Laboratory Experiments on Two Coalescing Axisymmetric Turbulent Plumes in a Rotating Fluid

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1 Introduction

Turbulent plumes from a point source can be observed at various scale. Eruption of a volcano, smoke from a chimney, and seafloor vents are good examples. Such kind of plumes are called axisymmetric turbulent plumes. The presence of ambient rotation and/or stratification affects the behavior of axisymmetric turbulent plumes, and various experiments were carried out (see Table 1 of Bush and Woods (1999) for a summary).

Fernando, Chen, and Ayotte (1998; FCA hereafter) investigated the development of a single axisymmetric turbulent plume in a homogeneous, rotating fluid. They showed experimentally that the rotational effects become important after the plume descends a vertical distance $z_f \approx 3.3(F_0/\Omega^3)^{\frac{1}{4}}$ for a time $0.75T_f$. Here F_0 is the buoyancy flux, Ω the angular velocity of background rotation, and $T_f = \pi/\Omega$ the inertial period. They also showed that the plume's front descends as $z \approx 1.7F_0^{\frac{1}{4}}t^{\frac{3}{4}}$ for $t \leq 0.75T_f$ and $z \approx 1.8F_0^{\frac{1}{4}}t^{\frac{3}{4}}$ for $t > 0.75T_f$.

Meanwhile, the coalescence of axisymmetric turbulent plumes to form a single plume is an interesting problem. Kaye and Linden (2004; KL hereafter) investigated this problem in a non-rotating, homogeneous fluid. They introduced a theoretical model for merging height of two coalescing axisymmetric turbulent plumes in a homogenous fluid. They considered both equal and unequal plumes' cases. Their theory for equal plumes found that the relation between the separation length x_0 and the merging height z_m is $z_m \approx (0.44/\alpha)x_0$, where α is the entrainment constant. They also carried out laboratory experiments and showed that their model is qualitatively correct but over predicts the merging height slightly. However, they mentioned that a quantitative prediction is highly dependent on the value of entrainment constant.

In this study, we investigate the behavior of two coalescing plumes in a homogeneous, rotating fluid. During my project at the GFD Summer School, three kinds of experiments were carried out: a filling tank experiment (Baines and Turner, 1969), two plumes experiments without rotation (KL), and two plumes experiments with rotation. The filling tank experiment was carried out to determine the values of the virtual origin height and the entrainment constant; this is explained in appendix A. The derivation of the theory of KL is explained in appendix B.

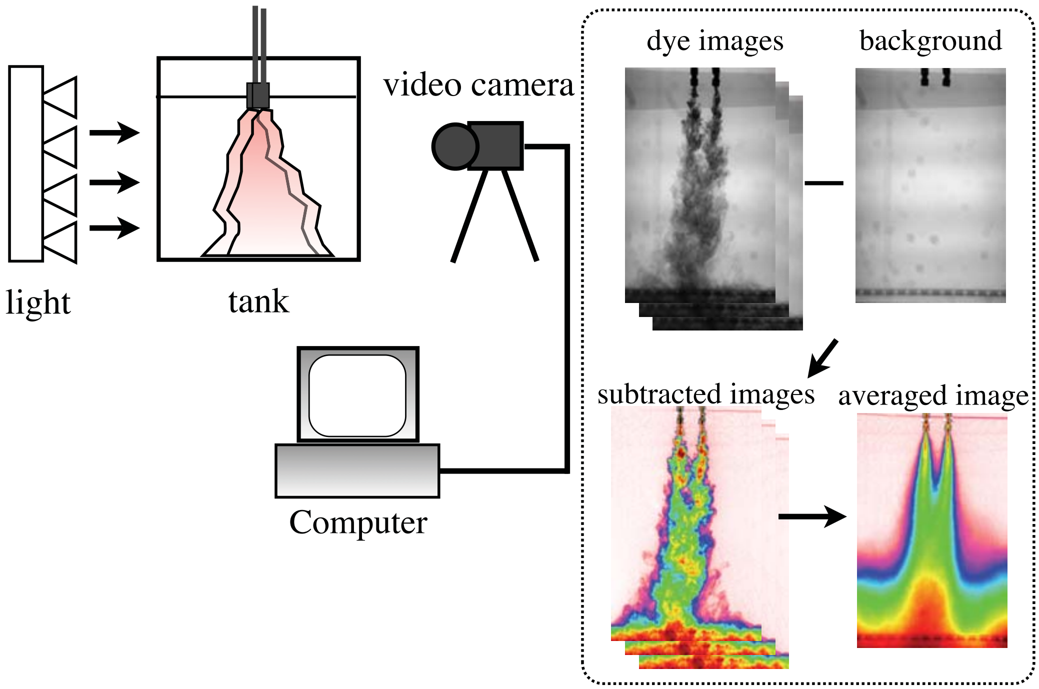

Figure 1: Schematic illustration of the dye attenuation technique.

2 Experimental set up

A transparent tank which has 60×60 cm^2 cross section was used. Fresh water (1.0 g/cm^3) was filled up to 45 cm depth for the ambient fluid, and dyed sea water (1.25 g/cm^3) was used for the fluid of the axisymmetric turbulent plume. The plume was generated by 5 mm diameter Cooper nozzle which was positioned just below the surface, and its constant flow rate was 1.7 cm^3/s, namely the buoyancy flux was $F_0 = 41$ cm^4/s^3.

A dye attenuation technique was used to visualize the plumes' behavior. Figure 1 shows a schematic illustration of this technique. A sequence of dye images were captured with side-view video recordings and the background image was subtracted from them using Digiflow. The color tone of the subtracted images shows the snapshots of distribution of dye concentration (i.e. distribution of buoyancy). This is because the intensity of the light recorded is associated with the dye concentration. Then the sequence of subtracted images were averaged to obtain the image of average distribution of buoyancy. The time interval between dye images is shown in table 1. The averaging period was 2 minutes for the non-rotating cases and $0.25T_f$ for the rotating cases. See KL for details of this technique.

For accuracy, we wrote and used an algorithm to determine the merging height. Our algorithm works as follows.

1. Smooths horizontally the time averaged image by box averaging every 7 pixels (if $x_0 > 6$ cm, 13 pixels) to remove noise.

2. Finds a large slope of concentration ($dc/dx > \delta$; c is concentration) starting from the left.

$f = 2\Omega$ (s^{-1})	time interval (s)
0 ($x_0 < 6$ cm)	1
0 ($x_0 > 6$ cm)	0.25
0.05	1
0.1	0.5
0.25	0.25
0.5	0.1
0.75	0.1
1	0.1

Table 1: Time interval between dye images.

3. From this point, finds the location where the slope is negative ($dc/dx < 0$), and defines this point as x_1.

4. Finds a large slope of concentration ($dc/dx < -\delta$) starting from the right.

5. From this point, finds the location where the slope is positive ($dc/dx > 0$), and defines this point as x_2.

6. Comparers x_1 with x_2 at every height, then $x_1 > x_2$ indicates that plumes are merged, $x_1 < x_2$ indicates that plumes are not merged.

In order to visualize the vortices generated by the plumes in the rotating experiments, the free surface was colored with powder dyes. The evolution of the resulting vortices were recorded with top-view video recordings. Note that it is reasonable to consider that the vortices are barotropic, because of the absence of stratification.

For the non-rotating cases, 23 experiments were carried out, and the separation length x_0 varied between 2.4 and 10.3 cm.

For the rotating cases, the Coriolis parameter $f = 2\Omega$, where Ω is the angular velocity of the background rotation, and the separation length x_0 are important parameters. We carried out rotating experiments with combinations of $f = 0.05, 0.1, 0.25, 0.5, 0.75, 1.0$ s^{-1} and $x_0 \approx 3, 5, 8, 10$ cm.

3 Experimental results

3.1 Without rotation

A summary of the two plumes experiments without rotation is shown in figure 2. The solid line and dashed line in figure 2 are the linear fit of our experimental data and the theoretical prediction, $z_m = (0.44/\alpha)x_0$, from the model of KL, respectively. Note that z_m is the merging height measured from the virtual origin. The virtual origin height $z_V = 1.0$ cm and the entrainment constant $\alpha = 0.12$ were determined by the filling tank experiment (Baines and Turner, 1969); see appendix A for details.

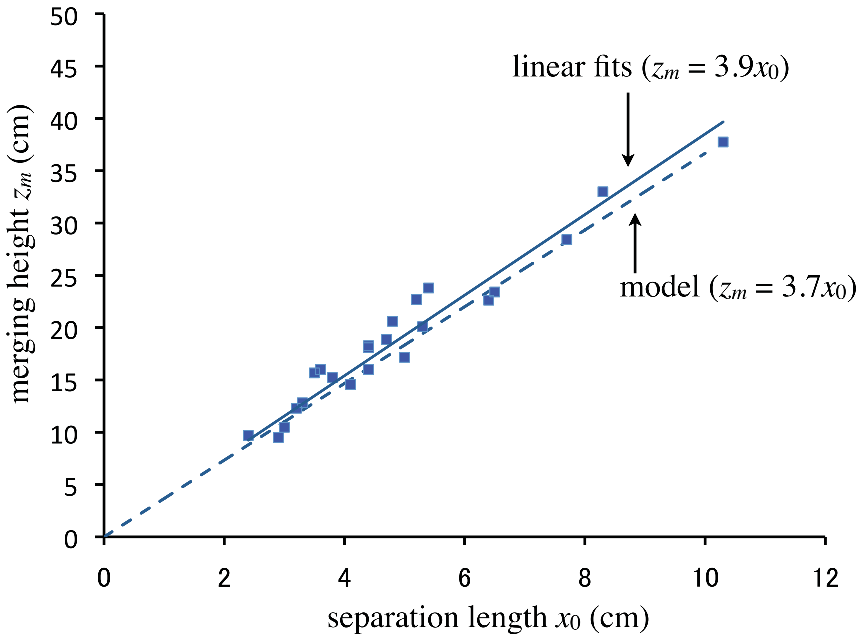

Figure 2: Plume merging height z_m plotted against the initial separation length x_0. The solid line is the linear fit of experimental data and the dashed line is the theoretical prediction, $z_m \approx (0.44/\alpha)x_0$, from the model of Kaye and Linden (2004).

Our experimental results show the linear relationship between x_0 and z_m, which is in agreement with the theoretical prediction. However it should be mentioned that the prediction $z_m = (0.44/\alpha)x_0$ is highly dependent on the value of α.

3.2 With rotation

Figure 3 shows the evolution of two plumes and the single vortex induced by the plumes for $x_0 = 5$ cm, $f = 0.25$ s^{-1}. It was observed that the axes of the plumes tilted off the vertical and a single vortex was generated at the center of two plumes for $t > 0.8T_f$. Figure 4 shows a similar sequence for $x_0 = 5$ cm, $f = 1$ s^{-1}. In this case, two vortices are generated, one from each plume. After $t \sim 3T_f$ the vortices started shedding from the sources (not shown).

In all cases, the plumes did not reach a steady state, because they were advected by the vortex/vortices which was/were generated by the plumes themselves. These processes are so complicated that, in this study, we do not consider the plumes after rotation dominates the dynamics but focus on the vortex/vortices generated by the plumes.

Types of vortex/vortices generated by the plumes for each case are shown in table 2.

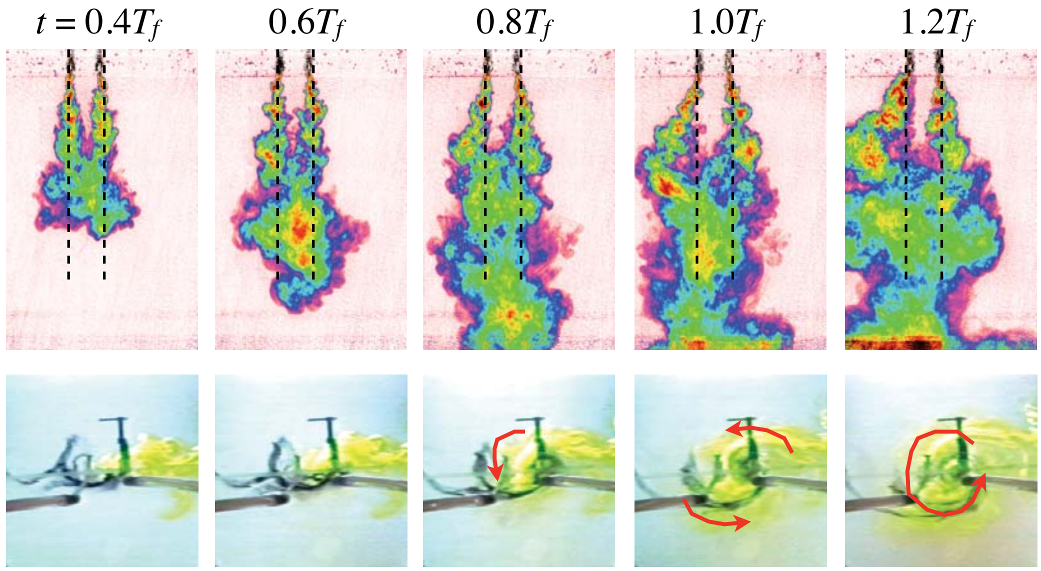

Figure 3: Evolution of two plumes with separation $x_0 = 5$ cm, rotation $f = 0.25\ \mathrm{s}^{-1}$. Upper pictures are snapshots of side-view dye concentration, and lower pictures are snapshots of top view. $T_f = 2\pi/f$ is the inertial period.

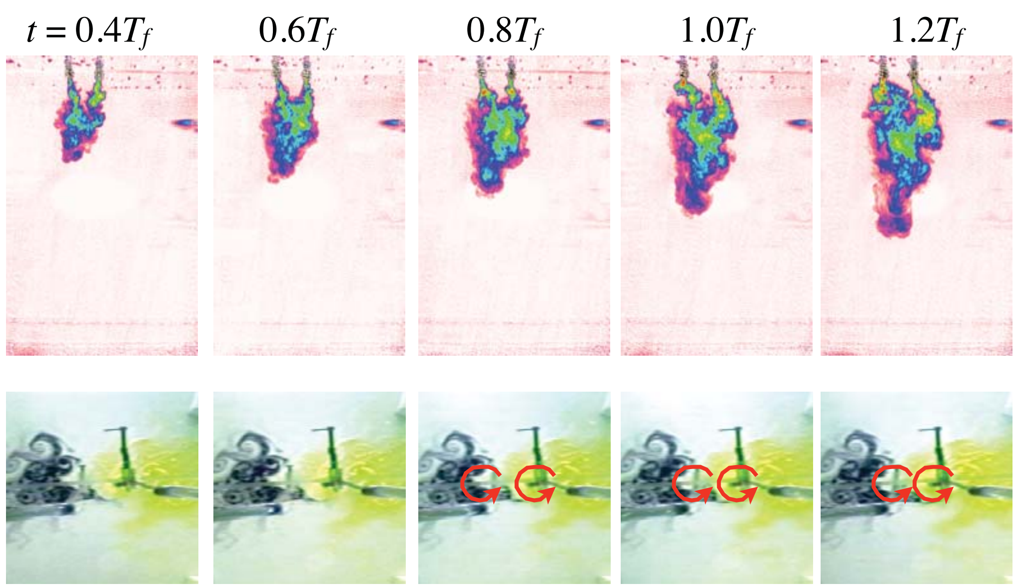

Figure 4: Same as figure 3 but for rotation $f = 1\ \mathrm{s}^{-1}$.

		x_0 (cm)			
		3	5	8	10
$f(\mathrm{s}^{-1})$	0.05	**1**	**1**	**1**	**1**
	0.1	**1**	**1**	**1**	**1**
	0.25	**1**	**1**	**1**	**2**
	0.5	**1**	**1**	**2**	**2**
	0.75	**1**	**2**	**2**	**2**
	1	**2**	**2**	**2**	**2**

Table 2: Types of vortices generated by the plumes. Number **1** denotes that a single vortex was generated at the center of two plumes, and **2** denotes that two vortices were generated, one from each plume.

4 Quantitative analysis

4.1 Merging height

Figure 5 shows the relation between z_m and x_0 for $0.5T_f < t < 0.75T_f$ for the rotating cases. It is shown that, in this time range, the two plumes do not feel the effect of rotation, so the merging height is same as in the non-rotating cases. In cases of (f, x_0) = (0.05, 10), (0.25, 3), and (0.5, 3), z_m obtained for $0.5T_f < t < 0.75T_f$ are far from the theoretical prediction, but those obtained for $0.25T_f < t < 0.5T_f$ agree well with the prediction. For (f, x_0) = (0.05, 10), this is because the averaging time interval was so long that z_m was affected by dyed water filling up the tank from the bottom. For (f, x_0) = (0.25, 3) and (0.5, 3), the effect of rotation may have appeared earlier.

Figure 6 shows the relation between z_m and x_0 for $0.75T_f < t < T_f$ for the rotating cases. For this averaging time interval, z_m is affected by rotation, and there is not a clear relation between z_m and x_0. The scatter S defined as

$$S = \sqrt{\frac{\sum_i^N (z_{m,i}^{\mathrm{theory}} - z_{m,i}^{\mathrm{experiment}})^2}{N}}, \tag{1}$$

where N is the number of experiments, $z_{m,i}^{\mathrm{theory}}$ the theoretical prediction of the merging height, $z_{m,i}^{\mathrm{experiment}}$ the experimental merging height, for the non-rotation experiments, rotating experiments for $0.5T_f < t < 0.75T_f$, and for $0.75T_f < t < T_f$ is 1.85, 2.89, and 10.98 cm, respectively.

From the above results, we deduce that the effect of rotation becomes important after $t > 0.75T_f$, in agreement with the result of FCA. Our experimental results also show that the merging height for $t < 0.75T_f$ agrees with the theoretical model for no rotation.

4.2 Vortex generated by the plumes

The process of generating a single or two vortices can be described as follows. When the separation length is small and the rotation rate is low, the plumes feel the effects of rotation after they merged and a single vortex will be generated by the resulting single plume. On

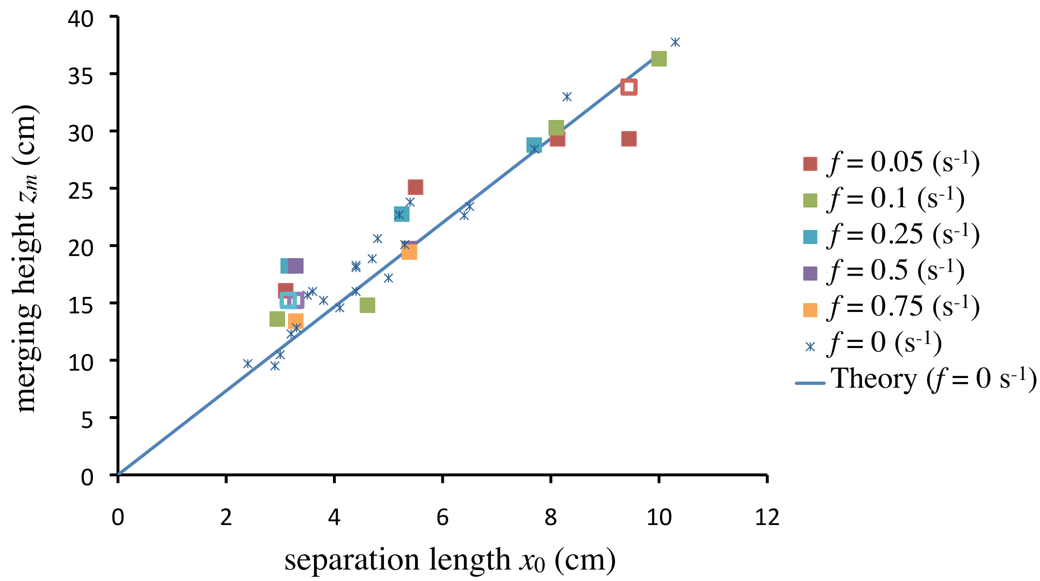

Figure 5: Plume merging height z_m plotted against the separation length x_0 for $0.5T_f < t < 0.75T_f$ (filled square), where T_f is the inertial period. Asterisks denote the results of experiments without rotation, solid line denotes the theoretical prediction for no rotation, $z_m = (0.44/\alpha)x_0$. Open squares are z_m for $0.25T_f < t < 0.5T_f$.

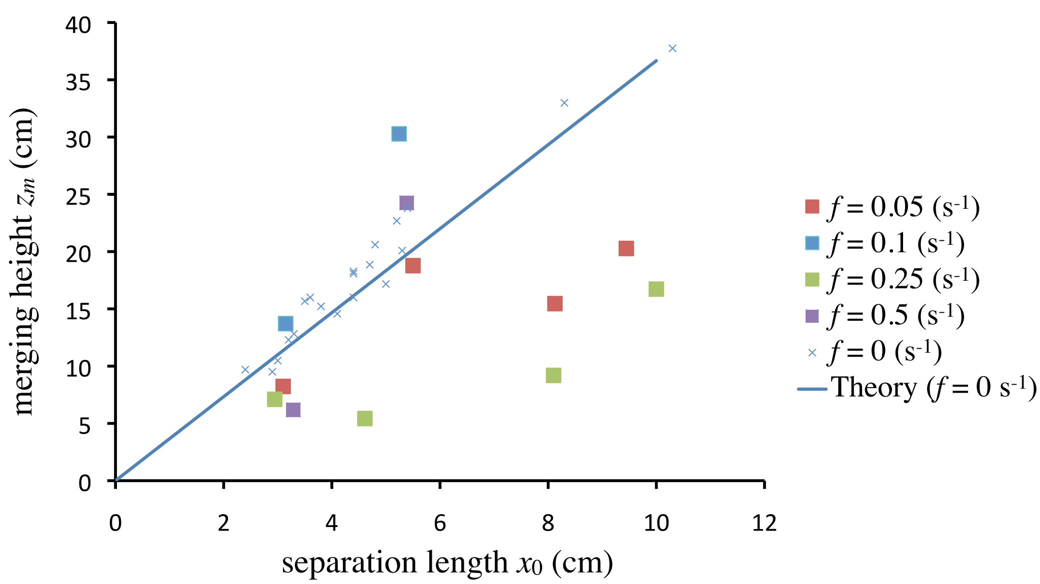

Figure 6: Same as figure 5 but for $0.75T_f < t < T_f$.

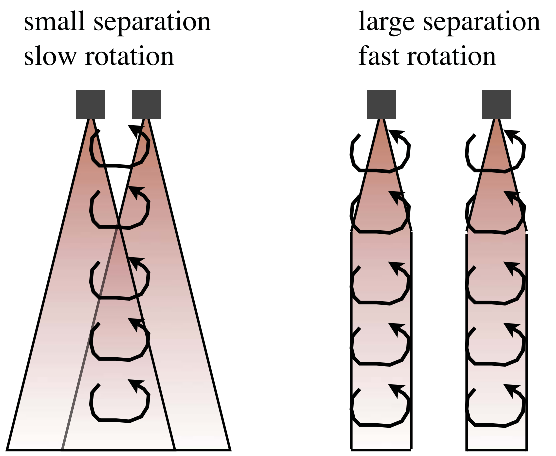

Figure 7: Schematic illustration of the two plumes generating a single vortex after merging (left) and two vortices before merging (right).

the other hand, when the separation length is large and the rotation rate is high, the plumes feel the effects of rotation before merging and two vortices are generated, one by each plume (figure 7).

Considering the process described above, we can predict the number of vortices generated by the two plumes using the results of FCA for the development of a single plume in a homogeneous, rotating fluid, and the theory of KL for the merging height in a non-rotating fluid. FCA showed experimentally that a single axisymmetric turbulent plume's front in a homogeneous, rotating fluid descends as

$$z \approx 1.7F_0^{\frac{1}{4}}t^{\frac{3}{4}} \quad \text{for} \quad t \leq 0.75T_f, \tag{2}$$

$$z \approx 1.8F_0^{\frac{1}{4}}t^{\frac{3}{4}} \quad \text{for} \quad t > 0.75T_f. \tag{3}$$

We measured the plume's front descent in the rotating and non-rotating experiments (figure 8). Our experimental results agree well with FCA.

Figures 5, 6, and 8 show that the time when rotational effects become important and the descent of the plume's front in our experiments agree with FCA. Therefore, the depth z_f where plumes feel rotation can be written as

$$z_f \approx 1.7F_0^{\frac{1}{4}}(0.75T_f)^{\frac{3}{4}} \approx 5.5F_0^{\frac{1}{4}}f^{-\frac{3}{4}}. \tag{4}$$

Remember that the theoretical prediction of the merging height

$$z_m \approx \frac{0.44}{\alpha}x_0, \tag{5}$$

can be applied to rotating cases for $t < 0.75T_f$. Therefore, we can predict that a single vortex will be generated if $z_f > z_m$, and two vortices will be generated if $z_f < z_m$. Figure 9 shows the number of generated vortices in the z_m-z_f plane. Our experimental results agree well with the prediction described above. Therefore, we can predict the number of vortices generated by the two plumes from the values of the entrainment constant (α), separation length (x_0), Coriolis parameter (f), and buoyancy flux (F_0).

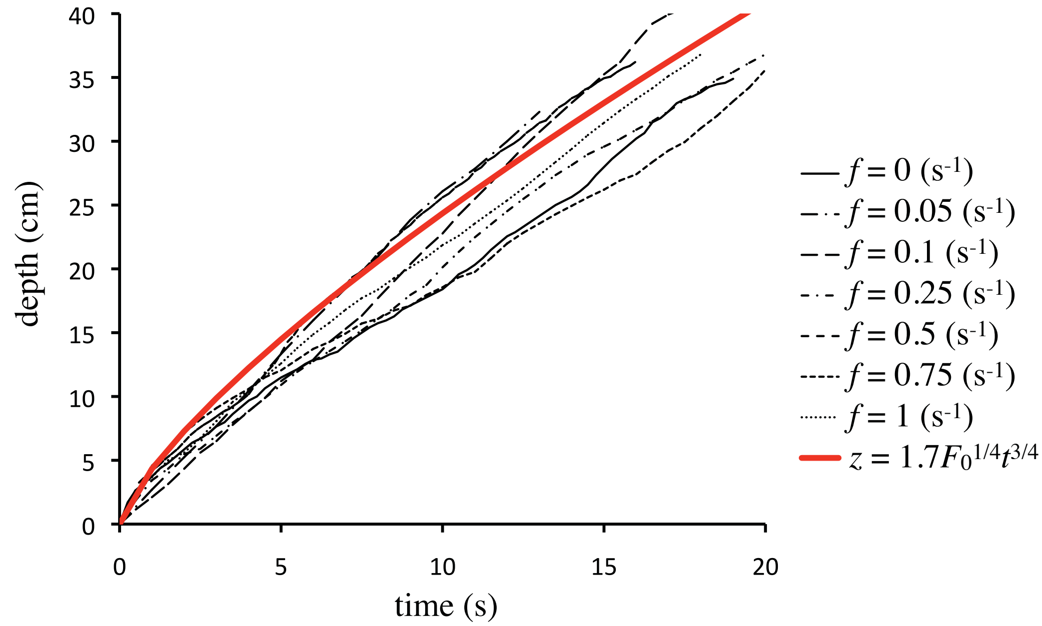

Figure 8: Depth of the plume's front as a function of time. Black lines are experimental results for various rotation rates and the red line is $z \approx 1.7F_0^{\frac{1}{4}}t^{\frac{3}{4}}$(Fernando, Chen, and Ayotte, 1998).

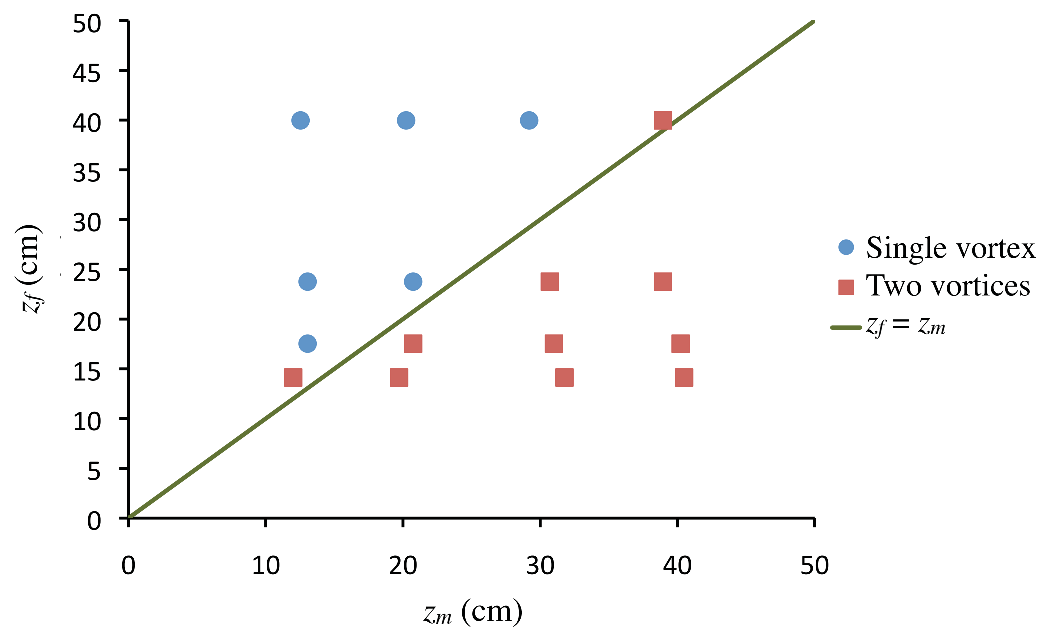

Figure 9: Number of vortices generated by the plumes, in the z_m-z_f plane. Blue circles denote single vortex and red squares denote two vortices. Solid line denotes $z_f = z_m$. The results for $f = 0.05, 0.1$ s^{-1} are not shown.

5 Summary

Laboratory experiments on two coalescing axisymmetric turbulent plumes in a homogeneous, non-rotating and rotating fluids were carried out. The merging height z_m is defined as the height from the virtual origin where the time averaged horizontal buoyancy profile has a single local maximum. The values of the virtual origin height $z_V = 1.0$ cm and entrainment constant $\alpha = 0.12$ were determined by the filling tank experiment (Baines and Turner, 1969).

For the non-rotating experiments, our experimental results (figure 2) agree well with the theory of Kaye and Linden (2004) predicting a merging height $z_m \approx (0.44/\alpha)x_0$, where x_0 is an initial separation length of the two plumes' sources. However, as mentioned in Kaye and Linden (2004), this prediction is highly dependent on the value of α.

For the rotating experiments, our experimental results show that the merging height z_m agrees with the theory of Kaye and Linden (2004) for no rotation, for $t < 0.75T_f$ (figure 5). However, for $t > 0.75T_f$, the plumes were not steady and z_m did not agree with the theory (figure 6). Namely, the effects of rotation became important after $t > 0.75T_f$; this is in agreement with the experimental results of Fernando, Chen, and Ayotte (1998) who studied the development of a single axisymmetric turbulent plume in a homogeneous, rotating fluid. Fernando, Chen, and Ayotte (1998) also obtained experimentally the height of the plume's front as a function of time t as $z \approx 1.7F_0^{\frac{1}{4}}t^{\frac{3}{4}}$, and our results are in agreement with this expression (figure 8).

In the rotating experiments, vortices were generated by the plumes. Two types of vortices were observed: a single vortex generated at the center of the two plumes, and two vortices generated, one from each plume. We showed that the number of vortices can be predicted using the theory for the merging height $z_m \approx (0.44/\alpha)x_0$ and the experimental prediction for the height where the plumes start feeling rotation $z_f \approx 5.5F_0^{\frac{1}{4}}f^{-\frac{3}{4}}$. When $z_f > z_m$, a single vortex is generated, and when $z_f < z_m$, two vortices are generated (figure 9).

Acknowledgments

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Appendices

A Filling tank experiment

In order to determine the values of the virtual origin height and the entrainment constant a filling tank experiment was carried out. Theoretically, the origin of an axisymmetric turbulent plume is assumed to be a point. However, in laboratory experiments, the real

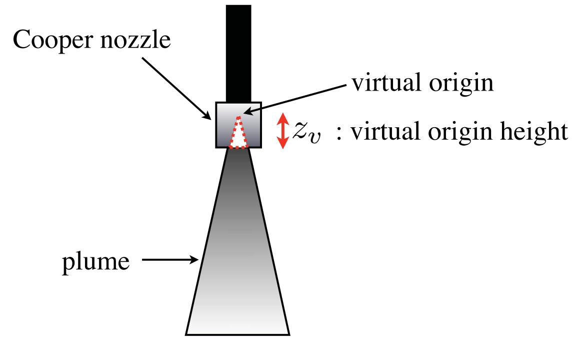

Figure 10: Schematic illustration of virtual origin.

origin of the plume (i.e. Cooper nozzle) has a finite cross section, so it is assumed that there is a virtual origin which is considered to be a point source of the plume (figure 10).

Turbulent plumes entrain the ambient fluid by turbulent mixing, and the horizontal velocity of the entrained flow u_e is assumed to be $u_e = -\alpha w$, where α is the entrainment constant and w is the vertical velocity. This assumption is called the entrainment assumption, and it is a good approximation (Turner, 1973). However the value of α has to be determined experimentally.

The filling tank experiment is a good method to determine both the virtual origin height and the entrainment constant.

A.1 Theory of the filling tank experiment

In this section, the theory of the filling tank experiment (Baines and Turner, 1969) is explained. First, some general properties of an axisymmetric turbulent plume are derived. Then the method for determining both the virtual origin height and the entrainment constant is explained.

A.1.1 General properties of an axisymmetric turbulent plume

It is assumed that the profiles of vertical velocity w and buoyancy g' of the plume are of Gaussian form,

$$w(r,z) = W(z)\exp\left(-\frac{r^2}{b(z)^2}\right), \tag{6}$$

$$g'(r,z) = g\left(\frac{\rho_A - \rho_P(r,z)}{\rho_R}\right) = G'(z)\exp\left(-\frac{r^2}{b(z)^2}\right). \tag{7}$$

Here z is the vertical coordinate, r the radial distance from the axis of the plume, g the gravitational acceleration , $W(z) = w(z,0)$, and $G'(z) = g'(z,0)$. Then ρ_A and ρ_P are the densities of the ambient fluid and the plume, respectively, ρ_R is the reference density for the system. Note that $b(z)$ is the radius where buoyancy and vertical velocity are reduced by a factor $1/e$ from those on the plume's axis.

Volume flux Q of the axisymmetric turbulent Gaussian plume is

$$Q = \int_0^{2\pi} \int_0^{\infty} w r dr d\theta = \pi b^2 W, \tag{8}$$

where θ is the angular coordinate.

The momentum flux M is

$$M = \int_0^{2\pi} \int_0^{\infty} w^2 r dr d\theta = \frac{\pi}{2} b^2 W^2. \tag{9}$$

Then b can be written as

$$b = \frac{1}{\sqrt{2}} \pi^{-\frac{1}{2}} \frac{Q}{M^{\frac{1}{2}}}. \tag{10}$$

The buoyancy flux F is

$$F = \int_0^{2\pi} \int_0^{\infty} g' w r dr d\theta = \frac{G'}{2} Q. \tag{11}$$

The velocity of the entrained flow, u_e, is now assumed to be horizontal,

$$u_e = -\alpha W, \tag{12}$$

where α is the entrainment constant.

The volume flux Q increases with z, because of the entrained flow, so

$$\begin{aligned} \frac{\partial Q}{\partial z} &= -2\pi b u_e, & (13)\\ &= 2\pi \alpha b W, & (14)\\ &= 2\sqrt{2}\pi^{\frac{1}{2}} \alpha M^{\frac{1}{2}}. & (15) \end{aligned}$$

Because the plume is accelerated by buoyancy, the momentum flux changes with z as

$$\begin{aligned} \frac{\partial M}{\partial z} &= \pi b^2 G', & (16)\\ &= \frac{(\pi G' b^2 W)/2 \cdot \pi b^2 W}{(\pi b^2 W^2)/2}, & (17)\\ &= \frac{FQ}{M}. & (18) \end{aligned}$$

We can set $F = F_0$ (constant) because the entrainment not only decrease the buoyancy g' but also increase the flow flux Q. Therefore, equation (18) can be written as

$$\frac{\partial M}{\partial z} = \frac{F_0 Q}{M}. \tag{19}$$

Now, we assume that Q and M can be expressed as

$$\begin{aligned} Q &= Q_0 (z + z_V)^q, & (20)\\ M &= M_0 (z + z_V)^m, & (21) \end{aligned}$$

where z_V is the height of virtual origin, and Q_0, M_0 are constants. This assumption implies that the radius of the plume, b, increases linearly from 0 at its virtual origin $z = -z_V$. Then (15) and (19) become

$$Q_0 q (z+z_V)^{q-1} = 2\sqrt{2}\pi^{\frac{1}{2}}\alpha M_0^{\frac{1}{2}}(z+z_V)^{\frac{m}{2}}, \tag{22}$$

$$M_0 m (z+z_V)^{m-1} = \frac{F_0 Q_0}{M_0}(z+z_V)^{q-m}. \tag{23}$$

From the above equations, we obtain

$$q = \frac{5}{3}, \tag{24}$$

$$m = \frac{4}{3}. \tag{25}$$

Substituting (24) and (25) into (22) and (23), with some algebra, we obtain

$$M_0 = \left(\frac{9\sqrt{2}\pi^{\frac{1}{2}}\alpha F_0}{10}\right)^{\frac{2}{3}}, \tag{26}$$

$$Q_0 = \frac{6\pi^{\frac{1}{2}}\alpha}{5}\left(\frac{18\pi^{\frac{1}{2}}\alpha F_0}{5}\right)^{\frac{1}{3}}. \tag{27}$$

Eventually, the volume flux Q and the momentum flux M are written in the buoyancy flux F_0 and the entrainment constant α.

A.1.2 Method to determine the virtual origin height and the entrainment constant

We consider a tank of cross section A_C filled with ambient fluid, and an axisymmetric turbulent plume which is denser than the ambient fluid and is dyed, descending from the top of the tank ($z = 0$), as in figure 11. After the front of the plume reaches bottom of the tank, a layer of dyed fluid forms at the bottom of the tank and starts filling up the tank. We set the time $t = 0$ when $z_F(t) = z_{F0}$, where z_F is the height of the dyed fluid's front which comes up from the bottom. It is convenient to choose z_{F0} as the height where we can measure z_F accurately enough.

Considering the balance of the volume flux of the dyed fluid at $z = z_F$, the time derivative of $(z_F(t) + z_V)$ can be written as

$$\begin{aligned} \frac{\partial}{\partial t}(z_F(t)+z_V) &\approx -\frac{Q(z_F)}{A_C}, \\ &= -\frac{Q_0}{A_C}(z_F+z_V)^{\frac{5}{3}}. \end{aligned} \tag{28}$$

Here it is assumed that the cross section of the plume is negligible. Integrating (28), we

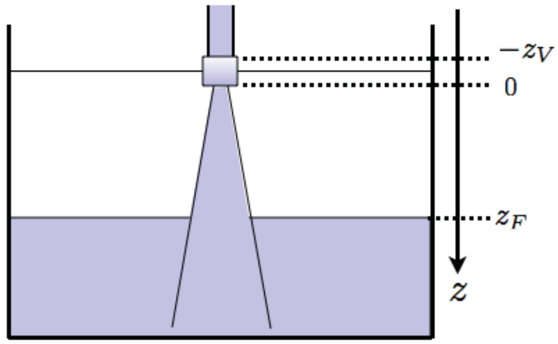

Figure 11: Schematic illustration of the filling tank experiment.

obtain

$$\int_{z_{F0}}^{z_F} (z_F + z_V)^{-\frac{5}{3}} d(z_F + z_V) = -\frac{Q_0}{A_C} \int_0^t dt, \tag{29}$$

$$-\frac{3}{2}\left[(z_F + z_V)^{-\frac{2}{3}} - (z_{F0} + z_V)^{-\frac{2}{3}}\right] = -\frac{Q_0}{A_C} t, \tag{30}$$

$$(z_F + z_V)^{-\frac{2}{3}} = \frac{2}{3}\frac{Q_0}{A_C} t + (z_{F0} + z_V)^{-\frac{2}{3}}. \tag{31}$$

In other words, there is a linear relation between $(z_F + z_V)^{-\frac{2}{3}}$ and t.

We can measure z_F and t from the experiment, so we can determine the value of z_V which produce a straight line in the plot of $(z_F + z_V)^{-\frac{2}{3}}$ against t. After determining z_V, we can estimate the value of α by choosing the value that makes fit equation (31) best to the experimental data with the determined z_V.

A.2 Experimental set up

Experimental equipments and their settings are the same as those explained in section 2, but the depth of the fresh water was 40 cm. We captured the digital images every minute and determined z_F by looking at the the horizontally averaged (excluding the region of the plume) height where the value of concentration of the dye is that of the interface between fresh water and dyed water.

A.3 Experimental results

Figure 12 shows measured values of $(z_F + z_V)^{-\frac{2}{3}}$ against t for various values of z_V. From this figure we can determine $z_V = 1.0$ cm, because this value gives the highest R^2 for the linear fit to the data, where R^2 is the coefficient of determination. Figure 13 shows the measured values of $(z_F + z_V)^{-\frac{2}{3}}$ against t with $z_V = 1.0$ cm, and the theoretical lines calculated from equation (31) with various values of α. From this figure we can estimate that the value $\alpha = 0.12$ fit best the experimental data.

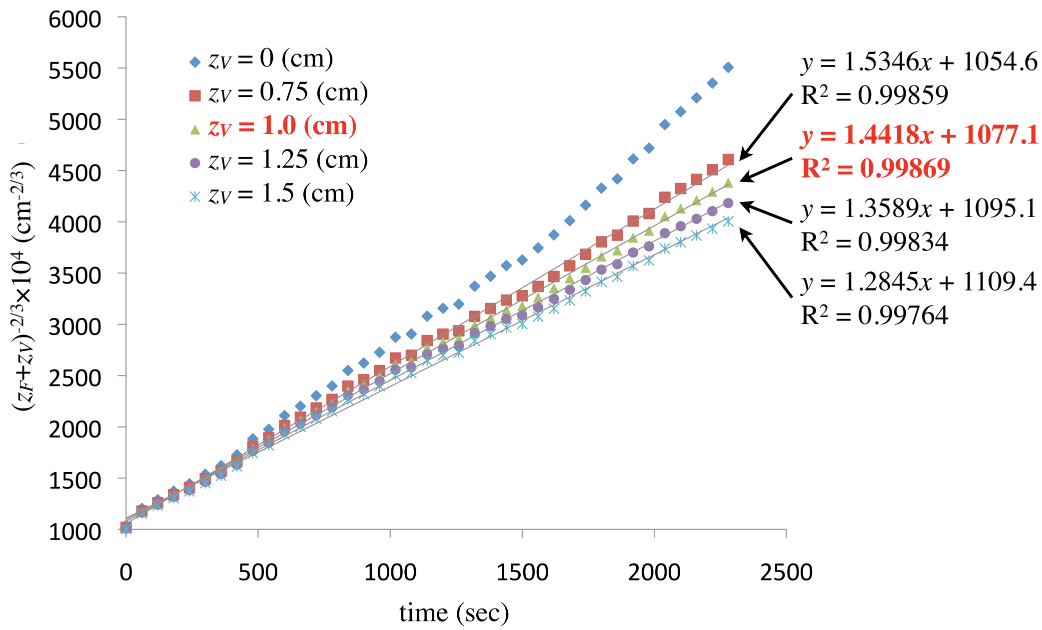

Figure 12: Measured values of $(z_F + z_V)^{-\frac{2}{3}}$ against t for various values of z_V. Solid lines are linear fits of each data set. This figure shows that the best linear fit (highest value of R^2) occurs for $z_V = 1.0$ cm.

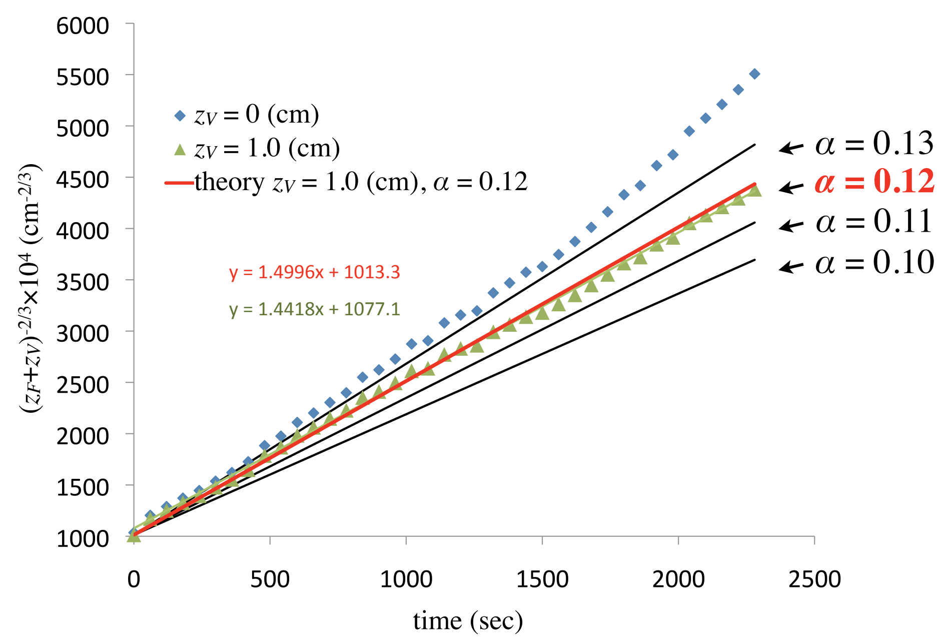

Figure 13: Measured values of $(z_F + z_V)^{-\frac{2}{3}}$ against t for $z_V = 1.0$ cm. Thick solid lines are theoretical lines calculated from equation (31) with various values of α. This figure shows that the value of $\alpha = 0.12$ fit best the experimental data.

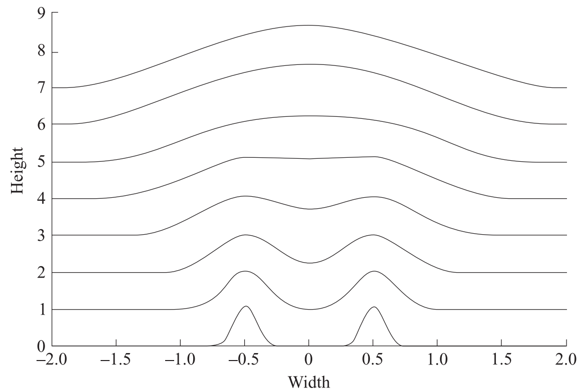

Figure 14: Merging Gaussian functions for a unit sources separation. (Kaye and Linden, 2004)

B Theory of coalescing axisymmetric turbulent equal plumes

In this section the theory of Kaye and Linden (2004) for equal plumes is summarized.

First, we define dimensionless variables as followings,

$$\lambda = \frac{z}{x_0}, \quad \phi = \frac{x}{x_0}, \quad \gamma = \frac{b}{x}, \tag{32}$$

where z is the height above the point plume sources (i.e. including the virtual origin), x_0 the separation of the plume sources, x the separation of the plume axes at any given height, and b the plume radius.

Consider two equal plumes with origins at the same height. The average buoyancy profile of a single turbulent plume can be taken as Gaussian, with a radius given by

$$b = \frac{6}{5}\alpha z, \tag{33}$$

where α is the entrainment constant. Assuming that the plumes do not interact, the buoyancy profile function is given by

$$g'(r,z) \sim G'(z)E(r), \tag{34}$$

$$E(r) = \exp\left[-\frac{(r-\frac{1}{2}x_0)^2}{b^2}\right] + \exp\left[-\frac{(r+\frac{1}{2}x_0)^2}{b^2}\right]. \tag{35}$$

Here r is the distance from the center of the two plumes and g' is the buoyancy.

The function (34) is plotted in figure 14 for $1 < \lambda < 8$ and $x_0 = 1$. Clearly the two Gaussians coalesce when λ is large enough. We define the merging height to be the height where the trough in the buoyancy profile disappears, that is,

$$\frac{d^2E}{dr^2} = 0 \quad \text{at} \quad r = 0. \tag{36}$$

For non-interacting plumes (i.e. $x(z) = x_0$), (36) is easily solved to give

$$x_0 = \sqrt{2} b_m. \tag{37}$$

Here the subscript m denotes the value at the point where plumes merge. Dividing (37) by x, we obtain

$$\gamma_m = \frac{1}{\sqrt{2}}, \tag{38}$$

so, in terms of the non-dimensional height we obtain

$$\hat{\lambda}_m = \frac{1}{\sqrt{2}} \frac{5}{6\alpha}. \tag{39}$$

Here $(\hat{\cdot})$ denotes the upper bound on the value. As shown by Bjorn and Neilsen (1995) this estimate of λ_m is poor. In order to model the drawing together of two equal plumes we need to consider the entrainment of one plume by another.

Based on experimental results (e.g. Rouse, Yih, and Humpherys, 1952) it is reasonable to take the velocity field outside the plumes, created by entrainment, as horizontal. The mean entrainment velocity field, over a horizontal plane across the two plumes, may be approximated by two sinks of strength $-m(z)$ placed at $(-\frac{1}{2}x, z)$ and $(\frac{1}{2}x, z)$. The complex velocity potential in this horizontal plane is given by

$$\Psi = -\frac{m}{2\pi} \left[\ln\left(Z + \frac{1}{2}x \right) + \ln\left(Z - \frac{1}{2}x \right) \right], \tag{40}$$

where Z is the complex variable $re^{i\theta}$. The velocity field is given by

$$U = \frac{\partial \Psi}{\partial Z} = -\frac{m}{2\pi} \left(\frac{1}{Z + \frac{1}{2}x} + \frac{1}{Z - \frac{1}{2}x} \right). \tag{41}$$

The sink strength of the plume is

$$m = \int_0^{2\pi} b\alpha w d\theta. \tag{42}$$

Along the line joining the sources of the plume ($\theta = 0$) the velocity is given by

$$U_{\theta=0} = -b\alpha w \left(\frac{1}{r + \frac{1}{2}x} + \frac{1}{r - \frac{1}{2}x} \right). \tag{43}$$

On the plume axis ($r = -\frac{1}{2}x, \frac{1}{2}x$), the value of horizontal entrained velocity due to that plume is zero, and only the term resulting from the other plume needs to be considered. Therefore, (43) gives an expression for the mean horizontal velocity u on the plume axis

$$u = -\frac{b\alpha w}{x}. \tag{44}$$

Assuming that each plume is passively advected by the entrainment field of the other, the rate of change of the plumes' separation with height is given by the ratio of the vertical and horizontal velocities at the plume axis. As both plumes are deflected equally, the rate is

$$\frac{dx}{dz} = -2\frac{1}{w}\frac{\alpha b w}{x}. \tag{45}$$

Substituting $b = 6\alpha z/5$, $\lambda = z/x_0$, and $\phi = x/x_0$ gives

$$\frac{d\phi}{d\lambda} = -\frac{12\alpha^2}{5}\frac{\lambda}{\phi}, \tag{46}$$

which can be integrated to give

$$\phi^2 - \phi_0^2 = \frac{12\alpha^2}{5}(\lambda_0^2 - \lambda^2), \tag{47}$$

where $\phi_0 = 1$ and $\lambda_0 = 0$. Using (38) to (47) we obtain

$$\lambda_m = \frac{1}{\alpha}\sqrt{\frac{25}{132}} \approx \frac{0.44}{\alpha}. \tag{48}$$

In other word, the merging height is predicted as

$$z_m \approx \frac{0.44}{\alpha}x_0. \tag{49}$$

Note that the use of (38) derived for the non-interacting model is appropriate here as the correction for drawing together of the plumes assumes that they are passively advected only. The radial growth rate of each plume is assumed to be unaffected by this process.

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