Linear Stability of Su-Gardner Solutions to Small Two-Dimensional Perturbations

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1 Introduction

Many wave phenomena of interest in the ocean are strongly affected by mathematically messy factors: variable bottom topography, shocks and wave breaking, and so on. The original motivation for this project begins with the 2004 Indian Ocean tsunami, which among many other difficult to model phenomena included examples of nearby villages (with tiny separations on the scale of the tsunami front) experiencing great differences in the severity of the arriving waves. Purely one-dimensional models of propagation are clearly insufficient for such problems, so from here the focus of the project narrowed to examining a particular model of waves in shallow water and the complications introduced by allowing variation along a second direction of propagation.

A large amount of the material in the principal lectures this summer involved the Korteweg-de Vries equation, a long wavelength and small amplitude treatment of waves in shallow water. Here we advance the treatment a step by using the Su-Gardner equations, which maintain the requirement of long wavelength but allow for larger amplitudes (and thus strongly nonlinear waves). The goal is to understand the one-dimensional system first, then extend it to examine the stability of Su-Gardner solutions to small transverse perturbations via the Green-Naghdi equations, which provide a two-dimensional formulation of Su-Gardner.

2 The 1D Problem

The Su-Gardner equations, which consist of shallow water plus several extra correction terms, are derived here along with their solutions. Before moving on to the two-dimensional case, we show that the Su-Gardner system reduces to the Korteweg-de Vries equation in the limit of small wave amplitude.

2.1 Derivation of Su-Gardner Equations

These equations were first derived by Serre in 1953 [6], more widely known under the later work of Su and Gardner [7]. The derivation below follows the latter (see also the appendix of [2]).

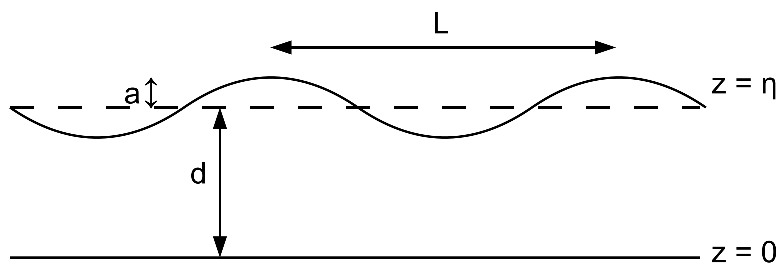

Figure 1: Flat bottom, with $\eta = \eta(x,t)$ or $\eta = \eta(x,y,t)$ for the 1D and 2D problems respectively (in the later case, we also must consider the relative size of the y wavelength, as discussed in 3.2).

We begin with the 2D Euler equations for inviscid and incompressible flow,

$$\begin{aligned} u_t + uu_x + wu_z &= -p_x \\ w_t + uw_x + ww_z &= -p_z - g \\ u_x + w_z &= 0 \end{aligned}$$

where we have assumed a constant fluid density ($\rho = 1$). We consider the domain between a flat bottom at $z = 0$ and a free surface at $z = \eta$ as sketched in Figure 1. The boundary conditions are

$$\begin{aligned} w = 0 &\quad \text{at} \quad z = 0 \\ w = \eta_t + u\eta_x &\quad \text{at} \quad z = \eta \\ p = 0 &\quad \text{at} \quad z = \eta \end{aligned}$$

representing respectively a flat impermeable bottom, and the kinematic and dynamic free surface conditions. As solutions we are seeking the surface height $\eta(x,t)$ and the depth-averaged horizontal velocity $\bar{u}(x,t)$.

From here we begin by integrating the incompressibility condition over the entire depth,

$$\int_0^\eta u_x \, dz + \int_0^\eta w_z \, dz = 0$$

where application of the Liebniz integral rule transforms this to

$$\left(\int_0^\eta u \, dz \right)_x - \eta_x u|_\eta + 0 \cdot u|_0 + w|_0^\eta = 0$$

Use of the kinematic free surface boundary condition gives

$$-\eta_x u|_\eta = \eta_t - w|_\eta$$

and the bottom boundary condition requires

$$w|_0 = 0$$

Applying these reduces the equation to

$$\left(\int_0^\eta u\,dz\right)_x + (\eta_t - w|_\eta) + w|_\eta = 0$$

which rearranges to the final form,

$$\eta_t + (\eta\bar{u})_x = 0$$

where the bar denotes a depth average,

$$\bar{u}(x,t) = \frac{1}{\eta}\int_0^\eta u\,dz$$

Next we integrate the horizontal momentum equation over the water depth. Going term by term, we have

$$\begin{aligned}
\int_0^\eta u_t\,dz &= \left(\int_0^\eta u\,dz\right)_t - \eta_t u|_\eta + 0\cdot u|_0 \\
&= (\eta\bar{u})_t - \eta_t u|_\eta \\
\int_0^\eta u u_x\,dz &= \frac{1}{2}\int_0^\eta (u^2)_x\,dz \\
&= \frac{1}{2}\left[\left(\int_0^\eta u^2\,dz\right)_x - \eta_x u^2|_\eta - 0\cdot u^2|_0\right] \\
\int_0^\eta w u_z\,dz &= \int_0^\eta (uw)_z\,dz - \int_0^\eta u w_z\,dz \\
&= (uw)|_0^\eta + \int_0^\eta u u_x\,dz \\
&= (uw)|_0^\eta + \frac{1}{2}\left[\left(\int_0^\eta u^2\,dz\right)_x - \eta_x u^2|_\eta\right] \\
\int_0^\eta p_x\,dz &= \left(\int_0^\eta p\,dz\right)_x - \eta_x \cancel{p}|_\eta + 0\cdot p|_0
\end{aligned}$$

and putting the pieces together with use of boundary conditions:

$$(\eta\bar{u})_t - (w - u\eta_x)u|_\eta + (uw)|_0^\eta + \left(\int_0^\eta u^2\,dz\right)_x - \eta_x u^2|_\eta = -\left(\int_0^\eta p\,dz\right)_x$$

This reduces to

$$(\eta\bar{u})_t + [\eta(\overline{u^2} + \bar{p})]_x = 0 \tag{1}$$

where the bars over u^2 and p denote depth averages, as above.

To deal with the averaged terms, we need to use the vertical momentum equation. Multiplying by z and then integrating over depth gives

$$\begin{aligned}
\int_0^\eta z\frac{dw}{dt}\,dz &= -\int_0^\eta z p_z\,dz - \int_0^\eta gz\,dz \\
&= -(pz)|_0^\eta + \int_0^\eta p\,dz - \frac{1}{2}g\eta^2
\end{aligned}$$

where the first term on the right hand side is zero when evaluated at the endpoints, and the rest rearranges to

$$\eta\bar{p} = \frac{1}{2}g\eta^2 + \int_0^\eta z\frac{dw}{dt}\,dz$$

After these integrations, we can rewrite equation 1 as

$$(\eta\bar{u})_t + \left(\eta\bar{u}^2 + \frac{1}{2}g\eta^2\right)_x + \left(\int_0^\eta z\frac{dw}{dt}\,dz + \eta\bar{u^2} - \eta\bar{u}^2\right)_x = 0$$

or the shallow water momentum equation plus correction terms.

Instead of neglecting the last set of terms in parentheses, we can continue further to approximate them using the additional assumptions of irrotationality ($u_z = w_x$) and long wavelength ($d \ll L$, see Figure 1). Beginning with the first of the correction terms, expand u and w in a Taylor series at the bottom:

$$\begin{aligned}
u(z) &= u(0) + z\left.\frac{\partial u}{\partial z}\right|_0 + \frac{1}{2}z^2\left.\frac{\partial^2 u}{\partial z^2}\right|_0 + \ldots \\
w(z) &= w(0) + z\left.\frac{\partial w}{\partial z}\right|_0 + \ldots
\end{aligned}$$

Starting with the second expression, the first term disappears from the bottom boundary condition, while the second can be rewritten using the incompressibility condition,

$$w(z) = -z\tilde{u}_x + \ldots$$

where $\tilde{u} = u(0)$. Looking at the horizontal velocity next, from irrotationality the second term is zw_x, but when evaluated at the bottom $w_x = 0$. This leaves

$$u(z) = \tilde{u} - \frac{1}{2}\tilde{u}_{xx}z^2 + \ldots$$

Now we can approximate the depth averaged velocity:

$$\begin{aligned}
\bar{u} &= \frac{1}{\eta}\int_0^\eta \left(\tilde{u} - \frac{1}{2}\tilde{u}_{xx}z^2 + \ldots\right)dz \\
&\approx \tilde{u} - \frac{1}{6}\tilde{u}_{xx}\eta^2
\end{aligned}$$

so $\tilde{u} \approx \bar{u} + \frac{1}{6}\bar{u}_{xx}\eta^2$, and we can write

$$\begin{aligned}
u(z) &= \tilde{u} - \frac{1}{2}\tilde{u}_{xx}z^2 + \ldots \\
&= \bar{u} + \frac{1}{6}\bar{u}_{xx}\eta^2 - \frac{1}{2}z^2(\bar{u}_{xx} + \ldots) \\
w(z) &= -z\tilde{u}_x + \ldots \\
&= -z(\bar{u}_x + \frac{1}{6}\bar{u}_{xxx}\eta^2 + \frac{1}{3}\bar{u}_{xx}\eta\eta_x) - \ldots
\end{aligned}$$

Finally, these expansions of u and w go into the integral we're working to approximate:

$$\begin{aligned}\int_0^\eta (z d_t w) dz &= \int_0^\eta z(w_t + uw_x + ww_z)\, dz \\ &\approx -\int_0^\eta z^2 \bar{u}_{xt}\, dz - \int_0^\eta z^2 \bar{u}\bar{u}_{xx}\, dz + \int_0^\eta z^2 (\bar{u}_x)^2\, dz \\ &\approx -\frac{1}{3}\eta^3[\bar{u}_{xt} + \bar{u}\bar{u}_{xx} - (\bar{u}_x)^2]\end{aligned}$$

By a similar method, it can be shown that $\bar{u^2} - \bar{u}^2$ is a higher-order correction term. Now the Su-Gardner equations, for mass and momentum conservation, are

$$\begin{aligned}\eta_t + (\eta\bar{u})_x &= 0 &(2)\\ (\eta\bar{u})_t + \left(\eta\bar{u}^2 + \frac{1}{2}g\eta^2\right)_x &= \left\{\frac{1}{3}\eta^3[\bar{u}_{xt} + \bar{u}\bar{u}_{xx} - (\bar{u}_x)^2]\right\}_x &(3)\end{aligned}$$

2.2 Su-Gardner solutions

To look for solutions, we begin with the momentum equation in conservation form (3). If we look for a traveling wave solution of speed c_0, then the dependence of the velocity and surface height becomes $u(x,t) \to u(x - c_0 t)$, $\eta(x,t) \to \eta(x - c_0 t)$, and the derivatives transform to $\partial_x \to \partial_\zeta$, $\partial_t \to -c_0\partial_\zeta$, where $\zeta = x - c_0 t$ is the moving coordinate. Now the equation can be integrated once:

$$\begin{aligned}-c_0(\eta u) + \eta u^2 + \frac{1}{2}g\eta^2 &= \frac{\eta^3}{3}[-c_0 u_{\zeta\zeta} + uu_{\zeta\zeta} - (u_\zeta)^2] + K \\ (\eta u)(u - c_0) + \frac{1}{2}g\eta^2 &= \frac{\eta^3}{3}[(u - c_0)u_{\zeta\zeta} - (u_\zeta)^2] + K\end{aligned}$$

(From this point forward, we work only with the depth-averaged velocity, and so drop the overbars.)

Next we step back to the continuity equation for a moment, where the same transformation gives

$$-c_0\eta_\zeta + (\eta u)_\zeta = 0$$

which we can integrate over ζ:

$$\begin{aligned}-c_0\eta + \eta u &= K' \\ u - c_0 &= \frac{K'}{\eta} &(4)\end{aligned}$$

where K', like K, is an integration constant. To make full use of this relation, first rewrite the momentum equation in terms of $u - c_0$,

$$(\eta u)(u - c_0) + \frac{1}{2}g\eta^2 = \frac{\eta^3}{3}\{(u - c_0)(u - c_0)_{\zeta\zeta} - [(u - c_0)_\zeta]^2\} + K$$

and then simplify the resulting η terms:

$$\begin{aligned}
K'u+\frac{1}{2}g\eta^2 &= \frac{\eta^3}{3}\left\{\frac{K'}{\eta}\left(\frac{K'}{\eta}\right)_{\zeta\zeta}-\left[\left(\frac{K'}{\eta}\right)_{\zeta}\right]^2\right\}+K \\
K'\left(\frac{K'}{\eta}+c_0\right)+\frac{1}{2}g\eta^2 &= \frac{\eta^3}{3}\left[\frac{K'}{\eta}\left(-\frac{K'\eta_\zeta}{\eta^2}\right)_{\zeta}-\left(-\frac{K'\eta_\zeta}{\eta^2}\right)^2\right]+K \\
K'c_0+\frac{K'^2}{\eta}+\frac{1}{2}g\eta^2 &= \frac{\eta^3}{3}\left[\frac{K'}{\eta}\left(-\frac{K'\eta_{\zeta\zeta}}{\eta^2}+2\frac{K'\eta_\zeta^2}{\eta^3}\right)-\frac{K'^2\eta_\zeta^2}{\eta^4}\right]+K \\
&= \frac{\eta^3}{3}\left(-\frac{K'^2\eta_{\zeta\zeta}}{\eta^3}+\frac{K'^2\eta_\zeta^2}{\eta^4}\right)+K \\
&= \frac{K'^2}{3}\left(\frac{\eta_\zeta^2}{\eta}-\eta_{\zeta\zeta}\right)+K
\end{aligned}$$

But the first term on the right hand side can be rewritten as another derivative,

$$K'c_0+\frac{K'^2}{\eta}+\frac{1}{2}g\eta^2=-\frac{K'^2\eta}{3}\left(\frac{\eta_\zeta}{\eta}\right)_{\zeta}+K$$

Now multiply both sides by η_ζ/η^2, and note that this lets us rewrite both sides as derivatives:

$$\begin{aligned}
-\frac{K'^2}{6}\left[\left(\frac{\eta_\zeta}{\eta}\right)^2\right]_{\zeta} &= \frac{\eta_\zeta}{\eta^2}\left(K'c_0-K+\frac{K'^2}{\eta}+\frac{1}{2}g\eta^2\right) \\
&= \left(-\frac{K'c_0-K}{\eta}\right)_{\zeta}+\left(-\frac{K'^2}{2\eta^2}\right)_{\zeta}+\left(\frac{1}{2}g\eta\right)_{\zeta}
\end{aligned}$$

Integrating to remove the ζ derivative, we have

$$\begin{aligned}
\left(\frac{\eta_\zeta}{\eta}\right)^2 &= -\frac{6}{K'^2}\left(-\frac{K'c_0-K}{\eta}-\frac{K'^2}{2\eta^2}+\frac{1}{2}g\eta+K''\right) \\
(\eta_\zeta)^2 &= -\frac{3g}{K'^2}\,\eta^3-\frac{6K''}{K'^2}\,\eta^2+\frac{6(K'c_0-K)}{K'^2}\,\eta+3 \\
\eta_\zeta &= \pm\sqrt{\frac{3g}{K'^2}\left(-\eta^3-\frac{2K''}{g}\eta^2+\frac{2(K'c_0-K)}{g}\,\eta+\frac{K'^2}{g}\right)} \qquad (5)
\end{aligned}$$

The solution, available in El et al. [3], is in the form of the Jacobian elliptic function $\mathrm{cn}(\zeta;m)$,

$$\eta(\zeta)=\eta_2+a\,\mathrm{cn}^2\left(\frac{1}{2}\sqrt{\frac{3g}{K'^2}(\eta_3-\eta_1)}\,\zeta;m\right) \qquad (6)$$

where ζ is the moving coordinate ($\zeta=x-c_0t$), the η_i are the roots of the cubic in (5), with $\eta_3\geq\eta_2\geq\eta_1>0$, and the elliptic modulus is $m=\frac{\eta_3-\eta_2}{\eta_3-\eta_1}$.

2.3 A limit of the equations

Equations 2 and 3, above, assume long wavelengths but place no restrictions on wave amplitude. In the small amplitude limit—in particular, when $a/d \sim (d/L)^2$—we demonstrate that Su-Gardner reduces to the well-known Korteweg-de Vries equation. Starting from (2) and (3), we follow the method of Su and Gardner [7] and switch to the moving frame, where $\xi = \epsilon^{\alpha}(x - c_0 t)$, $\tau = \epsilon^{\alpha+1} t$, and thus

$$\begin{aligned} \partial_t &= \epsilon^{\alpha+1}\partial_\tau - \epsilon^{\alpha} c_0 \partial_\xi \\ \partial_x &= \epsilon^{\alpha}\partial_\xi \end{aligned}$$

After making the transformation and canceling common powers of epsilon, we have

$$\epsilon\eta_\tau + (u - c_0)\eta_\xi + \eta u_\xi = 0$$

$$\epsilon(\eta u)_\tau - c_0(\eta u)_\xi + +\eta_\xi u^2 + 2\eta u u_\xi + g\eta\eta_\xi - \frac{1}{3}\epsilon^{2\alpha}\left\{\eta^3[\epsilon u_{\xi\tau} + (u - c_0)u_{\xi\xi} - u_\xi^2]\right\}_\xi = 0$$

If we now try expansions of the form $\eta = \eta_0 + \epsilon\eta^{(1)} + \ldots$ and $u = 0 + \epsilon u^{(1)} + \ldots$, where η_0 is the constant equilibrium height ($\eta_0 > 0$ in our coordinate system), then at leading order all terms are zero in both equations. At $O(\epsilon)$,

$$\begin{aligned} -c_0\eta_\xi^{(1)} + \eta_0 u_\xi^{(1)} &= 0 \\ -c_0\eta_0 u_\xi^{(1)} + g\eta_0\eta_\xi^{(1)} &= 0 \end{aligned}$$

These two equations together have a nontrivial solution only if $c_0^2 = g\eta_0$, so by integrating we just have $c_0\eta^{(1)} = \eta_0 u^{(1)}$. Now that we can eliminate $u^{(1)}$, take a look at $O(\epsilon^2)$:

$$\eta_\tau^{(1)} + u^{(1)}\eta_\xi^{(1)} - c_0\eta_\xi^{(2)} + \eta_0 u_\xi^{(2)} + \eta^{(1)} u_\xi^{(1)} = 0$$

$$\eta_0 u_\tau^{(1)} - c_0\eta_0 u_\xi^{(2)} - c_0\eta^{(1)} u_\xi^{(1)} - c_0\eta_\xi^{(1)} u^{(1)} + 2\eta_0 u^{(1)} u_\xi^{(1)} + g\eta_0\eta_\xi^{(2)} + g\eta^{(1)}\eta_\xi^{(1)} - \frac{1}{3}\epsilon^{2\alpha-1}\eta_0^3(-c_0 u_{\xi\xi}^{(1)}) = 0$$

or after eliminating $u^{(1)}$ (and canceling terms),

$$\eta_\tau^{(1)} + 2\frac{c_0}{\eta_0}\eta^{(1)}\eta_\xi^{(1)} - c_0\eta_\xi^{(2)} + \eta_0 u_\xi^{(2)} = 0$$

$$c_0\eta_\tau^{(1)} - c_0\eta_0 u_\xi^{(2)} + g\eta_0\eta_\xi^{(2)} + g\eta^{(1)}\eta_\xi^{(1)} + \frac{1}{3}\epsilon^{2\alpha-1}c_0^2\eta_0^2\eta_{\xi\xi\xi}^{(1)} = 0$$

Finally, multiply the first equation by c_0 and add it to the second to cancel the $\eta^{(2)}$ and $u^{(2)}$ terms, leaving

$$2c_0\eta_\tau^{(1)} + 3g\eta^{(1)}\eta_\xi^{(1)} + \frac{1}{3}\epsilon^{2\alpha-1}c_0^2\eta_0^2\eta_{\xi\xi\xi}^{(1)} = 0$$

Finally, we set $\alpha = 1/2$, so that the dispersion term comes in at the same order as the nonlinear term, which is the balance described by the KdV equation. (See the notes on Lecture 5 in this volume for more about the scaling of the small parameters.) This gives

$$\eta_\tau^{(1)} + \frac{3}{2}\frac{c_0}{\eta_0}\eta^{(1)}\eta_\xi^{(1)} + \frac{1}{6}c_0\eta_0^2\eta_{\xi\xi\xi}^{(1)} = 0 \tag{7}$$

which matches the KdV form of $\eta_\tau + \beta\eta\eta_\xi + \mu\eta_{\xi\xi\xi} = 0$. Note that up to $O(\epsilon^2)$, only one piece of the Su-Gardner correction to shallow water is relevant, specifically the $c_0 u_{\xi\xi\xi}$ term. As the wave amplitude increases, we expect the other parts of the correction term to become non-negligible, but in the KdV range of small amplitudes the two solutions should be indistinguishable.

3 The 2D Problem

With the 1D problem treated analytically, we want to examine the stability of those solutions to small disturbances along the direction perpendicular to propagation. Fortunately, the task of extending the Su-Gardner equations to two dimensions has already been accomplished by Green and Naghdi [4], so we can work from there to linearize around the 1D solutions.

3.1 The Green-Naghdi equations

Originally from Green and Naghdi in 1976 [4], using the updated form of Nadiga et al. [5], we have the Green-Naghdi equations for conservation of mass and momentum:

$$\eta_t + \nabla \cdot (\eta \mathbf{u}) = 0 \tag{8}$$

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + g\nabla\eta = \frac{1}{3\eta}\nabla\left[\eta^2(\partial_t + \mathbf{u}\cdot\nabla)(\eta\nabla\cdot\mathbf{u})\right] \tag{9}$$

Note that the velocity here is the depth-averaged quantity, so $\mathbf{u} = (u, v)$ and $\nabla = \hat{\mathbf{i}}\partial_x + \hat{\mathbf{j}}\partial_y$. It is not completely obvious from inspection that these become the Su-Gardner equations in one dimension,

$$\begin{aligned} \eta_t + (\eta u)_x &= 0 \\ u_t + uu_x + g\eta_x &= \frac{1}{3\eta}\left[\eta^3(u_{xt} + uu_{xx} - u_x^2)\right]_x \end{aligned}$$

but the verification is straightforward. Reducing to one dimension, $\mathbf{u} \to u$ and $\nabla \to \partial_x$. Applied to mass conservation (8), this simply becomes

$$\eta_t + (\eta u)_x = 0$$

while the momentum equation becomes

$$\begin{aligned} u_t + uu_x + g\eta_x &= \frac{1}{3\eta}\left[\eta^2(\partial_t + u\partial_x)(\eta u_x)\right]_x \\ &= \frac{1}{3\eta}\left[\eta^2(\eta_t u_x + \eta u_{xt} + \eta_x uu_x + \eta uu_{xx})\right]_x \\ &= \frac{1}{3\eta}\left[\eta^2(-\eta u_x^2 + \eta u_{xt} + \eta uu_{xx})\right]_x \end{aligned}$$

where the last line follows via substitution from the continuity equation. But this rearranges to

$$u_t + uu_x + g\eta_x = \frac{1}{3\eta}\left[\eta^3(u_{xt} + uu_{xx} - u_x^2)\right]_x$$

which is the second of the Su-Gardner equations, so the match is complete.

3.2 A limit of the equations

In the previous section, we looked at the behavior of the Su-Gardner equations in the small amplitude limit, and showed that they reduce to the KdV equation, providing a useful body of existing solutions to compare with numerical Su-Gardner results. It is reasonable to ask if a similar comparison can be found in two dimensions, and indeed the Kadomtsev-Petviashvili equation provides a weakly two-dimensional extension of KdV. In particular, the KP regime assumes scaling between the x and y derivatives such that the two highest-order spatial terms in the equation, u_{xxxx} and u_{yy}, come in at the same order. For this to happen we need not only the requirements of KdV,

$$\frac{a}{h} = O(\epsilon) \ , \ \left(\frac{h}{L_x}\right)^2 = O(\epsilon)$$

but additionally that the x and y wavelengths compare as

$$\left(\frac{L_x}{L_y}\right)^2 = O(\epsilon)$$

representing a y variation which is even slower than the already slow variation in the x-direction.

If these conditions are met, we conjecture that the Green-Naghdi solutions in this limit should recapture the known stability results for KdV solitons in KP: either linearly stable or unstable to small transverse perturbations, depending on relative signs of two highest derivatives (u_{xxxx} and u_{yy}) in the KP equation.

3.3 Linear stability of 1D solutions

To examine the stability of the known Su-Gardner solutions to transverse disturbances, add small perturbations to the variables of interest:

$$\begin{aligned} \eta &= \eta^{(0)} + \epsilon\eta^{(1)} + \ldots \\ u &= u^{(0)} + \epsilon u^{(1)} + \ldots \\ v &= 0 + \epsilon v^{(1)} + \ldots \end{aligned}$$

where $\eta^{(0)}(x,t)$, $u^{(0)}(x,t)$ represent the one-dimensional Su-Gardner solutions discussed above in (2.2). If we start with the simple case of $\eta^{(0)}$ and $u^{(0)}$ both constant, the equations then substantially reduce in complexity to

$$\begin{aligned} \eta_t^{(1)} + \eta^{(0)}u_x^{(1)} + u^{(0)}\eta_x^{(1)} + \eta^{(0)}v_y^{(1)} &= 0 \\ u_t^{(1)} + u^{(0)}u_x^{(1)} + g\eta_x^{(1)} &= \frac{1}{3\eta^{(0)}}\left[(\eta^{(0)})^3(\partial_t + u^{(0)}\partial_x)(u_x^{(1)} + v_y^{(1)})\right]_x \\ v_t^{(1)} + u^{(0)}v_x^{(1)} + g\eta_y^{(1)} &= \frac{1}{3\eta^{(0)}}\left[(\eta^{(0)})^3(\partial_t + u^{(0)}\partial_x)(u_x^{(1)} + v_y^{(1)})\right]_y \end{aligned}$$

Now look for normal modes, $\left\{\eta^{(1)}, u^{(1)}, v^{(1)}\right\} = \{\hat{\eta}, \hat{u}, \hat{v}\} \exp(\lambda t + ikx + ily)$, and the equations become

$$
\begin{aligned}
(\lambda + iku^{(0)})\hat{\eta} + \eta^{(0)}(ik\hat{u} + il\hat{v}) &= 0 \\
(\lambda + iku^{(0)})\hat{u} + ikg\hat{\eta} &= ik\frac{(\eta^{(0)})^2}{3}(\lambda + iku^{(0)})(ik\hat{u} + il\hat{v}) \\
(\lambda + iku^{(0)})\hat{v} + ilg\hat{\eta} &= il\frac{(\eta^{(0)})^2}{3}(\lambda + iku^{(0)})(ik\hat{u} + il\hat{v})
\end{aligned}
$$

which combines into

$$
\left[1 + \frac{1}{3}(\eta^{(0)})^2(k^2 + l^2)\right](\lambda + iku^{(0)})^2 + g\eta^{(0)}(k^2 + l^2) = 0
$$

We can rearrange this for the growth rate, λ, as

$$
\lambda = -iku^{(0)} \pm i\sqrt{\frac{g\eta^{(0)}(k^2 + l^2)}{1 + \eta^{(0)}(k^2 + l^2)/3}}
$$

so the growth rate is entirely imaginary, representing pure oscillation. Additionally, when moving from the system of three equations to the quadratic form above, we factored out an additional root, $\lambda = -iku^{(0)}$, so all roots of the system are purely imaginary. If we rewrite the quadratic root slightly in terms of the angular frequency, we have (with $\lambda = i\omega$ and $K^2 = k^2 + l^2$)

$$
\omega = -ku^{(0)} \pm K\sqrt{\frac{g\eta^{(0)}}{1 + \eta^{(0)}K^2/3}} \tag{10}
$$

For the special case of constant $\eta^{(0)}$ and $u^{(0)}$, then, the Su-Gardner system plus small transverse perturbations is linearly stable, with weak dispersion for two of the three modes.

Without the assumption of constant leading-order velocity and surface displacement, the $O(\epsilon)$ picture is considerably less pretty:

$$
\begin{aligned}
\eta_t^{(1)} &+ (u_0\eta^{(1)} + u^{(1)}\eta_0)_x + (\eta_0 v^{(1)})_y = 0 \\
(\eta^{(1)}u_0 + \eta_0 u^{(1)})_t &+ (2u_0u^{(1)}\eta_0 + u_0^2\eta^{(1)} + g\eta_0\eta^{(1)})_x + (u_0v^{(1)}\eta_0)_y \\
&= \frac{1}{3}\Big\{2\eta_0\eta^{(1)}(\partial_t + u_0\partial_x)(\eta_0u_{0x}) + \eta_0^2(u^{(1)}\partial_x + v^{(1)}\partial_y)(\eta_0u_{0x}) \\
&\quad + \eta_0^2(\partial_t + u_0\partial_x)(\eta^{(1)}u_{0x}) + \eta_0^2(\partial_t + u_0\partial_x)\left[\eta_0(u_x^{(1)} + v_y^{(1)})\right]\Big\}_x \\
(\eta_0v^{(1)})_t &+ (u_0\eta_0v^{(1)})_x + (g\eta_0\eta^{(1)})_y = \frac{1}{3}\{\ldots\}_y
\end{aligned}
$$

Even assuming normal modes of the form $\left\{\eta^{(1)}, u^{(1)}, v^{(1)}\right\} = \{\hat{\eta}(x), \hat{u}(x), \hat{v}(x)\}\exp(\lambda t + ily)$, these equations have not proved tractable to simplify to a dispersion relation comparable to (10) above. The next step is to treat them numerically, discussed below.

4 Numerical solutions

While the one-dimensional Su-Gardner system has been solved, the 2D case is much more difficult to reduce analytically, and to make substantial further progress it seems that numerical solutions will be required. To apply transverse perturbations to the cnoidal solutions of the Su-Gardner equations, we first need them in numerical form, and so we take a step back to the one-dimensional system to capture that starting point. The first stage of this work seeks solutions of the KdV equation, as a simpler test case which should be comparable with low-amplitude Su-Gardner. The tool used was a Newton-Raphson-Kantorovich solver previously developed by P. Garaud, which treats two-point boundary value problems for systems of ordinary differential equations by the method of relaxing to a solution.

4.1 KdV numerical solution

Starting with the KdV equation, to eliminate the τ derivative and express the problem as a set of ODEs, we shift to a frame that is moving with the speed of the soliton solution, V. (Note that this is the speed of the soliton with respect to the frame that is already moving with the linear wave speed, so that in the notation of the previous sections, the total solution speed in the rest frame is $c = c_0 + \epsilon V$.) Once this is done the problem can be expressed in terms of ordinary differential equations,

$$\begin{aligned} Y_1 &= \eta_1 \\ Y_2 &= \eta_1' \\ Y_3 &= \eta_1'' \\ Y_4 &= V \\ Y_5 &= \int_0^x \eta_1 \, dx \end{aligned}$$

where the primes denote spatial derivatives in the frame moving with speed c, the soliton speed V is solved for as an eigenvalue, and the fifth equation is used to impose a mass conservation boundary condition. The system fed into the solver, then, is

$$\begin{aligned} Y_1' &= Y_2 \\ Y_2' &= Y_3 \\ Y_3' &= Y_2 Y_4 - Y_1 Y_2 \\ Y_4' &= 0 \\ Y_5' &= Y_1 \end{aligned}$$

The boundary conditions supplied are as follows. At the first endpoint, $x_a = 0$, we require that $Y_1 = H$ (the surface height perturbation equals some specified amplitude), that $Y_2 = 0$, and that $Y_5 = 0$. At the second endpoint, $x_b = L$, we again fix the amplitude at $Y_1 = H$ (the solution should be periodic), and also require that $Y_5 = 0$ (the total mass integrated over the domain is zero, which should be the case since we are solving for the perturbation to the equilibrium height η_0).

Finally, we need a set of initial guesses at the solution. For Y_1, Y_2, and Y_3, a cosine and its first two derivatives are used, while for Y_4 and Y_5 nonzero constants are supplied. The system was integrated over a sample domain of horizontal extent $L = 10$ with amplitude $a = 1$ (see Figure 2).

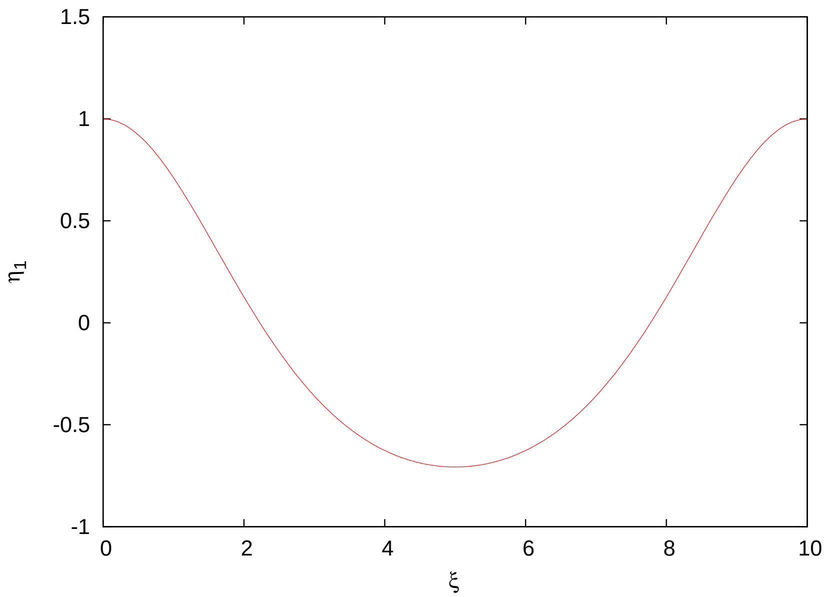

Figure 2: Sample integration of KdV equation for η_1 with the background amplitude η_0 subtracted.

4.2 Su-Gardner numerical solution

With the KdV numerical solution in hand, treatment of the Su-Gardner equations begins with the conservation form of equation 3, shifted to a frame moving with speed c. The continuity equation is integrated to give $u - c = K'/\eta$, but further integrations of the momentum equation in section 2.2 are not carried out. The momentum equation at this point is

$$-c(\eta u)_\zeta + (\eta u^2)_\zeta + \frac{1}{2}(g\eta^2)_\zeta = \left[\frac{1}{3}\eta^3(-cu_{\zeta\zeta} + uu_{\zeta\zeta} - u_\zeta^2)\right]_\zeta$$

where derivatives are in the frame of the moving coordinate, $\zeta = x - ct$. If we rewrite in terms of $(u - c)$,

$$\left[(u-c)\eta u + \frac{1}{2}g\eta^2\right]_\zeta = \frac{1}{3}\left\{\eta^3\left[(u-c)(u-c)_{\zeta\zeta} - (u-c)_\zeta^2\right]\right\}_\zeta$$

then substitute using the integrated continuity equation,

$$\left[\eta\frac{K'}{\eta}\left(\frac{K'}{\eta} + c\right) + \frac{1}{2}g\eta^2\right]_\zeta = \frac{1}{3}\left\{\eta^3\left[\frac{K'}{\eta}\left(\frac{K'}{\eta}\right)_{\zeta\zeta} - \left(\frac{K'}{\eta}\right)_\zeta^2\right]\right\}_\zeta$$

After expanding the derivatives on both sides this leaves

$$-K'^2\frac{\eta_\zeta}{\eta} + g\eta\eta_\zeta = \frac{K'^2}{3}\left(\frac{2\eta_\zeta\eta_{\zeta\zeta}}{\eta} - \frac{\eta_\zeta^3}{\eta^2} - \eta_{\zeta\zeta\zeta}\right)$$

which rearranges to

$$-3\eta_\zeta + \frac{3g}{K'^2}\eta^3\eta_\zeta = 2\eta\eta_\zeta\eta_{\zeta\zeta} - \eta_\zeta^3 - \eta^2\eta_{\zeta\zeta\zeta}$$

Note that η here is the total quantity, so to compare with the KdV solution above, we rewrite in terms of $\eta_1 = \eta - \eta_0 = \eta - 1$ (normalizing the equilibrium level η_0 to 1). The equation is now

$$(\eta_1 + 1)^2\eta_{1\zeta\zeta\zeta} = 2(\eta_1 + 1)\eta_{1\zeta}\eta_{1\zeta\zeta} - \eta_{1\zeta}^3 + 3\eta_{1\zeta} - \frac{3}{K'^2}(\eta_1 + 1)^3\eta_{1\zeta}$$

where we have also set $g = 1$ for convenience. Finally we have the first system of equations tried numerically for Su-Gardner,

$$\begin{aligned} Y_1 &= \eta_1 \\ Y_2 &= \eta_{1\zeta} \\ Y_3 &= \eta_{1\zeta\zeta} \\ Y_4 &= \int_0^x \eta_1\,dx \end{aligned}$$

and its derivatives

$$\begin{aligned} Y_1' &= Y_2 \\ Y_2' &= Y_3 \\ (Y_1 + 1)^2 Y_3' &= 2(Y_1 + 1)Y_2Y_3 - Y_2^3 + 3Y_2 - (3/K^2)(Y_1 + 1)^3 Y_2 \\ Y_4' &= Y_1 \end{aligned}$$

With four unknowns we require four boundary conditions, which are to specify the amplitude $\eta_1 = H$ at both endpoints, and to set the first derivative and the integral of η_1 to zero at the first endpoint. The same initial guesses and endpoint amplitude as above were used, and various values of the integration constant K' were tried, but the system failed to converge. A second attempt was made by additionally solving for the eigenvalue $Y_4 = K'$ (making Y_5 the mass integral, as in the KdV case above), but this also did not converge.

This numerical work was begun in the later stage of the summer, and was not successfully completed by the end. To continue in this direction, clues to the lack of convergence might be sought in the recent work of Carter and Cienfuegos [1], who numerically examine the stability of Su-Gardner solutions to small perturbations along the direction of propagation and find that only solutions of sufficiently small amplitude and steepness are stable.

5 Conclusion

By the end of this project, I hoped to have an answer, either analytic or numerical, for the stability of Su-Gardner solutions to small transverse perturbations. The desired end result was a better understanding of large-amplitude waves in shallow water, specifically the stability of known 1D solutions to small transverse perturbations. Toward this goal I made the most progress in the one-dimensional analytics, which are now well understood. Analysis of the two-dimensional case was also begun but not yet complete, with linear stability results obtained for the simplest case of constant η_0 and u_0, and the more general case most likely needing a numerical treatment. Numerical results in 1D for the purposes of extending to the 2D problem were begun but not completed, with several avenues remaining to explore to track down the non-convergence of the code. To continue from this point will necessitate further attention to the numerical solutions in one dimension, to compare with recent results [1] before extending along the second direction of propagation.

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