Two-dimensional Vortex Shedding From a Corner

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1 Introduction

The understanding of the unsteady separation of high-Reynolds number flow past a pointed edge is of interest to several fluid-mechanical problems, including flow past an aircraft wing or flow past a coastline. Much work has been done on the special case of flow past a semi-infinite line, notably by Rott [1], but much of that work has not been extended to less sharp points.

In this paper, we consider a two-dimensional approximation of high Reynolds number vortex shedding from a corner initially at rest in a motionless, unbounded, incompressible fluid. If the corner is sufficiently sharp, that is if it has exterior angle greater than π, vorticity must be shed when the fluid begins to move in order to maintain regularity at the edge. We consider a flow with a vortex sheet emanating from the corner at $t = 0$. As it moves away from the corner, the vortex sheet rolls up at its end producing an effect that is approximately that of a concentrated vortex. Therefore, we model the shed vorticity by a single point vortex.

The vortex created at $t = 0$ moves in the flow created by itself, it's image vortex, and the forcing flow. The magnitude of the vortex may not decrease because this would represent the unraveling of the sheet. Therefore, if it reaches some maximum at $t = t'$, a new vortex must be created in order to maintain regularity at the corner. The magnitude of the first vortex will remain constant while the the magnitude of the second vortex increases and both move in the flow created by the forcing and both vortices and their images. This process could continue ad infinitum each time the most recently created vortex reaches a maximum.

2 Mathematical Formulation

Let us consider a body with a (not necessarily bounded) boundary $\mathcal{C}$ in the physical plane, which we shall denote as the z-plane with $z = x+iy$ with $x, y \in \mathrm{Re}$. Now suppose $F : z \to \zeta$ is a conformal mapping from the exterior of $\mathcal{C}$ in the physical plane to the plane $\mathrm{Im}(\zeta) > 0$.

We create this image plane because we expect to be able to satisfy the boundary condition at the body more easily in this plane than in the physical plane. In the physical plane, we require that the velocity be parallel to the boundary. In the mapped plane, this is equivalent to

$$\mathrm{Im}\Phi'\Big|_{\mathrm{Im}\zeta=0} = 0. \tag{1}$$

Since a point vortex has a logarithmic singularity, we require

$$\Phi \sim \frac{\Gamma i}{2\pi} \log(\zeta - \zeta_0) + const. \tag{2}$$

near $\zeta = \zeta_0$, where Γ is the strength of the vortex. To satisfy the boundary condition, we place an "image vortex" with strength $-\Gamma$ at position $\zeta = \overline{\zeta_0}$ within the body in the lower half plane. The potential of the two vortices is

$$\Phi(\zeta) = \frac{i\Gamma}{2\pi} \left[\log(\zeta - \zeta_0) - \log(\zeta - \overline{\zeta_0})\right], \tag{3}$$

which satisfies the boundary condition (1).

To calculate the potential in the ζ-plane due to n vortices with strengths $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ at positions $\zeta_1, \zeta_2, \ldots, \zeta_n$ respectively, we need only superimpose the potentials due to the vortices and their images. Therefore, the potential including n vortices and the forcing flow, $U(t)$, is

$$\Phi(\zeta) = U(t)\zeta + \frac{i}{2\pi} \sum_{j=1}^{n} \Gamma_n \left[\log(\zeta - \zeta_j) - \log(\zeta - \overline{\zeta_j})\right]. \tag{4}$$

Therefore, since $\Phi(\zeta)$ is analytic in the upper half plane and F is a conformal map, $\Phi(F^{-1}(z))$ is the complex potential in the physical plane which is created by n vortices at $z_1 = F(\zeta), z_2 = F(\zeta), \ldots, z_n = F(\zeta)$, their images, and a uniform flow. Thus, we can write the complex velocity in the physical plane, in terms of the variables in the ζ-plane. We have

$$\frac{d\Phi}{dz} = \frac{d\zeta}{dz}\Phi'(\zeta) \tag{5}$$

$$= \frac{1}{F'(\zeta)}\Phi'(\zeta). \tag{6}$$

Without loss of generality, we can choose the center of coordinates of the z-plane and F so that the corner in the physical plane is at $z = 0$ and so that $F(0) = 0$. Therefore, if $F'(0) = 0$, the only way to make the velocity at the origin nonsingular is to force $\Phi'(0) = 0$. This condition is known as the Kutta condition and can be expressed as

$$U(t) + \frac{i}{2\pi} \sum_{j=1}^{n} \Gamma_j \left[\frac{1}{\overline{\zeta_j}} - \frac{1}{\zeta_j}\right] = 0. \tag{7}$$

As each new vortex is created, n increases. Note that this condition only depends on variables defined in the ζ-plane.

We now derive the equation of motion for the vortices in the flow. Recall, that the velocity near a vortex has a singular part due to its strength and a nonsingular part due to the flow and all of the other vortices and images. Saffman [2] shows that the balance of the pressure force on a small circle around the vortex and the change of momentum through the circle requires that the vortex move with the nonsingular part of the flow. In the physical plane that means

$$\begin{aligned} \frac{d\overline{z_j}}{dt} &= \lim_{z \to z_j} \frac{d}{dz}\left[\Phi\left(F^{-1}(z)\right) - \frac{i\Gamma_j}{2\pi}\log(z - z_j)\right] \\ &= \lim_{\zeta \to \zeta_j} \frac{1}{F'(\zeta)} \frac{d}{d\zeta}\left[\Phi(\zeta) - \frac{i\Gamma_j}{2\pi}\log(F(\zeta) - F(\zeta_j))\right] \end{aligned}$$

Removing the terms which are not singular at $\zeta = \zeta_j$ and combining the singular logarithms, we have

$$\frac{d\overline{z_j}}{dt} = \frac{1}{F'(\zeta_j)} \left\{ U(t) + \frac{i}{2\pi} \sum_{k \neq j}^{n} \left[\frac{\Gamma_k}{\zeta_j - \zeta_k} - \frac{\Gamma_k}{\zeta_j - \overline{\zeta_l}} \right] - \frac{i\Gamma_j}{2\pi \left(\zeta_j - \overline{\zeta_j} \right)} - \frac{i\Gamma_j}{2\pi} \lim_{\zeta \to \zeta_j} \frac{d}{d\zeta} \log \left(\frac{F(\zeta) - F(\zeta_j)}{\zeta - \zeta_j} \right) \right\} \tag{8}$$

Finally, we expand $F(\zeta)$ in a Taylor series around ζ_j to find that

$$\begin{aligned} \log \left(\frac{F(\zeta) - F(\zeta_j)}{\zeta - \zeta_j} \right) &= \log \left(F'(\zeta_j) \left[1 + \frac{F''(\zeta_j)}{2F'(\zeta_j)} (\zeta - \zeta_j) + o\left((\zeta - \zeta_j)^2 \right) \right] \right) \\ &= \log F'(\zeta_j) + \frac{F''(\zeta_j)}{2F'(\zeta_j)} (\zeta - \zeta_j) + o\left((\zeta - \zeta_j)^2 \right) \end{aligned} \tag{9}$$

Substituting equation (9) into equation (8) reveals that

$$\frac{d\overline{z_j}}{dt} = \frac{1}{F'(\zeta_j)} \left\{ U(t) + \frac{i}{2\pi} \sum_{k \neq j}^{n} \left[\frac{\Gamma_k}{\zeta_j - \zeta_k} - \frac{\Gamma_k}{\zeta_j - \overline{\zeta_l}} \right] - \frac{i\Gamma_j}{2\pi \left(\zeta_j - \overline{\zeta_j} \right)} - \frac{i\Gamma_j}{4\pi} \frac{F''(\zeta_j)}{F'(\zeta_j)} \right\}. \tag{10}$$

The last term in equation (10) is known as the Routh correction and takes into account the self advection of the vortex. For convenience, we will study the evolution in the ζ-plane, so we convert equation (10) by conjugating both sides and multiplying on both sides by $d\zeta_j/dz_j = 1/F'(\zeta_j)$ to find that for each $1 \leq j \leq n$

$$\frac{d\zeta_j}{dt} = \frac{1}{|F'(\zeta_j)|^2} \left\{ U(t) - \frac{i}{2\pi} \sum_{k \neq j}^{n} \left[\frac{\Gamma_k}{\overline{\zeta_j} - \overline{\zeta_k}} - \frac{\Gamma_k}{\overline{\zeta_j} - \zeta_k} \right] - \frac{i\Gamma_j}{2\pi \left(\zeta_j - \overline{\zeta_j} \right)} + \frac{i\Gamma_j}{4\pi} \frac{\overline{F''(\zeta_j)}}{\overline{F'(\zeta_j)}} \right\}. \tag{11}$$

For completeness, we mention here that some authors have used equations other than equation (11) to study the shedding of vortices in two-dimensional flows. Brown and Michael [3] include a correction term on the left-hand side of equation (11) which is proportional to $d\Gamma_n/dt$ to balance the force on a hypothetical line of vorticity stretching from the corner to the vortex, which feeds the vortex allowing it to grow in strength. Howe [4] also considers a line of vorticity, but he further requires that the correction term account for balancing the torque on the sheet. We will discuss Brown and Michael's equation briefly in section 3, but will not consider Howe's equation because it is significantly more complicated and even according to his own results only adds a very small correction.

3 The Evolution of the First Vortex

As discussed in Section 2, at $t = 0$, i.e. just as the fluid starts moving, we expect a vortex to be shed from the corner in the physical plane. Since there is only one vortex in the fluid initially, we need only consider equation (11) for $j = n = 1$ where Γ_1 allows the Kutta condition, i.e. equation (7), to be satisfied. Solving equation (7) for Γ_1 yields

$$\Gamma_1 = 2\pi i \frac{|\zeta_1|^2}{\zeta_1 - \overline{\zeta_1}} U(t). \tag{12}$$

Therefore, we have

$$\begin{aligned}\frac{d\zeta_1}{dt} &= \frac{1}{|F'(\zeta_1)|^2}\left[U(t) - \frac{i\Gamma_1}{2\pi\left(\zeta_1 - \overline{\zeta_1}\right)} + \frac{i\Gamma_1}{4\pi}\frac{\overline{F''(\zeta_1)}}{\overline{F'(\zeta_1)}}\right] \\ &= \frac{U(t)}{|F'(\zeta_1)|^2}\left[1 - \frac{|\zeta_1|^2}{4(\mathrm{Im}\zeta_1)^2} + i\frac{\overline{F''(\zeta_1)}}{\overline{F'(\zeta_1)}}\frac{|\zeta_1|^2}{4\mathrm{Im}\zeta_1}\right],\end{aligned} \tag{13}$$

with the initial condition that $\zeta_1(0) = 0$.

3.1 An Exact Solution for the Infinite Wedge

To proceed, we must choose the boundary in the physical plane and calculate the conformal map F. An infinite wedge with its tip at $z = 0$ is of particular interest because in the region very near the tip, any corner is approximated by the infinite wedge. Therefore, let us choose the boundary to be an infinite wedge with exterior angle $\alpha\pi$ that is bisected by the imaginary axis. The corresponding conformal map is therefore given by

$$F(\zeta) = e^{-i(\alpha-1)\frac{\pi}{2}}\zeta^{\alpha}, \tag{14}$$

so that equation (13) becomes

$$\frac{d\zeta_1}{dt} = \frac{U(t)}{\alpha^2|\zeta_1|^{2(\alpha-1)}}\left[1 - \frac{|\zeta_1|^2}{4(\mathrm{Im}\zeta_1)^2} + i\frac{(\alpha-1)\zeta_1}{4\mathrm{Im}\zeta_1}\right]. \tag{15}$$

Writing $\zeta = \xi + i\eta$ with ξ, η real, we separate real and imaginary parts in equation (15) to find coupled equations for $d\xi_1/dt$ and $d\eta_1/dt$. That is

$$\frac{d\xi_1}{dt} = \frac{U(t)}{\alpha^2\left(\xi_1^2 + \eta_1^2\right)^{\alpha-1}}\left[1 - \frac{\alpha}{4} - \frac{\xi_1^2}{4\eta_1^2}\right], \tag{16}$$

and

$$\frac{d\eta_1}{dt} = \frac{U(t)}{\alpha^2\left(\xi_1^2 + \eta_1^2\right)^{\alpha-1}}\left[\frac{\alpha-1}{4}\frac{\xi_1}{\eta_1}\right]. \tag{17}$$

To solve equations (16) and (17), we note that the only explicit time dependence occurs in a prefactor that multiplies the right side of both equations. Therefore, by dividing equation (16) by equation (17) we are left with a differential equation for ξ_1 as a function of η_1. Solving this equation with the initial condition $\xi(\eta = 0) = 0$ yields gives us

$$\xi_1 = \pm\sqrt{\frac{4-\alpha}{\alpha}}\eta_1. \tag{18}$$

Now, we can solve for $\eta_1(t)$ in equation (17) by substituting equation (18) for ξ_1 recalling that $\eta_1 \geq 0$. This yields an exact solution for η_1, which in turn gives us an exact solution for ζ_1. We find

$$\zeta_1(t) = \left[\frac{(\alpha-1)(2\alpha-1)}{4\alpha^{5/2}}\left|\int_0^t U(\tau)\,d\tau\right|\right]^{\frac{1}{2\alpha-1}}\left(\frac{\sqrt{\alpha}}{2}i + sign\left(\int_0^t U(\tau)\,d\tau\right)\frac{\sqrt{4-\alpha}}{2}\right). \tag{19}$$

This indicates that the first vortex moves away from the origin along a ray in the ζ-plane.

3.2 Preparing to Shed the Second Vortex

As discussed in Section (2), we expect the vortex to continue along its path as determined by equation (19) until such time that the magnitude of its strength reaches a maximum. From equation (12), we compute

$$\Gamma_1(t) = \frac{2\pi U(t)}{\sqrt{\alpha}} \left[\frac{(\alpha-1)(2\alpha-1)}{4\alpha^{5/2}} \left| \int_0^t U(\tau)\, d\tau \right| \right]^{\frac{1}{2\alpha-1}}. \tag{20}$$

Without loss of generality, we can take U to be nonnegative for small t since replacing U with $-U$ and ξ_1 with $-\xi_1$ leaves equations (16) and (17) unchanged. With this assumption, upon differentiating equation (20) by t, we see that Γ_1 reaches a local extremum at $t = \hat{t}$ if and only if

$$U'\left(\hat{t}\right) \int_0^{\hat{t}} U(\tau)\, d\tau + \frac{U^2\left(\hat{t}\right)}{2\alpha - 1} = 0. \tag{21}$$

3.3 An Exact Solution with the Brown and Michael Model

As discussed in Section 2, Brown and Michael [3] added a correction term to equation (10) to account for an unbalanced force on a vortex sheet leading up to the vortex coming out of the origin. For the first vortex, the corrected equation is

$$\frac{d\overline{z_1}}{dt} = -\overline{z_1}\frac{1}{\Gamma_1}\frac{d\Gamma_1}{dt} + \frac{1}{F'(\zeta_j)} \left\{ U(t) - \frac{i\Gamma_j}{2\pi\left(\zeta_j - \overline{\zeta_j}\right)} - \frac{i\Gamma_j}{4\pi}\frac{F''(\zeta_j)}{F'(\zeta_j)} \right\}. \tag{22}$$

Cortelezzi [5] gives an exact solution for the infinite wedge in the case that $\alpha = 2$. However, we can solve equation (22) for any value of α, $1 < \alpha \leq 2$. In terms of ζ_1 equation (22) can be rewritten

$$\alpha\overline{\zeta_1}^{\alpha-1}\dot{\overline{\zeta_1}} = -\overline{\zeta_1}^{\alpha} \left(\frac{\dot{U}}{U} + \frac{\dot{\overline{\zeta_1}}}{\overline{\zeta_1}} + \frac{\dot{\zeta_1}}{\zeta_1} - \frac{\dot{\zeta_1} - \dot{\overline{\zeta_1}}}{\zeta_1 - \overline{\zeta_1}} \right) + \frac{U(t)}{\alpha\zeta_1^{\alpha-1}} \left(1 + \frac{|\zeta_1|^2}{\left(\zeta_1 - \overline{\zeta_1}\right)^2} + \frac{\alpha-1}{2}\frac{\overline{\zeta_1}}{\zeta_1 - \overline{\zeta_1}} \right). \tag{23}$$

Changing variables by setting $\zeta = \rho e^{i\theta}$ reduces equation (23)to equations for $\dot{\rho}$ and $\dot{\theta}$. They are

$$\dot{\theta} = \frac{\alpha - 4\sin^2\theta}{4\alpha^2 \sin\theta}\frac{U(t)}{\rho^{2\alpha-1}}, \tag{24}$$

and

$$\dot{\rho} = -\frac{\rho}{\alpha+1}\frac{\dot{U}}{U} + \frac{\alpha-1}{\alpha^2(\alpha+1)}\frac{U\cos\theta}{\rho^{2\alpha-2}}. \tag{25}$$

We now solve equations (24) and (25) using the following change of variables:

$$\lambda = U^{(2\alpha-1)/(\alpha+1)}\rho^{2\alpha-1}, \quad \mu = \cos\theta, \quad \tilde{t} = \textstyle\int_0^t U^{3\alpha/(\alpha+1)}(\tau)\, d\tau. \tag{26}$$

After some algebra, we have

$$\tfrac{d\lambda}{d\tilde{t}} = \tfrac{(2\alpha-1)(\alpha-1)}{\alpha^2(\alpha+1)}\mu, \quad \tfrac{d\mu}{d\tilde{t}} = \tfrac{4-\alpha-4\mu^2}{4\alpha^2\lambda} \tag{27}$$

with the initial conditions that $\lambda(0) = 0$ and $\mu(0) = \mu_0$. Combining equations (27) to form a single differential equation for $u = \lambda^2$ gives

$$\frac{d^2u}{d\tilde{t}^2} = \frac{(2\alpha-1)(\alpha-1)(4-\alpha)}{2\alpha^4(\alpha+1)} - \frac{\alpha(2-\alpha)}{(2\alpha-1)(\alpha-1)}\frac{1}{u}\left(\frac{du}{d\tilde{t}}\right)^2. \tag{28}$$

To solve this equation, we set $du/d\tilde{t} = f(u)$ since there is no explicit time dependence. After some manipulation, we find that

$$\lambda\left(\tilde{t}\right) = \pm\frac{(2\alpha-1)(\alpha-1)}{2\alpha^2(\alpha+1)}\sqrt{4-\alpha}\,\tilde{t} \tag{29}$$

and that

$$\mu = \pm\frac{\sqrt{4-\alpha}}{2}. \tag{30}$$

Putting all of this together, we find that the exact solution for the first vortex with the Brown and Michael correction is

$$\zeta_1 = \left(\frac{(2\alpha-1)(\alpha-1)\sqrt{4-\alpha}}{2\alpha^2(\alpha+1)}\right)^{\frac{1}{2\alpha-1}}\left(\frac{1}{U(t)}\right)^{\frac{1}{\alpha+1}}\left(\int_0^t U(\tau)^{3\alpha/(\alpha+1)}\,d\tau\right)^{\frac{1}{2\alpha-1}}\left(\pm\frac{\sqrt{4-\alpha}}{2} + i\frac{\sqrt{\alpha}}{2}\right). \tag{31}$$

Therefore,

$$\Gamma_1 = \frac{2\pi}{\sqrt{\alpha}}\left(\frac{(2\alpha-1)(\alpha-1)\sqrt{4-\alpha}}{2\alpha^2(\alpha+1)}\right)^{\frac{1}{2\alpha-1}}\left(U^{\frac{\alpha(2\alpha-1)}{\alpha+1}}\int_0^t (U(\tau))^{\frac{3\alpha}{\alpha+1}}\,d\tau\right), \tag{32}$$

which implies that the first vortex increases in strength until $t = t_1$ satisfies

$$\frac{\alpha(2\alpha-1)}{\alpha+1}U'(t_1)\int_0^{t_1}(U(\tau))^{\frac{3\alpha}{\alpha+1}}\,d\tau + U(t_1)^{\frac{4\alpha+1}{\alpha+1}} = 0. \tag{33}$$

This agrees with the solution found by Cortelezzi for $\alpha = 2$. In addition, equation (31) agrees with our solution without the vortex sheet, equation (19), in its angle of departure.

4 The Second Vortex for the Infinite Wedge

If for some $t_1 > 0$, equation (21) is satisfied, a second vortex will be shed from the corner of the infinite wedge, and the strength of the first vortex will be fixed for all times $t \geq t_1$. For example, if $U = \sin t$, $t_1 = \cos^{-1}(-1/2\alpha)$. The system of equations given by (11) with $n = 2$ is

$$\frac{d\zeta_1}{dt} = \frac{1}{\alpha^2|\zeta_1|^{2\alpha-2}}\left\{U(t) - \frac{i}{2\pi}\left[\frac{\Gamma_2}{\overline{\zeta_1}-\overline{\zeta_2}} - \frac{\Gamma_2}{\overline{\zeta_1}-\zeta_2}\right] - \frac{i\Gamma_1}{2\pi\left(\zeta_1-\overline{\zeta_1}\right)} + (\alpha-1)\frac{i\Gamma_1}{4\pi\overline{\zeta_1}}\right\}, \tag{34}$$

and

$$\frac{d\zeta_2}{dt} = \frac{1}{\alpha^2|\zeta_2|^{2\alpha-2}}\left\{U(t) - \frac{i}{2\pi}\left[\frac{\Gamma_1}{\overline{\zeta_2}-\overline{\zeta_1}} - \frac{\Gamma_1}{\overline{\zeta_2}-\zeta_1}\right] - \frac{i\Gamma_2}{2\pi\left(\zeta_2-\overline{\zeta_2}\right)} + (\alpha-1)\frac{i\Gamma_2}{4\pi\overline{\zeta_2}}\right\}, \tag{35}$$

with the initial conditions that ζ_1 is continuous at t_1 and that $\zeta_2(t_1) = 0$.

From the Kutta condition, we can solve for Γ_2 yielding

$$\Gamma_2 = 2\pi i \frac{|\zeta_2|^2}{\zeta_2 - \overline{\zeta_2}} \left[U(t) + \frac{i\Gamma_1}{2\pi} \left(\frac{1}{\overline{\overline{\zeta_1}}} - \frac{1}{\zeta_1} \right) \right]. \tag{36}$$

Plugging equation (36) into equation (35), we find

$$\frac{d\zeta_2}{dt} = \frac{1}{\alpha^2 |\zeta_2|^{2\alpha-2}} \left\{ U(t) - \frac{i}{2\pi} \left[\frac{\Gamma_1}{\overline{\overline{\zeta_2}} - \overline{\zeta_1}} - \frac{\Gamma_1}{\overline{\zeta_2} - \zeta_1} \right] + \left[U + \frac{i\Gamma_1}{2\pi} \frac{\zeta_1 - \overline{\zeta_1}}{|\zeta_1|^2} \right] \left[-\frac{|\zeta_1|^2}{4(\mathrm{Im}\zeta_2)^2} + i(\alpha - 1)\frac{\zeta_2}{4\mathrm{Im}\zeta_2} \right] \right\} \tag{37}$$

Since $0 < |\zeta_2| \ll |\zeta_1(t)|$ for $t - t_1 \ll 1$, we can expand the second term inside the braces in equation (37). Defining

$$K_1(t) = U + \frac{i\Gamma_1}{2\pi} \frac{\zeta_1 - \overline{\zeta_1}}{|\zeta_1|^2}, \tag{38}$$

and

$$\epsilon_2 = \frac{1}{\alpha^2 |\zeta_2|^{2\alpha-2}}, \tag{39}$$

we are able to write

$$\frac{d\zeta_2}{dt} = \epsilon_2 \left\{ K_1(t) \left[1 - \frac{|\zeta_2|^2}{4(\mathrm{Im}\zeta_2)^2} + i\frac{(\alpha-1)\zeta_2}{4\mathrm{Im}\zeta_2} \right] - 2\mathrm{Re}(g_1)\overline{\zeta_2} - 2\mathrm{Re}(h_1)\overline{\zeta_2}^2 + o(\zeta_2^3) \right\}, \tag{40}$$

where

$$g_1 = \tfrac{i\Gamma_1}{2\pi\zeta_1^2}, \quad h_1 = \tfrac{i\Gamma_1}{2\pi\zeta_1^3}. \tag{41}$$

4.1 The Size of the Terms in Equation (40)

Recall that we defined

$$K_1(t) = U + \frac{i\Gamma_1}{2\pi} \frac{\zeta_1 - \overline{\zeta_1}}{|\zeta_1|^2} \tag{42}$$

for all times t. From the Kutta condition, $K_1 \equiv 0$ for all $t < t_1$. Therefore,

$$\begin{aligned} \frac{d}{dt}K_1 &= U' + \frac{i\Gamma_1}{2\pi}\frac{d}{dt}\left(\frac{\zeta_1 - \overline{\zeta_1}}{|\zeta_1|^2}\right) + \frac{i\Gamma_1'}{2\pi}\frac{\zeta_1 - \overline{\zeta_1}}{|\zeta_1|^2} \\ &= 0 \end{aligned} \tag{43}$$

for all $t < t_1$. However, t_1 is defined to be the location of the first local extremum of Γ_1 so that $\Gamma_1'(t_1) = 0$. Therefore,

$$\begin{aligned} \lim_{t \to t_1^+} \frac{dK_1}{dt}(t) &= U'(t_1) + \frac{i\Gamma_1(t_1)}{2\pi}\frac{d}{dt}\left(\frac{\zeta_1 - \overline{\zeta_1}}{|\zeta_1|^2}\right)\bigg|_{t=t_1} \\ &= 0, \end{aligned} \tag{44}$$

since U and ζ_1 are continuous functions at $t = t_1$. This means that the Taylor expansion of K_1 for $t > t_1$ is of the form

$$K_1(t) = \frac{k_1}{2}(t - t_1)^2 + o\left((t - t_1)^3\right), \tag{45}$$

where

$$k_1 = \lim_{t \to t_1^+} \frac{d^2 K_1}{dt^2}(t) \tag{46}$$

is, in general, a nonzero constant.

Now if we suppose that $\zeta_2 \sim (t - t_1)^\beta$, equating exponents of the leading terms on both sides of equation (40), we find that

$$\beta - 1 = \min(2, \beta) - 2\beta(\alpha - 1) \tag{47}$$

so that

$$\beta = \begin{cases} \frac{1}{2\alpha - 2} & \alpha \geq \frac{5}{4} \\ \frac{3}{2\alpha - 1} & \alpha \leq \frac{5}{4}. \end{cases} \tag{48}$$

4.2 Case 1: $\alpha > 5/4$

From equation (48), we see that if $\alpha > 5/4$, then to leading order in $(t - t_1)$, equation (40) is

$$\frac{d\zeta_2}{dt} = -\frac{2\mathrm{Re}(g_1(t_1))}{\alpha^2 |\zeta_2|^{2\alpha - 2}} \overline{\zeta_2}. \tag{49}$$

Therefore, setting $\zeta_2 \sim a(t - t_1)^{1/(2\alpha - 2)}$, we find an equation for a:

$$\frac{a}{2\alpha - 2} = -2\frac{\overline{a}}{\alpha^2 |a|^{2\alpha - 2}} \mathrm{Re}(g_1(t_1)). \tag{50}$$

Solving for a, we find

$$a = \left(\frac{4\alpha - 4}{\alpha^2} |\mathrm{Re}(g_1(t_1))|\right)^{2\alpha - 2} \sqrt{sign(-\mathrm{Re}(g_1(t_1)))}. \tag{51}$$

Therefore, if $\mathrm{Re}(g_1) > 0$,

$$\zeta_2 \sim i\left(\frac{4\alpha - 4}{\alpha^2} |\mathrm{Re}(g_1(t_1))|\right)^{2\alpha - 2} (t - t_1)^{1/(2\alpha - 2)} \tag{52}$$

and

$$\Gamma_2 \sim \pi\frac{k_1}{2}\left(\frac{4\alpha - 4}{\alpha^2} |\mathrm{Re}(g_1(t_1))|\right)^{2\alpha - 2} (t - t_1)^{2 + \frac{1}{2\alpha - 2}}. \tag{53}$$

However, if $\mathrm{Re}(g_1(t_1)) < 0$, we would have a contradiction because the vortex would be trying to stay on the wedge so that the boundary condition cannot be satisfied. Therefore, we must determine $sign(\mathrm{Re}(g_1))$. We calculate,

$$\begin{aligned} sign(\mathrm{Re}(g_1)) &= sign\left(\frac{i\Gamma_1}{2\pi\zeta_1^2}\right) \\ &= sign(\Gamma_1)\, sign\left(\mathrm{Im}\left(\zeta_1^2\right)\right). \end{aligned} \tag{54}$$

We know that if $signU > 0$, then ζ_1 is in the first quadrant so that ζ_1^2 is in the upper half plane; however, ζ_1 is in the second quadrant if $signU < 0$, which implies that ζ_1^2 is in the lower half plane. Therefore, $sign\left(\mathrm{Im}\left(\zeta_1^2\right)\right) = signU$, and

$$\begin{aligned} sign\left(\mathrm{Re}\left(g_1\right)\right) &= sign(\Gamma_1) sign(U) \\ &= (sign(U))^2 > 0, \end{aligned} \tag{55}$$

which implies that the second vortex can always be released. Once the vortex moves away from the singular point at the origin, the motion of both vortices can be analyzed by numerically integrating equations (34) and (35).

4.3 Case 2: $\alpha < 5/4$

Setting $\zeta_2 = \xi_2 + i\eta_2$, we see that equation (40) is, to leading order in $(t - t_1)$,

$$\frac{d\zeta_2}{dt} = \frac{1}{\alpha^2 \left|\zeta_2\right|^{2\alpha-2}} \left\{ \frac{k_1}{2} (t-t_1)^2 \left(1 - \frac{\alpha}{4} - \frac{\xi_2^2}{4\eta_2^2} + (\alpha+1)\, i \frac{\xi_2}{\eta_2}\right)\right\}. \tag{56}$$

This is equivalent to equation (15) with ζ_1 replaced with ζ_2 and $U(t)$ replaced with $k_2 (t-t_1)^2/2$. Therefore,

$$\zeta_2 \sim \left[\frac{(\alpha-1)(2\alpha-1)}{12\alpha^{5/2}} \left|k_1\right| (t-t_1)^3\right]^{\frac{1}{2\alpha-1}} \left(\frac{\sqrt{\alpha}}{2} i + sign(k_1) \frac{\sqrt{4-\alpha}}{2}\right), \tag{57}$$

and

$$\Gamma_2 \sim \frac{\pi k_1 (t-t_1)^2}{\sqrt{\alpha}} \left[\frac{(\alpha-1)(2\alpha-1)}{12\alpha^{5/2}} \left|k_1\right| (t-t_1)^3\right]^{\frac{1}{2\alpha-1}} \tag{58}$$

shortly after the second vortex is created. After it moves away from the origin, we can again use numerical integration to explore the dynamics of the system.

5 More Vortices for the Infinite Wedge

Now suppose that at some $t_2 > t_1$, the magnitude of the second vortex reaches a maximum. If we again assume that, once this maximum is reached, the strength of the second vortex is fixed, a third vortex must be shed from the origin to satisfy the Kutta condition. Furthermore, under certain conditions which will be discussed in this section, this process could repeat itself several (perhaps, infinitely many) more times. Therefore, for the rest of the section let us suppose that n vortices have been released from the origin and they are located at $z_1, z_2, \ldots, z_n$ respectively in the physical plane (corresponding to $\zeta_1, \zeta_2, \ldots, \zeta_n$ in the image plane). We further assume that each of the ζ_j are nonzero and that there exist times $0 < t_1 < t_2 < \cdots < t_n$ at which each of the n vortices have reached a maximum, respectively. We now consider the dynamics as the next vortex is released, i.e. for $t > t_n$.

Using equation (11) for a system of $n+1$ vortices and equation (14) we have for the newest vortex

$$\frac{d\zeta_{n+1}}{dt} = \epsilon_{n+1} \left\{ U(t) - \frac{i}{2\pi} \sum_{k=1}^{n} \left[\frac{\Gamma_k}{\overline{\zeta_{n+1} - \zeta_k}} - \frac{\Gamma_k}{\overline{\zeta_{n+1}} - \zeta_k}\right] - \frac{i\Gamma_{n+1}}{2\pi\left(\zeta_{n+1} - \overline{\zeta_{n+1}}\right)} + i\frac{(\alpha-1)\Gamma_{n+1}}{4\pi\overline{\zeta_{n+1}}}\right\}, \tag{59}$$

while for each j, $1 \leq j \leq n$,

$$\frac{d\zeta_j}{dt} = \epsilon_j \left\{ U\left(t\right) - \frac{i}{2\pi} \sum_{k \neq j}^{n+1} \left[\frac{\Gamma_k}{\overline{\zeta_j} - \overline{\zeta_k}} - \frac{\Gamma_k}{\overline{\zeta_j} - \zeta_k} \right] - \frac{i\Gamma_j}{2\pi \left(\zeta_j - \overline{\zeta_j}\right)} + i \frac{\left(\alpha - 1\right) \Gamma_j}{4\pi \overline{\zeta_j}} \right\}, \tag{60}$$

where we have defined, in agreement with section 4,

$$\epsilon_l = \frac{1}{\alpha^2 \left|\zeta_l\right|^{2\alpha - 2}} \tag{61}$$

for each l, $1 \leq l \leq n+1$. These equations must be solved with the initial conditions

$$\zeta_{n+1}\left(t_n\right) = 0, \quad \Gamma_{n+1}\left(t_n\right) = 0, \tag{62}$$

for the new vortex and that each of the other vortices must move in a continuous manner, holding their strength constant. From the Kutta Condition, i.e. equation (7) with $n+1$ vortices, we can solve for Γ_{n+1}, finding

$$\Gamma_{n+1} = 2\pi i \frac{\left|\zeta_{n+1}\right|^2}{\zeta_{n+1} - \overline{\zeta_{n+1}}} \left[U\left(t\right) + \sum_{k=1}^{n} \frac{i\Gamma_k}{2\pi} \left(\frac{1}{\overline{\zeta_k}} - \frac{1}{\zeta_k} \right) \right]. \tag{63}$$

We can substitute equation (63) into equation (59) and expand the resulting equation to leading order in ζ_{n+1} since, for $t - t_n \ll 1$, $\zeta_{n+1} \ll \zeta_j$, for all $1 \leq j \leq n$. The resulting equation of motion is

$$\frac{d\zeta_{n+1}}{dt} = \epsilon_{n+1} \left\{ K_n\left(t\right) \left[1 - \frac{\left|\zeta_{n+1}\right|^2}{4\left(\mathrm{Im}\zeta_{n+1}\right)^2} + i \frac{\left(\alpha - 1\right)\zeta_{n+1}}{4\mathrm{Im}\zeta_{n+1}} \right] - 2\mathrm{Re}\left(g_n\right) \overline{\zeta_{n+1}} - 2\mathrm{Re}\left(h_n\right) \overline{\zeta_{n+1}}^2 + o(\zeta_{n+1}^3) \right\}, \tag{64}$$

with

$$K_n\left(t\right) = U\left(t\right) + \sum_{k=1}^{n} \frac{i\Gamma_k}{2\pi} \frac{\zeta_k - \overline{\zeta_k}}{\left|\zeta_k\right|^2}, \tag{65}$$

and

$$g_n = \textstyle\sum_{k=1}^{n} \frac{i\Gamma_k}{2\pi\zeta_k^2}, \quad h_n = \sum_{k=1}^{n} \frac{i\Gamma_k}{2\pi\zeta_k^3}. \tag{66}$$

Note that equation (64) has exactly the same form as equation (40). Furthermore, the same reasoning used in section 4.1 to show that $K_1 \sim \left(t - t_1\right)^2$ can be reapplied to show that

$$K_n\left(t\right) \sim \frac{k_n}{2}\left(t - t_n\right)^2 \tag{67}$$

for $t - t_n \ll 1$ with

$$k_n = U'\left(t_n\right) + \frac{d}{dt}\left(\sum_{k=1}^{n} \frac{i\Gamma_k(t_n)}{2\pi} \frac{\zeta_k - \overline{\zeta_k}}{\left|\zeta_k\right|^2} \right)\Bigg|_{t=t_n}. \tag{68}$$

Supposing that $\zeta_{n+1} \sim \left(t - t_n\right)^\beta$, equating the exponents of the leading terms in equation (64) reveals that

$$\beta = \begin{cases} \frac{1}{2\alpha - 2} & \alpha \geq \frac{5}{4} \\ \frac{3}{2\alpha - 1} & \alpha \leq \frac{5}{4} \end{cases} \tag{69}$$

as before.

5.1 Revisiting $\alpha > 5/4$

Following section 4.2, we find that setting $\zeta_{n+1} \sim a\,(t - t_n)^{1/(2\alpha-2)}$ and substituting in equation (64) gives us an equation for a which can be solved to give

$$a = \left(\frac{4\alpha - 4}{\alpha^2} \left|\mathrm{Re}(g_n\,(t_n))\right|\right)^{2\alpha-2} \sqrt{sign(-\mathrm{Re}(g_n(t_n)))} \tag{70}$$

which still leaves us with a contradiction if $\mathrm{Re}(g_n(t_n)) < 0$. However, with more than two vortices it is actually possible for this contradiction to manifest itself. In fact, for the infinite line ($\alpha = 2$), considered by Cortelezzi and others, and $U = \sin t$, this contradiction arises at t_2, so that the third vortex cannot be shed. If we choose

$$U\,(t) = \begin{cases} \frac{2-2\pi\tau}{\tau}t + \frac{2\pi\tau-1}{\tau^2}t^2, & t \leq \tau \\ 1 + \frac{1}{10}\cos(20\pi t), & t > \tau, \end{cases} \tag{71}$$

with $\tau = .075$ following Cortelezzi [5], $g_n\,(t_n)$ is always positive so that vortices can be shed indefinitely.

5.2 Vorticity Dipoles

To handle the contradiction just discussed, we must consider what happens when vorticity is held near the boundary. The contradiction arises because the newly shed vortex and its image are at the same point in space when the vortex is on the boundary. Rott [1] suggests that one possibility would be to consider a boundary layer of vorticity around the body. A simpler idea is to treat the vortex and its image together as a vorticity dipole with strength $\vec{D}$ chosen to be parallel to the boundary. However, this approximation has a few complications. First, the strength of the dipole is not the same in the physical and mapped planes so that the shedding condition either cannot be treated as only dependent on variables in the mapped plane or does not correspond to the strength of the dipole reaching a maximum. Second, there is some ambiguity as to which direction along the boundary the dipole travels; that is, two possible trajectories appear to satisfy the Kutta condition at the origin and the equations of motion. These complications have not been worked out yet and will be the subject of future work.

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