GFD 2012 Lecture 2: Coherent Structures in 2D Fluid Dynamics

Jeffrey B. Weiss; notes by Yuan Guo and Bevin Maultsby

June 19, 2012

1 Introduction

Structures in the atmosphere and ocean such as hurricanes, storms and the path of jet streams that have large human impact like They provide insight into better sub-grid-scale parameterization. Most structures cannot be derived from underlying PDE's and to learn about their details requires observations and numerical simulations. There are several interesting and important structures present in two-dimensional fluid dynamics. One major motivation for examining such 2D fluid dynamics is to further understand the anisotropy that is present in atmosphere and ocean. These lecture notes are organized as follows:

In section 2, we introduce the point vortex model. This idealized model provides mathematical and physical insight. The equations for point vortex dynamics define a Hamiltonian system. However, their singular nature gives no insight into the dynamics of vortex shape, and filters out processes that depend on the shape dynamics.

Thus in section 3, we present a model of compact and well-separated vortex. The vortex moments are defined and the equations of the motion of the centroid are given. Asymptotic analysis leads to an infinite system of coupled ordinary differential equations for physical-space moments of the individual regions. If truncated to a finite number of moments, a self-consistent closed model is obtained at any order. Nonzero 2nd order moments yields the "elliptical moment model".

In section 4, we examine an important process in fluid mechanics: same-sign vortex merger. A threshold for the merger of equal-sized vortices is given. We also discuss the situation when diffusion is present, as well as the interesting phenomenon of the onset of chaotic motion in the elliptical moment model. Statistical mechanics give some predictions, but there are some constrains in actual fluids that prevent vortices from exploring the phase space as described by statistical mechanics.

In section 5, we briefly introduce the cascade theory of turbulence, focusing on 2D turbulence. We describe four different types of vortex interactions in 2d decaying turbulence: two-vortex merger, dipole propagation, vortex scattering, and tripole merger. The conservation laws of energy and enstrophy are crucial in the study of cascade theory. We also examine the relationship of cascade and vortex merger and the role of energy and enstrophy fluxes.

In section 6, we look into structure-based temporal scaling theory, which is quite different from traditional cascade theory. Our goal is to construct scaling theory guided by vortex statistics from numerical simulations. We make several assumptions in the theory resulting in relationships between the exponents of the power law evolution of vortex properties.

2 Point Vortices

First we will examine point vertices in greater detail; the setting here is inviscid dynamics. A point vortex system is closed: the vorticity of the system will always remain concentrated at the point vortices. This no longer holds, however, with the addition of viscosity, which causes the vorticity to spread out across the system. We will examine this case later.

Recall, the vorticity evolution equations

$$\begin{aligned} \partial_t \omega + J[\psi, \omega] &= \nu \nabla^2 \omega \\ \omega &= \nabla^2 \psi, \end{aligned}$$

while the definition of a point vortex is

$$\omega(\vec{x}) = \sum_{i=1}^{N} \Gamma_i \delta(\vec{x} - \vec{x}_i).$$

This expression may depend on time t. From the above, we obtain

$$\frac{D\omega_t}{Dt} = \sum_{i=1}^{N} [\partial_t \Gamma_i \delta(\vec{x} - \vec{x}_i) - \Gamma_i \nabla \delta(\vec{x} - \vec{x}_i) \cdot \dot{\vec{x}}_i] + \sum_{i=1}^{N} \vec{U} \cdot \nabla(\Gamma_i \delta(\vec{x} - \vec{x}_i)) = 0.$$

The gradient of a delta-function is infinite. Reinterpreting these equations for a small but finite-size Gaussian vortex of size r_0 renders these terms $O(1/r_0)$. Considering only the order one term yields

$$\partial_t \Gamma_i = 0,$$

while the $O(1/r_0)$ term gives

$$\dot{\vec{x}}_i = \vec{U}(\vec{x}_i),$$

and the point vortex is advected by the velocity field.

Another important fact to observe is that a collection of point vortices is Hamiltonian. Denoting the Hamiltonian by H, we can write

$$H = -\sum_{i \neq j} \frac{\Gamma_i \Gamma_j}{4\pi} \ln |\vec{x}_i - \vec{x}_j|.$$

(In the above, the natural logarithm is a result of Green's function for the Laplacian in two dimensions.) The equations of motion for this system are

$$\Gamma_i \begin{pmatrix} \dot{x}_i \\ \dot{y}_i \end{pmatrix} = \begin{pmatrix} -\dfrac{\partial H}{\partial y_i} \\ \dfrac{\partial H}{\partial x_i} \end{pmatrix}.$$

A physical interpretation of the Hamiltonian is that of the "interaction energy for an infinite domain."

As this is a Hamiltonian system, we can observe both regular and chaotic dynamics. Depending on the boundary conditions, there may be other invariants as well, such as translation symmetries that yield linear momentum conservation, and rotation symmetries that yield angular momentum conservation. Exploration in this direction reveals that a system on an infinite domain with three vortices is regular, while a system with four is chaotic.

3 Elliptical vortices

Elliptical vortices exhibit particular interesting behavior, and there is a lot of literature on this subject. One of the first papers on the subject is the 1986 paper on vortex interactions by Melander, Zabusky and Styczek [1].

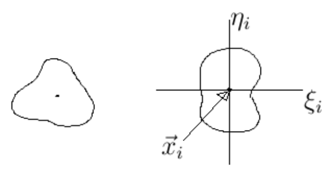

Figure 1: Two compact, separated vortices.

In this setting we are considering compact, well separated vortices, such as the two vortices above. In such systems, the vorticity is zero outside of these compact structures. Each vortex has its own constant vorticity ω_i, area A_i, and circulation Γ_i, defined by

$$\Gamma_i := \omega_i A_i.$$

As shown in the elliptical vortex on the right in the above illustration, we can put local coordinates (ξ_i, η_i) on each vortex. The centroid $\vec{x}_i$ of each vortex is then given by

$$A_i \vec{x}_i = \int_{\text{vortex } i} \vec{x}\, d\xi_i\, d\eta_i$$

Each vortex also has moments, which are given by

$$J_i^{(m,n)} = \int_{\text{vortex } i} \xi_i^m \eta_i^n\, d\xi_i\, d\eta_i.$$

We can glean various bits of information by choosing m and n in specific ways. For example, setting $m = n = 0$ yields the area of the vortex. If $m = 1$ and $n = 0$, we obtain the x component of the centroid, while if $m = 0$ and $n = 1$, we obtain the y component of the centroid. If $m = 2$ and $n = 0$, we obtain the variance in ξ of the size of the vortex, while if $m + n = 2$ then we obtain the covariance matrix for the size of the vortex. The sum $m + n$ defines the order of moments.

There are a couple of assumptions which are key to the development of these ideas. In particular, we have a small parameter

$$\varepsilon = \frac{\text{size of a vortex}}{\text{separation between vortices}} << 1.$$

In other words, we assume that the separation between vortices is $O(1)$, with the size of each vortex much smaller. As the size of each vortex is small, the moments bring in power of ε:

$$J^{(m,n)} \sim O(\varepsilon^{m+n+2}),$$

where the additional 2 in the exponent comes from the integration in the definition of J_i. Therefore, each higher moment is higher order in ε than lower order moments.

The above definitions are focused on a single vortex within the system. We have not yet addressed the interaction of the vortices. It is important to note that we do not need any assumption that vortices maintain the same shape; instead we describe vortices in terms of their moments.

Since the area A_i is constant, the motion of the centroids is given by

$$\begin{aligned} A_i \dot{\vec{x}}_i &= \frac{d}{dt} \int_{\text{vortex } i} \vec{x}\, d\xi_i\, d\eta_i \\ &= \int_{\text{vortex } i} \vec{U}(\vec{x})\, d\xi_i\, d\eta_i. \end{aligned}$$

The velocity $\vec{U}(\vec{x})$ is the velocity induced by all of the other vortices, which we recall are far away from our basic assumption on the separation distance between vortices. Asymptotic analysis can show that in the same way that point vortices do not have any self advection, the centroid position does not have any self advection.

We can now Taylor expand the velocity $\vec{U}(\vec{x})$ around the centroid of a vortex. Here $\vec{x}_i$ is in vortex i, and its velocity induced by vortex j becomes

$$\dot{\vec{x}}_i = \frac{1}{A_i} \sum_{q=0}^{\infty} \sum_{p=0}^{q} \frac{1}{p!(q-p)!} J_i^{(p,q-p)} \frac{\partial^p}{\partial x^p} \frac{\partial^{q-p}}{\partial y^{q-p}} \vec{U} \bigg|_{\vec{x}_i}.$$

Recall that

$$U(\vec{x}) = \sum_j U_j(\vec{x}),$$

where $U_j(\vec{x})$ is the velocity induced by vortex j at $\vec{x}$.

One can repeat similar steps to get an equation for $\frac{\partial}{\partial t} J_i^{(m,n)}$, from which one can perform asymptotic analysis. We do not go into details here, see Melander *et al.* 1986 [1] for more information.

A result of this analysis is that we can truncate at any order and get a closed system. In other words, if we keep all moments through the k^{th} order in the initial condition, then the time evolution of those moments up through the k^{th} order will not generate moments of order greater than k.

If we keep only the centroid, then this is exactly the point vortex model. The first order moment is zero by definition. For the second order moments, we need $m + n = 2$. In other words, we set the pair (m, n) to be $(2, 0)$, $(1, 1)$, and $(0, 2)$.

Here we can get a good idea of what it means for a system of point vortices to be closed. If we truncate at order two, then closure implies that all higher order moments depend completely on the 2nd order moments. As long as the second order moments are nonzero, this yields an ellipse, called the "elliptical moment model." In addition to the position and location parameters we have already seen, the elliptical model yields two additional parameters, illustrated below.

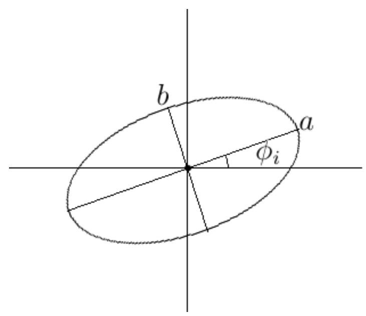

Figure 2: Two additional parameters for the elliptical moment model.

One of the additional parameter is the angle ϕ_i, while the other parameter is the ellipticity ratio $\frac{a}{b}$ of lengths of the major and minor axes. These parameters completely describe the ellipse. The result is a Hamiltonian dynamical system with four degrees of freedom (as the area is fixed).

4 Same-sign vortex merger

Same-sign point vortices co-rotate. We will see that two finite-size vortices with the same sign which are "close enough" will merge together. Such same-sign merger can be illustrated with numerical simulations and observed in laboratory experiments.

There is a threshold for merger which can make this notion of "close enough" more precise. In general, if the separation between two vortices of equal size is larger than 3.3 times the radius of those vortices, then they will rotate around each other for all t and never merge. However, their behavior changes dramatically once the separation distance is less than 3.3 times their radius; in this case, they will merge quite quickly.

If there is diffusion present in the system, then vortices will alway merge. This behavior happens because the radii will grow slowly on a diffusive time scale. As a result, the ratio between their radii and the separation is eventually small enough to cross the threshold.

One interesting phenomenon is the onset of chaos in an elliptical model. If a system begins with two elliptical vortices whose separation is above the threshold, as expected they rotate around each other. Visually, they appear to wiggle as their ellipticity rotates, and they become essentially circular. If however, their separation is below the threshold, the distance between their centers quickly collapses. As a result, the ellipticity of the system

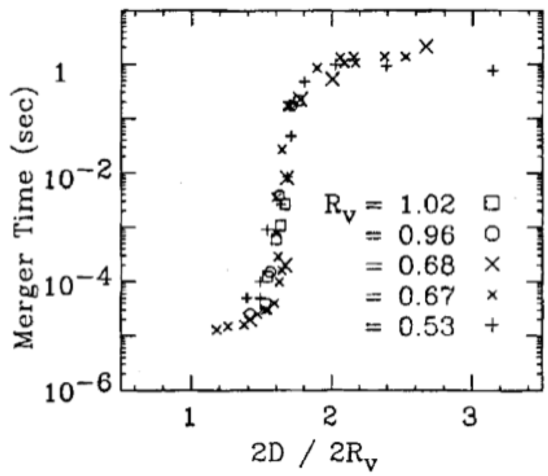

Figure 3: Measured two-vortex merger time as a function of separation $2D$ normalized to vortex diameter $2R_v$, by C. F. Driscoll, *et al.* 1991 [4].

becomes infinite. As the equation for the model has a term of $\frac{1}{r}$, where r denotes the separation between the vortices, the model blows up. This singular behavior is analogous to vortex merger.

There are several theories that give *correlations* with merger, yet do not seem to fully explain the threshold. One can argue that the threshold is still something of a mystery. It is interesting to note that statistical mechanics predicts that vortices always merge if the system is allowed to fully explore the phase space. According to this prediction, a system with an initial condition of two vortices will always result in a single vortex as the most probable state. However, there are constraints in the actual fluid dynamics that prevent vortices from exploring the phase space as described by statistical mechanics. But if the fluid dynamics allows the vortices to merge, then statistical mechanics calculations give the correct predictions.

We can connect the ideas of the last three sections with the following observation. Point vortices do not exhibit merger at all, whereas the elliptical moment model, which allows shape oscillations gives the signature for vortex merger. Hence the elliptical model is the simplest inviscid model that provides information about this important dissipative process.

5 Cascade Theory

5.1 2d turbulence cascades

We now give a brief overview of two-dimensional turbulence cascades. First we note that fluids in this context have small viscosity v, and large dimensionless Reynolds number given

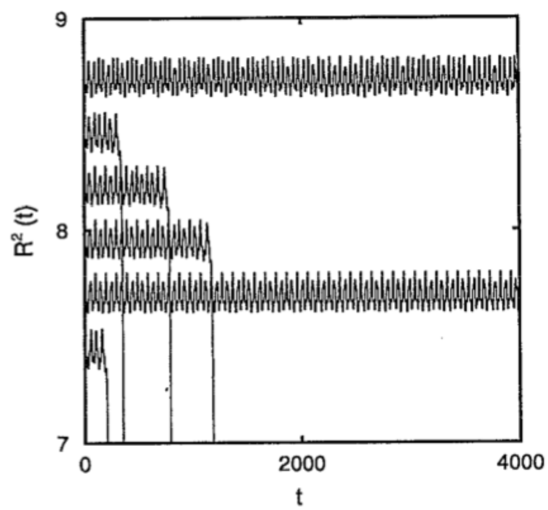

Figure 4: Vortex separation squared $R^2(t)$ from the elliptical-moment model for initially circular vortices with $\Gamma_1 = \pi/4$, $\Gamma_2 = \pi$ and initial separations $R_0 = 2.746 - 2.751$ in steps of 0.001. The trajectories are successively offset by $\triangle R^2 = 0.25$, by J. B. Weiss and J. C. McWilliams, 1993 [2].

by

$$\mathrm{Re} = \frac{UL}{v} >> 1,$$

where U is the velocity and L is the length scale. From the Laplacian dissipation, we get a term of

$$-\frac{k^2}{\mathrm{Re}},$$

in wavenumber space, where k is the wavenumber.

Classical cascade theories are based on physical models of how energy flows through wavenumber space coupled with dimensional analysis of Navier-Stokes equations. There are a few typical assumptions for cascade theory. The first is to assume that a system has some forcing concentrated along a particular forcing scale, k_f. Due to the form of the dissipation operator, dissipation occurs at large wavenumbers. There is an inertial range (perhaps more than one inertial range, in fact) determined by values of k where forcing and dissipation are both small. Lastly, the fundamental concept of local cascade theory is that energy is transferred locally in scale.

5.2 3D Cascades

When we consider 3d homogeneous isotropic turbulence, we assume that we have forcing at large scales and dissipation at small scales.

Let ε represent the energy flux of the system. If we assume that we are in a statistically steady state, then ε moves from large scales to small scales. In a statistically steady state

ε for a turbulent cascade must be constant in inertial range as there is no large forcing or dissipation terms. Dimensional analysis yields the Kolmogorov scaling relation

$$E(k) \sim k^{-5/3},$$

which represents a direct transfer from large scales to small scales.

5.3 2D Cascades

5.3.1 Conservation of Energy and Enstrophy

Two-dimensional homogeneous isotropic turbulence is different from the 3D case because there is conservation of enstrophy. Denote enstrophy by Z, then

$$Z = \frac{1}{2}\int d^2\vec{x}|\vec{\omega}|^2.$$

Statistically, enstrophy represents the mean square of the vorticity. It can also be seen though as analogous to the kinetic energy of the system.

Now let us examine the role that energy conservation plays. First, note that the time derivative of energy E depends on the enstrophy:

$$\frac{dE}{dt} = -2\nu Z.$$

A fact about 3d turbulence is that the energy dissipation is constant as the viscosity goes to zero because the enstophy of the system grows. The resulting vortex stretching is crucial in 3D. The major difference when we switch to 2D is the lack of a vortex stretching term. (Later, we will see quasigeostrophic dynamics in 3D that will resemble 2D as there is a similar lack of vortex stretching.)

The time derivative of the enstrophy is

$$\frac{dZ}{dt} = \left\langle \omega_i\omega_j\frac{\partial U_i}{\partial x_j}\right\rangle - 2\nu P,$$

where $\omega_i\omega_j$ is the vortex stretching term, and P is the "palinstrophy" given by

$$P = \frac{1}{2}\int dx|\nabla\times\vec{\omega}|^2.$$

One may wonder why the vortex stretching term $\omega_i\omega_j$ is zero in two dimensions. This is due to the fact that the vorticity and the velocity are always in perpendicular directions, which prevents vortex stretching. As a result, the enstrophy time derivative is always negative. Taking the limit as viscosity vanishes, we see that the enstrophy cannot grow. Hence the energy is conserved in 2D.

5.3.2 Cascade Theory and Mergers

Now we examine what happens if we add energy and enstrophy fluxes to the system. Due to the conservation of energy, there are separate energy and enstrophy cascade regimes with cascades in opposite directions. Energy cascades to large scales, while enstrophy cascades to small scales, where it then dissipates. Dimensional analysis gives the slope of spectrum in these cascades, and the inverse energy cascade results in large scale structures.

Several typical numerical simulations of decaying 2d turbulence have shown that in relatively short times periods, individual vortices self organize into a collection of coherent vortices. Afterwards, the vortices advect each other around, and same-sign vortices merge, whence there are fewer vortices. After a much longer period of time, the system ends with a dipole which slowly decays through diffusion.

The dominant dissipative mechanism in decaying 2d turbulence is vortex merger. Most merger events are two-vortex mergers, which are often catalyzed by a third vortex, but occasionally three-vortex mergers occur. Conservative vortex interactions include dipole propagation and vortex scattering. The idea of scattering is illustrated in the following example: imagine two dipoles, A translating with opposite sign vortex A', and B translating with opposite-sign vortex B', and the dipoles propagate to bring them close together. When A and B are close together, they may be near an unstable co-rotating periodic state. Then they will "switch partners" so that A is now rotating with B, and A' is rotating with B'. The trajectories of these new pairs depart near unstable manifolds of the are near unstable orbits of the periodic state. Varying the impact parameters, which governs how the pairs approach, changes how close the incoming dipoles are the stable manifold of the periodic orbit. The closer the dipoles approach the periodic orbit, the longer they remain in its vicinity, and the more they co-rotate before leaving its neighborhood. In this case, the angle at which they exit becomes sensitive to the impact parameter and is unpredictable. This example is a case of "chaotic scattering."

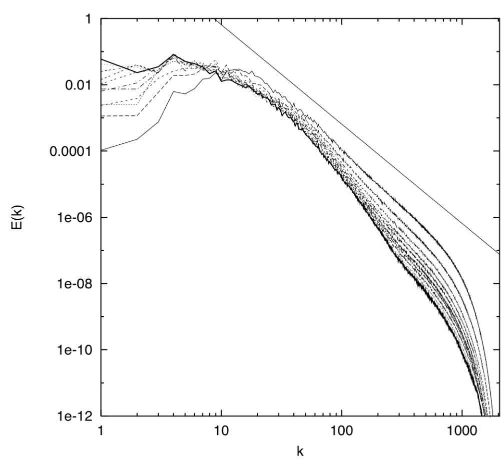

Figure 5: Energy spectra for the solution at high Re for times t=1,2,...,11. Solid line shows the k^{-3} classical prediction, by A. Bracco *et al.* 2000 [4].

6 Structure-Based Temporal Scaling Theory

Structure-based scaling theory address the properties of the vortex population, as well as the global quantities like energy and enstrophy. This differs from traditional cascade theory which ignores the coherent structures. Simulations of 2d decaying turbulence exhibit spectra that are steeper and an enstrophy time decay that is slower than cascade theory predicts.

In numerical simulations, we see three phases: vortex formation, vortex interaction, and the final dipole. The first phase is poorly understood compared to the vortex interaction phase. The goal here is to construct a scaling theory guided by vortex statistics from numerical simulations. First, however, we must measure vortex statistics, which is inevitably based on a subjective census algorithm (as there is no entirely precise definition of a vortex). The output this census is the number of vortices and the distribution of their properties, such as vortex size and enstrophy, over the course of the simulation.

6.1 Vortex Scaling Theory

For scaling theory we have a few assumptions. The first assumption, based on inviscid dynamics, is that energy is conserved. Let ω_p denote the peak vorticity, as illustrated below.

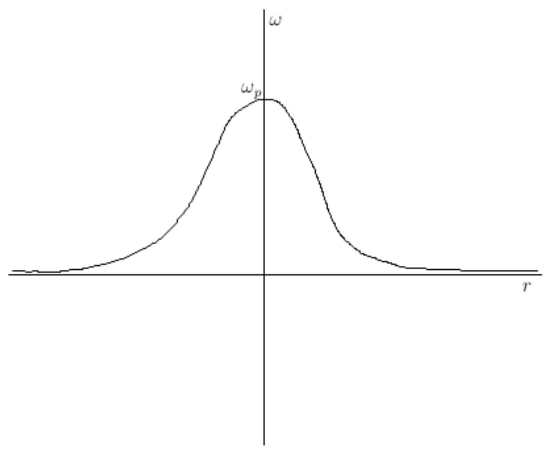

Figure 6: Typical vortex shape indicating the peak vorticity.

The second assumption, also based on inviscid dynamics, is that ω_p is conserved. Moreover, we assume that all of the vorticity is inside the coherent, well-formed vortices and the vorticity outside these vortices is zero.

Observations from numerical simulations show that the number of vortices N decays with a power law:

$$N \sim t^{-\xi}, \qquad \xi \approx 0.72.$$

Each individual vortex is characterized by a location $\vec{x}_i$, a size r_i (the notion of size can be made precise), and a vorticity ω_i. The population of vortices is then characterized by a probability density function

$$p(r, \omega, t),$$

which represents the probability if finding a vortex of size r and with vorticity ω at time t.

For convenience we assume that r and ω are independent. With this assumption we can write

$$p(r, \omega, t) = p_r(r, t) p_\omega(\omega, t).$$

In fact this assumption is not necessary, but it simplifies the following equations, as we reduce the number of variables from two to one.

The final assumption is that the probability density functions evolve self-similarly. In other words, the time dependence of the pdf depends only on the time dependence of the average

$$\langle r \rangle (t).$$

Thus if we define a new variable

$$X := \frac{r}{\langle r \rangle (t)},$$

then the assumption is that $p(X)$ is independent of time. (This could be done for the joint distribution as well.)

The assumption of self-similarity allows one to write the moments of the pdf in terms of the average. In particular, the average of the n^{th} power of the radius is equal to the n^{th} power of the average multiplied by a constant depending on n:

$$\langle r^n \rangle (t) = c_n \langle r \rangle^n (t).$$

Thus the time dependence of all moments can be related to the time dependence of the average.

References

[1] M. V. Melander, N. J. Zabusky and A. S. Styczek, 1986. A moment model for vortex interactions of the two-dimensional Euler equations. Part 1. Computational validation of a Hamiltonian elliptical representation. *J. Fluid Mech.* 167, pp 95-115.

[2] J. B. Weiss and J. C. McWilliams, 1993. Temporal Scaling Behavior of Decaying Two-Dimensional Turbulence. *Phys. Fluids A*, Vol. 5, No. 3, 608.

[3] G. F. Carnevale, J. C. McWilliams, Y. Pomeau, J. B. Weiss, and W. R. Young, 1991. Evolution of Vortex Statistics in Two-Dimensional Turbulence. *Physical Review Letters*, 66, 2735.

[4] A. Bracco, J. C. McWilliams, G. Murante, A. Provenzale and J. B. Weiss, 2000. Revisiting 2D Turbulence at Millenial Resolution. *Physics of Fluids*, 12, 2931-2941.

[5] K. S. Fine, C. F. Driscoll, J. H. Malmberg, and T. B. Mitchell, 1991. Measurements of Symmetric Vortex Merger. *Physical Review Letters*, 67, 588-591.