\right).\end{aligned} \tag{110}$$

6 Fresnel Region

We return to our governing equation

$$\nabla^2 \psi + k_\infty^2 \psi + \frac{2i\epsilon k_\infty a}{r^2} \frac{\partial \psi}{\partial \theta} = 0. \tag{111}$$

Note the substitution $\psi = \varphi \mathrm{e}^{-i\epsilon k_\infty a\theta}$ yields simply

$$\nabla^2 \varphi + k_\infty^2 \varphi = 0, \tag{112}$$

the Helmholtz equation for φ to $O(\epsilon^2)$. This reflects what we have been seeing in our ray calculations so far: the effect of the $O(\epsilon)$ mean flow is simply a phase shift determined by the angular procession of rays around the cylinder.

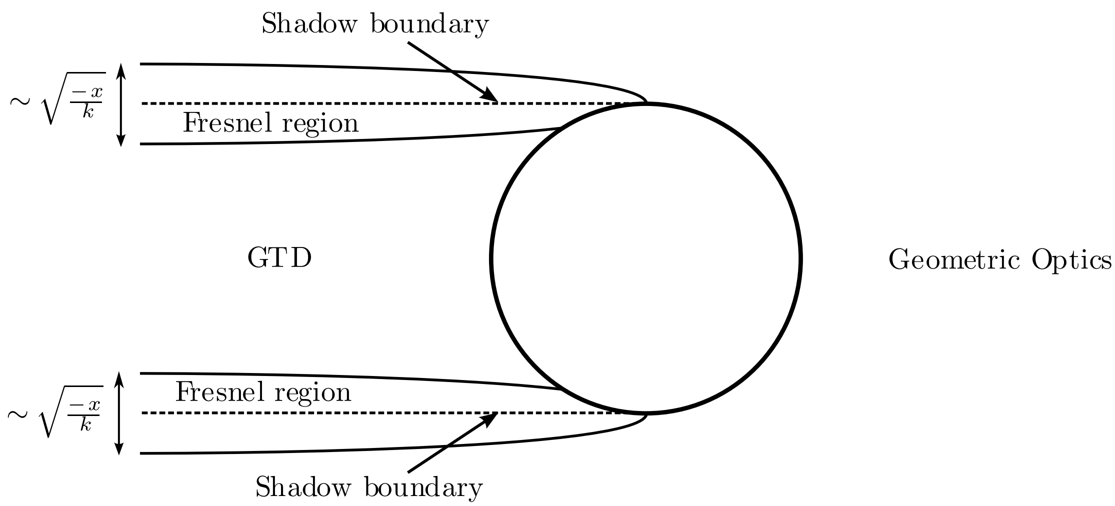

Figure 7: Cartoon of the different asymptotic regions. There is a further asymptotic region in the neighbourhood of the point at which the grazing ray hits (the Fock-Leontovich region) which we have not solved for.

At the shadow boundary the geometric optics field is discontinuous. Moreover, the GTD solution blows up at the shadow boundary. On the shadow boundary the solution takes the form of a Fresnel integral which we now derive (Figure 7). Solving for the Fresnel region is equivalent to solving for the field due to an incident wave past a flat screen. It is important to note that the Fresnel solution has no knowledge of the curvature of the surface.

Consider the ray which hits the top of the cylinder defining the shadow boundary. Far above this shadow boundary the field is dominated by the incident field, which takes the form $\psi^i = \mathrm{e}^{-ik_\infty x - i\epsilon k_\infty a\theta}$. This motivates searching for solutions to (111) of the form

$$\phi = v\mathrm{e}^{-ik_\infty x - i\epsilon k_\infty a\theta} \tag{113}$$

along the shadow boundary. We are solving in the left half plane for a wave coming from the top, so θ must be defined so that $\pi/2 < \theta < 3\pi/2$. Substituting in to (111) yields

$$v_{xx} - 2ik_\infty v_x + v_{yy} = 0. \tag{114}$$

Introduce boundary layer variables $x' = -x$, $y' = k_\infty^{1/2}(y - a)$. This boundary layer scaling implies that far away enough from the cylinder the Fresnel regions fill in the shadow region. (114) becomes

$$v_{x'x'} + 2ik_\infty v_{x'} + k_\infty v_{y'y'} = 0. \tag{115}$$

Hence expanding for large k_∞ we find the leading order term v_0 is given by

$$2iv_{0x'} + v_{0y'y'} = 0. \tag{116}$$

This is the paraxial wave equation. We introduce a similarity variable $\eta = y'/{x'}^{1/2}$ and let $v_0 = f(\eta)$. The above then reduces to

$$f'' - i\eta f' = 0. \tag{117}$$

As we go out from the boundary layer we want the solution to match on to the incident field for $y > a$ and to go to zero for $y < a$ in the shadow region. This allows us to give the final solution for the top Fresnel region as

$$\psi^f_{\text{top}} = \mathrm{e}^{-ik_\infty x - i\epsilon k_\infty a\theta}\left(\frac{1}{2} + \frac{\mathrm{e}^{-i\pi/4}}{\sqrt{2}}\,\mathrm{Fr}\left(\frac{k_\infty^{1/2}(y-a)}{\sqrt{-\pi x}}\right)\right), \tag{118}$$

where $\pi/2 < \theta < 3\pi/2$ and Fr is the Fresnel integral defined by

$$\mathrm{Fr}(z) = \int_0^z \mathrm{e}^{i\pi t^2/2}\,\mathrm{d}t. \tag{119}$$

A similar derivation for the bottom shadow boundary leads to

$$\psi^f_{\text{bot}} = \mathrm{e}^{-ik_\infty x - i\epsilon k_\infty a\theta}\left(\frac{1}{2} + \frac{\mathrm{e}^{-i\pi/4}}{\sqrt{2}}\,\mathrm{Fr}\left(\frac{k_\infty^{1/2}(-a-y)}{\sqrt{-\pi x}}\right)\right), \tag{120}$$

where it is important to note that θ is defined in this expression so that $-3\pi/2 < \theta < -\pi/2$.

We now have three asymptotic expansions which are valid in three different regions: the geometric optics solution is valid outside the shadow region, the GTD solution is valid inside the shadow region, and the Fresnel solution is valid in a neighbourhood of the shadow boundary. From these we can construct a uniformly valid solution by forming a composite expansion, and such a solution is plotted in Figures 8 and 9.

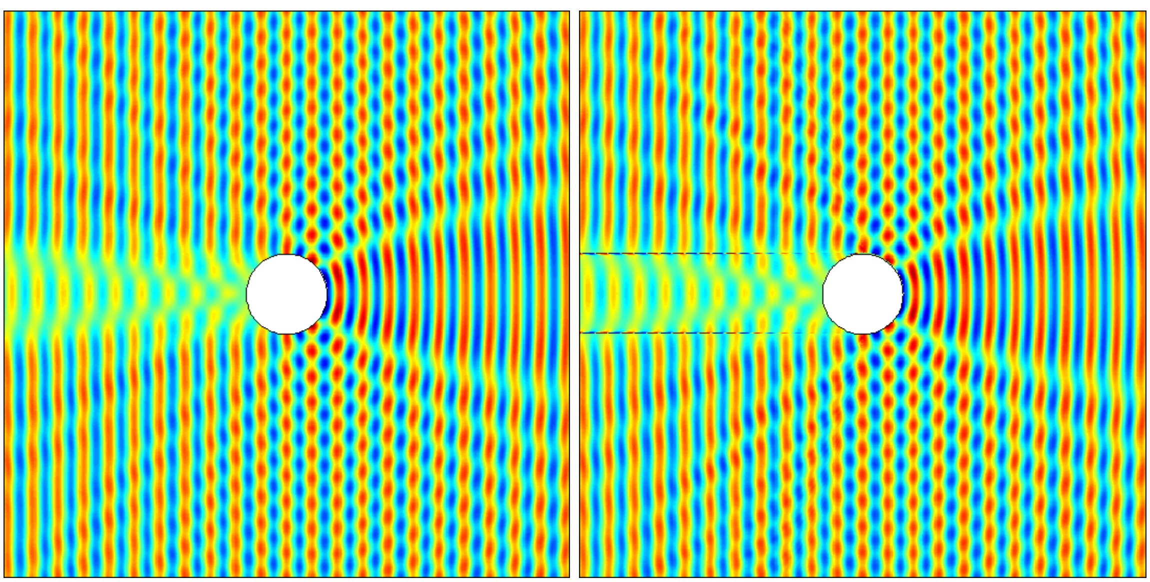

Figure 8: Plot of potential ϕ with $\epsilon = 0$, $k_\infty a = 10$. Left picture shows the eigenfunction solution, right the ray tracing solution. The two plots are remarkably similar, demonstrating the effectiveness of the ray tracing approximation. In the ray tracing solution there is a small numerical problem along the shadow boundary.

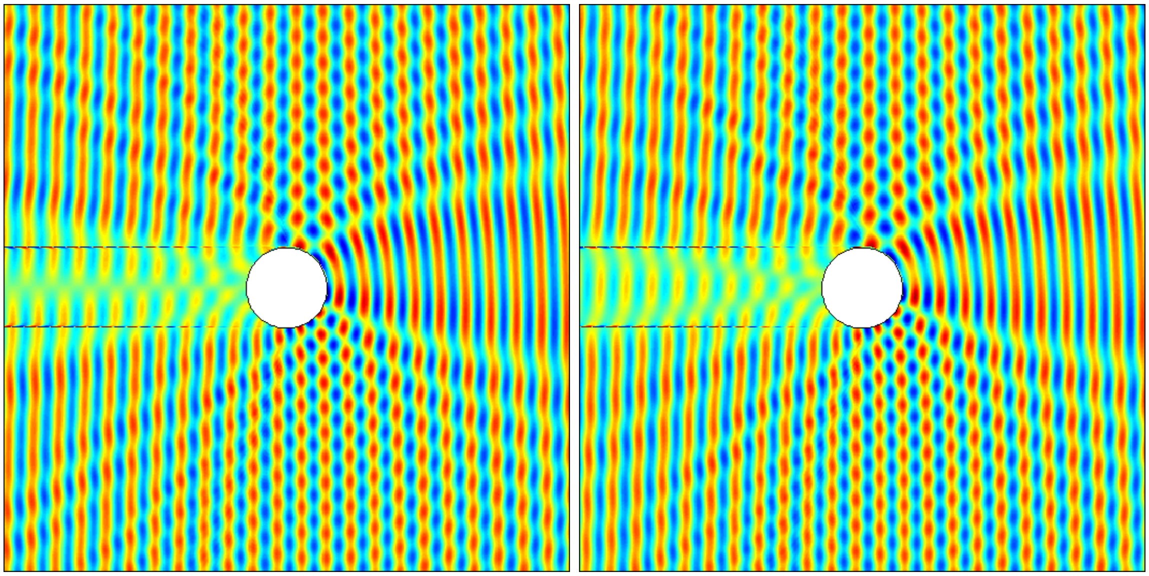

Figure 9: Plot of potential ϕ with $\epsilon = 0.25$ (left picture) and $\epsilon = 0.3$ (right picture) and $k_\infty a = 10$. Notice that refraction causes wavefronts to be out of phase on the left of the cylinder on the left picture, but in phase on the right picture.

7 Force on the cylinder

We are interested in calculating the time averaged force $\overline{\mathbf{F}}$ on the cylinder. This is simply given by integrating the time averaged pressure around the cylinder

$$\overline{\mathbf{F}} = -\int_{S_a} \overline{p}\mathbf{n}\, \mathrm{d}S = -\int_{-\pi}^{\pi} \overline{p}(\cos\theta, \sin\theta) a\, \mathrm{d}\theta. \tag{121}$$

where $\overline{p}$ is given by (28). The previous sections were devoted to solving for ϕ_1', but to find the force at second order in wave amplitude η we must also specify $\overline{\phi_2}$. To the order we are concerned with $\overline{\phi_2}$ is the solution to Laplace's equation. The boundary condition on $r = a$ is simply $\overline{\mathbf{u_2}} \cdot \mathbf{n} = 0$, but the question of what boundary condition to apply at ∞ is harder to answer. We could apply $\overline{\mathbf{u_2}} = 0$ at ∞, i.e. no Eulerian mean velocity at second order. However, this would imply that there is still an $O(\eta^2)$ mass flux past the cylinder given by $\eta^2 \overline{h_1' \mathbf{u}_1'}$; the Lagrangian mean velocity is non-zero. This phenomenon of net mass flux is known as Stokes drift. A perfectly acceptable alternative way of setting the problem up would be to demand no mass flux past the cylinder, i.e. no Lagrangian mean velocity at second order. There is no single "right way" of choosing the boundary condition on $\overline{\mathbf{u_2}}$ at ∞. Once we are given the constant velocity at infinity, $\overline{\mathbf{u_2}} \sim \overline{u_{2\infty}}(\cos\alpha, \sin\alpha)$ as $r \to \infty$, Laplace's equation has the classical solution

$$\overline{\phi_2} = \overline{u_{2\infty}} \left(r + \frac{a^2}{r} \right) \cos(\theta - \alpha). \tag{122}$$

7.1 Force calculation by ray tracing

It is straightforward to calculate the force from the ray tracing solution. The dominant contribution to ψ comes from the side of the cylinder on which the wave is incident ("the bright side"), and there the field is given by geometric optics as the sum of the reflected field and the incident field. On the cylinder the reflected field has the same phase and amplitude as the incident field. Hence on the bright side of the cylinder

$$\psi = 2\frac{c^2}{\omega}\mathrm{e}^{-ik_\infty a(\cos\theta+\epsilon\theta)}, \tag{123}$$

where we have inserted c^2/ω, an appropriate dimensional factor. We calculate each term in the time averaged pressure (28) from (122) and (123):

$$\frac{1}{2c^2}\overline{\left(\frac{\partial\phi_1'}{\partial t}\right)^2} = \frac{1}{4c^2}\,\mathrm{Re}\left(\frac{\partial\phi_1'}{\partial t}^*\frac{\partial\phi_1'}{\partial t}\right) = \frac{\omega^2}{4c^2}|\psi|^2 = c^2, \tag{124}$$

$$-\frac{1}{2}\overline{|\nabla\phi_1'|^2} = -\frac{1}{4}\,\mathrm{Re}\left(\nabla\phi_1'^*\cdot\nabla\phi_1'\right) = -\frac{1}{4}|\nabla\psi|^2 = c^2\left(-\sin^2\theta + 2\epsilon\sin\theta\right), \tag{125}$$

$$\overline{\frac{1}{c^2}\frac{\partial\phi_1'}{\partial t}\mathbf{U}\cdot\nabla\phi_1'} = \frac{1}{2c^2}\,\mathrm{Re}\left(\frac{\partial\phi_1'^*}{\partial t}\mathbf{U}\cdot\nabla\phi_1'\right) = \frac{1}{2c^2}\,\mathrm{Re}\left(i\omega\psi^*\mathbf{U}\cdot\nabla\psi\right) = -2c^2\epsilon\sin\theta, \tag{126}$$

$$-\mathbf{U}\cdot\nabla\overline{\phi_2} = 2\epsilon c\overline{u_{2\infty}}\sin\left(\theta-\alpha\right), \tag{127}$$

where we have used the identity $\overline{AB} = \mathrm{Re}(A^*B)/2$. Thus the time averaged pressure on the bright side of the cylinder is

$$\overline{p} = \text{constant} + \eta^2 Hc^2\left(\cos^2\theta + 2\epsilon\frac{\overline{u_{2\infty}}}{c}\sin\left(\theta-\alpha\right)\right) + O(\eta^3), \quad -\pi/2 < \theta < \pi/2. \tag{128}$$

On the dark side of the cylinder there is no leading order contribution from the linear waves, but there is still a contribution from the $\mathbf{U}\cdot\nabla\overline{\phi_2}$ term:

$$\overline{p} = \text{constant} + \eta^2 Hc^2\left(0 + 2\epsilon\frac{\overline{u_{2\infty}}}{c}\sin\left(\theta-\alpha\right)\right) + O(\eta^3), \quad \begin{array}{c}-\pi < \theta < -\pi/2,\\ \pi/2 < \theta < \pi.\end{array} \tag{129}$$

Integrating the time averaged pressure around the cylinder we find the $O(\eta^2)$ time averaged force is given by

$$\overline{\mathbf{F}} = -\int_{-\pi}^{\pi}\overline{p}(\cos\theta,\sin\theta)a\;\mathrm{d}\theta = \eta^2 Hc^2 a\left((-4/3,0) - 2\epsilon\pi\frac{\overline{u_{2\infty}}}{c}(-\sin\alpha,\cos\alpha)\right), \tag{130}$$

or in terms of the circulation $\Gamma = 2\pi\epsilon ca$,

$$\overline{\mathbf{F}} = \eta^2\left(Hc^2a(-4/3,0) - H\overline{u_{2\infty}}\Gamma(-\sin\alpha,\cos\alpha)\right), \tag{131}$$

where the last term can be recognised as the usual expression for the Magnus force due to flow past a cylinder.

7.2 Force calculation by eigenfunction solution

We can also calculate the force using the eigenfunction solution (48). After some algebra, we find

$$\overline{\mathbf{F}} = \eta^2 \left(Hc^2 a(\mathrm{S}(k_\infty a), 0) - H\overline{u_{2\infty}}\Gamma(-\sin\alpha, \cos\alpha) \right), \tag{132}$$

where $S(z)$ is a real valued function defined by

$$\mathrm{S}(z) = \frac{2i}{\pi z^4} \sum_{m \in \mathbb{Z}} \frac{z^2 - m(m-1)}{H_m'^{(1)}(z) H_{m-1}'^{(2)}(z)}. \tag{133}$$

Note that $S(z)$ is independent of ϵ so that the only $O(\epsilon)$ contribution to the force is still the Magnus force term. Also, by comparison with the ray tracing solution we have $\mathrm{S}(z) \to -4/3$ as $z \to \infty$. The form of $S(z)$ agrees with a similar calculation for the acoustic force on an elastic cylinder given by [4].

8 Conclusions

To $O(\epsilon)$ we have found expressions for the field resulting from the scattering useful for both small $k_\infty a$ (the eigenfunction solution) and for large $k_\infty a$ (the ray tracing solution). The effect of the $O(\epsilon)$ circulation is simply to add phase shifts in appropriate places in the calculation, and importantly it does not change the diffraction coefficients.

The $O(\epsilon)$ force has been calculated. It is important to note that we now have to specify parts of the $O(\eta^2)$ problem which we didn't have to consider in the no mean flow case. However, the only $O(\epsilon)$ contribution to the force turns out to be a Magnus force due to the mean Eulerian flow $\eta^2\overline{u_{2\infty}}$ past the cylinder.

To solve the $O(\epsilon^2)$ irrotational problem requires a lot more work. Firstly the governing partial differential equation is no longer separable. When calculating diffraction coefficients comparison with the eigenfunction solution was essential, and so this is an important stumbling block for application of the geometrical theory of diffraction. To $O(\epsilon^2)$ the rays are no longer straight, and so also we lose a lot of the geometrical considerations which made the $O(\epsilon)$ mean flow problem so similar to the no mean flow problem.

A probably tractable generalisation of this problem would be to look at the case of a rotational mean flow at $O(\epsilon)$. Here the rays are still no longer straight, but the ray curvature can be expressed simply in terms of the vorticity of the mean flow [3, 6].

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