Traversing the edge: a study of turbulent decay

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Abstract

In this work the "edge of chaos" is studied to increase our understanding of turbulence in shear flows. The "edge" is a hypersurface in phase space which separates conditions which return to the laminar state from those which engage in turbulent dynamics. We tackle the subject of the geometry of the edge, and its involvement during the return to the laminar state. Here studying plane-Couette flow we observe the death of the self-sustaining process during decay and identify the processes which govern the decay rate. The report concludes with tests on the validity of edge geometry observed in low dimensional models.

1 Introduction

The study of Newtonian fluid flow through through a straight circular pipe was first carried out in the 19th century, where Hagen [1] and Poiseuille [2] separately studied the laminar flow which now carries both their names. This work was continued by Reynolds [3] who studied the transition from laminar flow to flow which is both temporally and spatially disordered, called turbulent flow. The Reynolds number $Re := UD/\nu$ governs this transition, where U is the mean speed, D the diameter and ν the kinematic viscosity. The laminar Hagen-Poiseuille flow is linearly stable for all values of Re, meaning that a finite amplitude disturbance is required to generate turbulent behaviour. The energy that the disturbance requires to trigger turbulence has been studied for many years, and depend sensitively upon the shape of the disturbance and the Reynolds number. The minimum value of the Reynolds number at which turbulence is seen varies between experiments but appears to lie in the range 1750-2300. It is thought that in this range of Re that there exists a chaotic saddle responsible for the dynamics, which may transition to a chaotic attractor for larger Re [4]. In this regime the lifetime of turbulence can vary strongly, so mean dynamics described by probability functions are used to demonstrate the behaviour. Faisst et al. [4] amongst others showed that probability of turbulence surviving depends exponentially on the ratio of time, t, to a mean lifetime which depends upon Re. The relationship between Re and the mean lifetime is still a subject of research, both experimental and computational. In that work the discussion centres around the existence of a finite Re for which the probability of decay back to the laminar state is zero.

Pipe flow is one member of a class of shear flows which also include plane Couette flow, Taylor-Couette flow and boundary layer flow. Plane Couette flow (PCF) is the flow between two infinite plates, which are driven at constant speed in opposite directions. The flow shares all of the features discussed about (although with different critical Re), but for research has two advantages over pipe flow. The symmetry of the system allows the existence of fixed points solutions, whereas the simplest structure in pipe flow are travelling waves. The Cartesian geometry of PCF is computationally simpler than the cylindrical coordinate system of pipe flow.

The linear stability of the laminar states makes finding new solutions to the Navier-Stokes equations challenging, however recently new solutions have been found ([5], [6], [7], [8], [9], [10], [11], [12]). These solutions are fixed points (in PCF only), travelling waves which are steady under a translating reference frame and periodic orbits, many of which exhibit symmetries. These solutions are unstable but have both stable and unstable manifolds and are therefore saddle points in phase space. Kerswell [13] and others propose that these, and more solutions form a "skeleton" for the dynamics which can guide trajectories around the turbulent portion of phase space.

Within the last 10 years a new method has been implemented to find solutions in both pipe flow and PCF. The method, pioneered by Itano et al. [14], is to track the "Edge of Chaos", the hyper-surface surface which separates separates conditions which simply relaminarize from those which are subjected to turbulence. The edge therefore provides a minimum on the energy required to trigger turbulence, however there is no known method to use the edge to find this minimum energy point. By reducing the dynamics to only evolve along the edge, new structures have been found in both pipe flow and PCF ([15], [16], [17]). By evolving along the edge only, one unstable direction is removed, meaning solutions embedded in the edge with just one unstable direction in the full dynamics become local attractors within the dynamics of the edge. As discussed earlier, at low Reynolds number turbulence is transient and initial conditions experience sudden decay back to the laminar state. The edge can therefore not be considered a boundary for the basin of attraction of the laminar state, as conditions either side of the edge will decay. This therefore raises a question into the understanding of phase space in these systems. How do initial conditions on the "turbulent side" of the edge pass back to the laminar state and does there exist a unique point (or a small number of points) where this passing occurs? In particular geometries the edge contains simple attracting states, such as fixed points or travelling waves. In some of these situations, evidence points to a single global attracting state in the edge, to which all initial conditions on the edge converge, called the edge state. A secondary question of this work concerns the dynamical significance of this edge state during relaminarization. The point is significant within dynamics on the edge, and all initial conditions must, in some manner, pass by this edge, therefore the edge state may be important in this relaminarization process. We will attack this problem on two fronts. The first will examine the statistics of decay, and look for evidence of a unique "crossing" point. The second half will look to the results of low-dimensional models and attempt to draw parallels between these and the full dynamics. In the next sections we will discuss the set-up used to investigate this problem, and present some statistical work on this problem. We will then look at previous work using low dimensional models to consider this problem, and compare these to the full dynamics. Finally we shall draw conclusions and discuss further work into this problem.

2 Methods

To examine the questions considered above we choose plane Couette flow as the shear flow for our investigation. This has been chosen for its simple geometry and the evidence for a single fixed point attracting edge state in a particular geometry (Schneider private communication 2011). As with pipe flow discussed above, no-slip boundary conditions at the wall are used, with periodic boundaries in the two remaining directions. The laminar flow is linearly stable and takes the form

$$\bar{\mathbf{u}} = Uy\,\hat{\mathbf{x}}, \tag{1}$$

where convention dictates that x takes the direction of the wall motion, y the wall normal direction and z the spanwise direction. The DNS is carried out in a Fourier by Chebyshev by Fourier domain, with an adaptive 3rd order Semi-implicit Backwards Differentiation timestep code written by Gibson ([23] [24]). The Reynolds number in this system is $Re := Uh/\nu$, where U is the wall speed, h is half the wall separation and ν the kinematic viscosity. For simplicity both U and h remain equal to one. We will study Reynolds numbers in the range $[340, 380]$ where turbulence exists but turbulent lifetimes are short (order 1000 time units). The domain is $[0, 4\pi] \times [-1, 1] \times [0, 2\pi]$, chosen for the existence of a single edge state, a member of the "Nagata" solution family [5], which is visualised in figure 1. We shall use the notation breaking the velocity field, $\bar{\mathbf{u}}$, into the laminar and perturbation parts, $\bar{\mathbf{u}} = y\,\hat{\mathbf{x}} + \mathbf{u}$. This study will begin with a statistical investigation into decaying turbulence, where trajectories will be aligned to decay at the same point in a new time t^* which is defined for each simulation as

$$t^* := t_{lam} - t, \tag{2}$$

where t_{lam} is the time such that

$$\int_V \mathbf{u} \cdot \mathbf{u}\, dV < 0.005. \tag{3}$$

This finds the time where the flow is sufficiently close to the laminar state to be considered as laminar. The results of this section are not qualitatively affected by the precise choice of this distance from the laminar state. For brevity we shall refer to this as the relaminarization time. We begin by simulating a large number of DNS runs from turbulent initial conditions until they reach the laminar state. Once aligned by their relaminarization times we can find the mean and standard deviation of the L2 norm of $\mathbf{u}$, and plot this against the relaminization time. In figure 2 we carry out this procedure for 100 evolutions at a Reynolds number of 380. Several observations can be made from this figure in answering the questions posed previously. The decay from the turbulent state begins approximately 500 time units before relaminarization time, and before this a statistically steady state is observed with approximately constant variance. During decay, while a decrease of variance is observed, trajectories do not converge until just prior to $t^* = 0$. This simple observation suggests an answer to one of the questions postulated above: does there exist a unique point for passing by the edge? If a unique point existed, one would expect to see the trajectories converge at an L2 norm value associated with the edge (~ 0.2). The result was robust to using a range of metrics to align the decaying trajectories, including E_{3D}, vorticity, and downstream vorticity. With all of these metrics no patterns in the decay emerged, therefore suggesting that this hypothesis is false.

3 Statistical analysis

We can use this approach to examine the physical properties of the flow during decay, and confirm the features expected from analytical and previous computational work. In figure 3 we plot the mean evolution of the L2 norm of 4 physical quantities during the last 1000 time units before decay, the velocity, the vorticity, the downstream vorticity and the "3D velocity". These first two are related to the energy and dissipation of the system respectively, and begin to decay simultaneously with similar relative gradients. The L2 norm of the downstream vorticity provides a measure for the downstream vorticies, or rolls, which redistribute the mean shear. This then creates downstream streaks, which can develop instabilities. These instabilities feedback into the rolls. This is called the self-sustaining process (SSP [18]), and plays a crucial role in the maintenance of turbulence. Therefore if the rolls are removed then the sustaining process is broken, and turbulent cannot be maintained. The results in frame (c) of figure 3 show that the L2

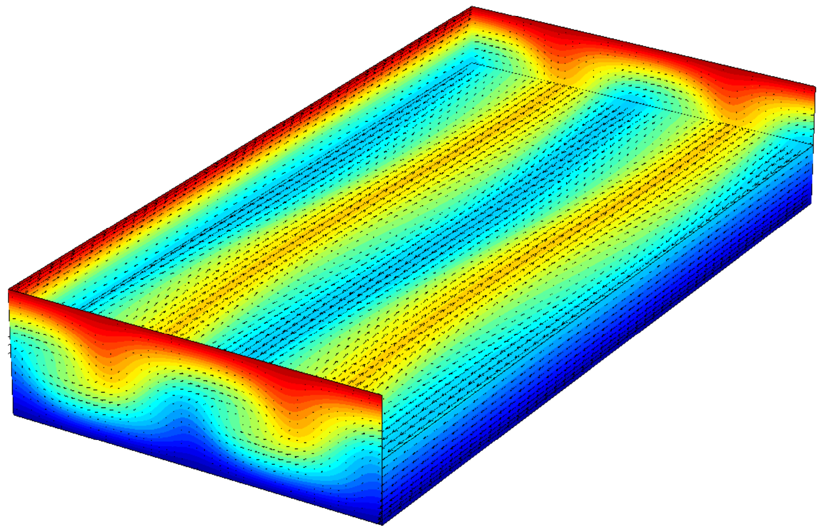

Figure 1: Visualization of the fixed point embedded in edge, which is an attractor when the dynamics are restricted to the edge. Colours indicate downstream velocity, with red flowing into the page and blue out of the page. Arrows plot cross stream velocity field.

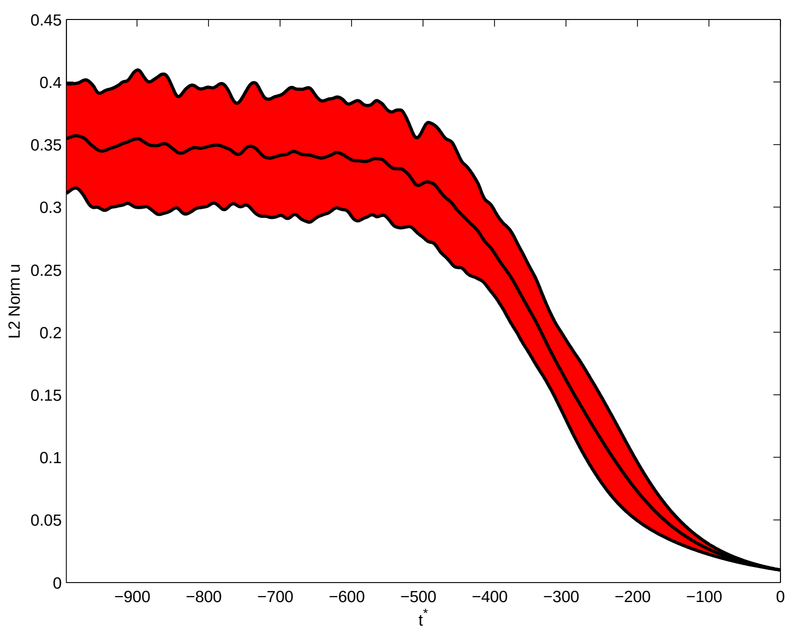

Figure 2: L2 Norm of perturbation velocity field **u** against t^*, the time before relaminarization. Plotted are the mean $\pm$ the standard deviation of 100 turbulent evolutions at $Re = 380$. Initial conditions generated from turbulence at slightly larger Re.

norm of the downstream vorticity begins to decay at the same time as the previous two metrics, however it decreases at greater relative rate. This results in no rolls existing for the final 200 time units of the decay. The final measure considered is the "3D velocity", $\mathbf{u}'$, the part of the velocity field which depends upon x.

$$\mathbf{u} = \frac{1}{L_x} \int \widetilde{\mathbf{u}} \, dx + \mathbf{u}' \tag{4}$$

The quantity is dynamically important to the maintenance of turbulence, the energy in this part of the velocity field we shall denote as E_{3D}. It can be shown in shear flows that a 2D perturbation cannot to lead to turbulence. From the view of the SSP this quantity measures the instabilities which complete the process and feedback upon the rolls. Frame (d) of figure 3 shows the evolution of this quantity before relaminarization. The beginnings of decay are observed in line with the other 3 quantities, with decay occurring at the same relative rate as downstream vorticity. Therefore we observe that the for the last 200 time units of decay the flow is two-dimensional with no downstream rolls, this leaves only downstream streaks (and a small amount of cross-stream flow). These findings therefore agree with the previous work, which suggested that downstream rolls and 3D flow are the first parts of the flow field to fully decay. After these two quantities have decayed the streak decay will govern the overall decay rate, which we shall now study.

Understanding the structure of the streaks during this decay will explain the variety of decays rates observed, as these are the only feature remaining during the final part of decay. In figure 4 frames (a) & (b) show the x-averaged[1] velocity for two different decay trajectories approximately 200 time units before relaminarization. Trajectories were chosen for displaying slow and fast decay respectively, i.e. shallow and steep decay rates during the final 200 time units of decay. Beyond this choice these trajectories are generic within their respective class (slow or fast decay). Obvious from the figure are the two different streak structures involved in the flow field, where streaks are indicated by waviness in the downstream velocity contours. Frame (a) has two streaks, one fast and one slow, whereas frame (b) has four streaks with two of each sign. The different length-scales involved with these flow fields explain the decay rates involved. In this regime the diffusion operator dominates the evolution, meaning that structures with small length-scales involved will decay at a faster rate compared to those with larger length-scales. The flow at this stage in decay is independent of x, and all flow-fields have similar dependence upon y leaving the z structure to set this rate. To further this work we consider the evolution of simple structures carrying the two and four streak pattern, members of the "Nagata" family of solutions. As previously discussed one member of this family, which has a four streak pattern (figure 1), is the edge state in the chosen geometry. However there exists a two streak member of the family, which also lies on the edge but has two unstable directions (and is therefore not an attracting structure on the edge). The x-averaged flow field for these solutions are plotted in frames (c) and (d) of figure 4. The comparisons between the decaying fields and solution fields can be easily seen, but the solutions have sharper streak structure and retain downstream vorticity. We can study the length-scales involved in these fixed points and the decaying trajectories by by representing the z dependence of the flow-field through a Fourier decomposition. As these solutions belong to the same family, but are effectively solutions from two different box widths their dependence on the first few Fourier modes differs. The two streak solution contains a large amount of energy in the first mode, whereas the four streak solution contains none. The second Fourier mode will be the dominant term for the 4-streak solution, but be of lesser importance in the 2-streak solution. When these two solutions are perturbed in the correct unstable direction (to the "laminar side" of

[1] Recall final decay is independent of x, therefore 2D visualization displays all flow features.

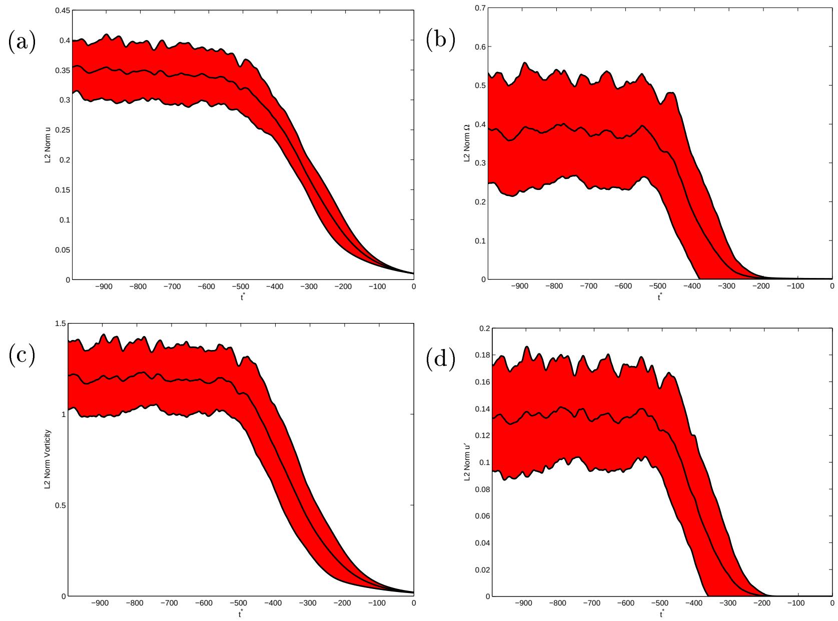

Figure 3: Evolution of the mean ± standard deviation for the L2 norm of 4 flow field quantities, (a) Velocity, (b) downstream vorticity, (c) vorticity & (d) "3D velocity". All experience decay approximately 500 time units before relaminarization, with faster relative decay rates for downstream vorticity and 3D velocity.

the edge) the solutions will smoothly decay to the laminar state. The decay of these states, with their different spanwise spectra will be a useful comparison for decaying turbulence. In figure 5 100 decaying trajectories are plotted, alongside the decay from the two states discussed all at the same Reynolds number. The striking feature of this figure is that the decay from two fixed points almost bounds the decay from turbulence. We can understand this by studying the spanwise Fourier modes during the last 200 time units of decay. Those decaying at a similar rate as the two streak solution, will have more energy in the first spanwise Fourier mode and little in the second. Whereas those decaying with the four streak solution will have little in the first spanwise Fourier mode and the majority in the second. Trajectories decaying at rates between these two "extremes" will have energy in both these modes in varying proportions which match the decay rate. An observation to made from this figure is that no decaying trajectory in our sample decayed at a significantly greater rate than the four streak fixed point. It appears within this domain all turbulence (in this range of Reynolds number) decays through a very simple streak structure. How this behaviour would change if a wider domain was used, or a larger Reynolds number set, remains a topic for further research. The author suggests that if the domain was sufficiently widened then turbulence would decay through a six streak structure, in addition to the two and four structures. It is not obvious that all decaying turbulence in this geometry should have streaks with such similar y-dependence, although the tight constraints of the domain might again be responsible. Studying the decay of other fixed points in this geometry would make for an interesting comparison with those discussed above. Are these two solutions special in the way they almost bound the decay, or is this a feature of fixed points?

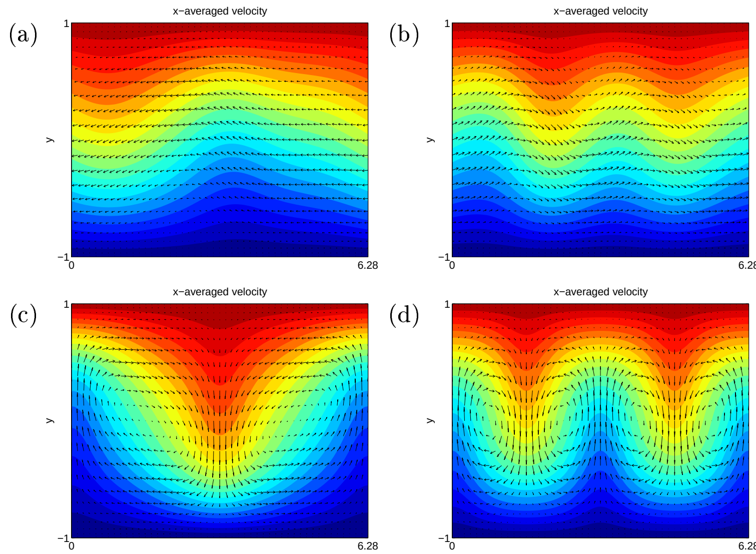

Figure 4: Frames (a), (b) show x-averaged flow fields 200 time units before decay. Solutions selected for demonstrating slow and fast relaminarization rates respectively. Frame (a) has two streaks, one fast at the centre of the domain, and one slow at the left edge. Frame (b) has four streaks, two fast and two slow. To be compared with frames (c) and (d) x-averaged flow field for members of the Nagata solution family. Frame (c) shows the longest spanwise wavelengths family member. Solution in frame (d) contains two copies of the Nagata solution in spanwise direction.

4 Edge geometry

Through examining the statistics of the decay from turbulence we have gained insight into the processes involved, and evidence that a unique route past the edge does not exist. Beyond this fact we have learnt little about how the decaying trajectories pass the edge. In this section we will examine this issue with the aid of low dimensional models. Attempts to study low dimensional models for shear turbulence have been used in recent years with limited success. Waleffe [18] took the ideas behind his self-sustaining process to construct both eight and four mode ODE models, however these were limited by representing turbulence with a fixed point. These models did show that the physical processes behind turbulence could be captured in a small number of well chosen modes. Extensions to a nine mode model [19] and an eight mode PDE model [20] have made progress in capturing more detail but more modes makes the analysis more complex. In order to understand the edge Lebovitz examined the edge structure in Waleffe's four mode model [21]. He subsequently designed a two dimensional system which captured the same edge topology [22]. It is this two dimensional system which we will compare with the edge structure in the full system. The equations of the system are

$$\begin{aligned} \dot{x}_1 &= -\delta x_1 + x_2 + x_1 x_2 - 3x_2^2 \\ \dot{x}_2 &= -\delta x_2 - x_1^2 + 3x_1 x_2, \end{aligned} \tag{5}$$

where δ is the control parameter and surrogate for Reynolds number, which we will fix at 0.4. For this value of δ the system has 3 fixed point solutions. One is stable and located at $(0, 0)$ which will be the surrogate for the laminar state. One is an unstable saddle, called the lower branch (LB), which is the surrogate for the edge state. The final fixed

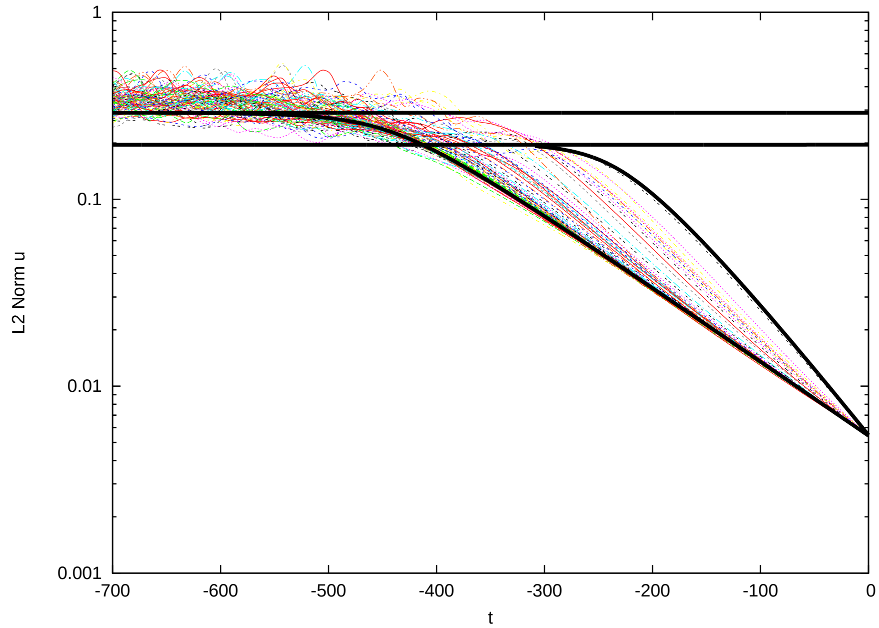

Figure 5: Coloured lines - L2 norm of the $\mathbf{u}$ against time for decaying trajectories. Black horizontal lines - indicate the value of L2 norm for the two solutions from figure 4 (higher line - two streak solution). Black lines - decay from fixed point solutions to the laminar state after perturbation. All decaying lines have been aligned by decay time.

point is unstable, called the upper branch (UB) and is the surrogate for the turbulent state. In figure 6 the three fixed points points are plotted alongside the manifolds of the edge state and a typical decay from near the upper branch point. The stable manifold of the edge state forms the edge in this system; near the edge state initial conditions below the edge decay to the laminar state, whereas points above the edge visit an area in phase space further from the laminar state before being attracted the laminar state. The feature captured by this model, and Waleffe's four order model, is that while the edge goes out to infinity in one direction, in the other it spirals infinitely many times around the upper branch point. An initial condition near the upper branch point will spiral outwards before passing around the edge on the way to the laminar state. It is this spiral feature that provides the route from the "turbulent" part of phase space to the "laminar state".

In this section we wish to answer the following. Does a higher dimensional equivalent of this behaviour occur in the full dynamics? It is obvious that we cannot simply plot the phase space of the full dynamics, so we need a test to compare the model with the full dynamics. We shall introduce the test in the reduced model before carrying out the same analysis in PCF. We begin with a trajectory spiraling out from the unstable equilibrium, and select several time points along the trajectory. At these points we shall calculate a new initial condition

$$\mathbf{x}_{in} = \lambda \mathbf{x} \tag{6}$$

in the model for a range of λ around $\lambda = 1$, the condition that recovers the original point on the relaminarizing trajectory. For each value of λ we will evolve the new initial condition and study the dynamics. A condition close to the surrogate for turbulence will be "above" the edge, therefore there exists a $\lambda \in [0, 1]$ for which a new condition lies

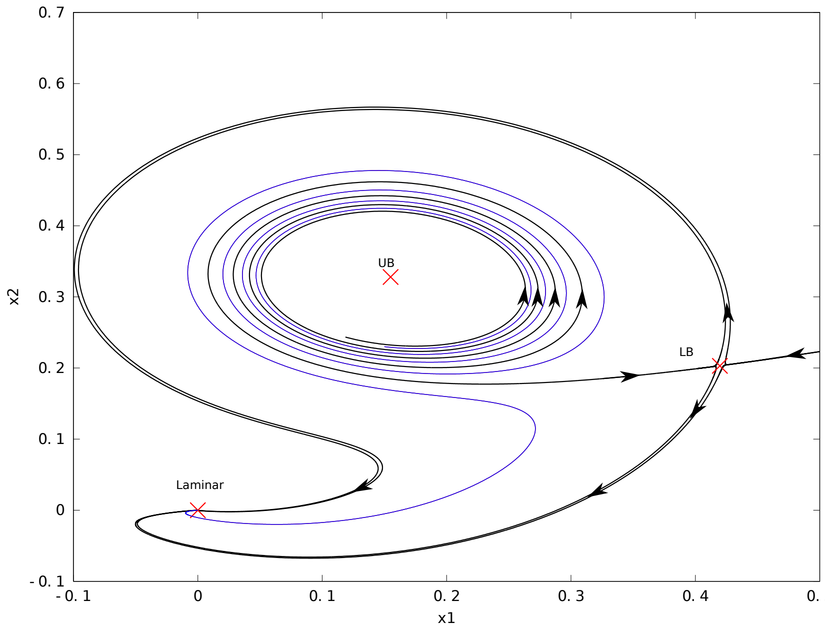

Figure 6: Phase space for 2D model. Red crosses indicate fixed points (increasing x_1) Laminar surrogate, Turbulence surrogate and Edge state surrogate. Black lines show the unstable and stable manifolds of the edge state. Stable manifold is the edge in this system. Blue line an example decaying trajectory beginning "near" turbulence.

below the edge and will quickly relaminarize. The subsequent behaviour is measured by a time average of the L2 norm of $\mathbf{x}$, (called the T measure), taken over a suitable time. The T measure should be considered a surrogate for the lifetime of the flow. Considering how this T measure changes with λ will locate the edge relative to the relaminarizing trajectory. A sudden and large change in the value of T denotes a transition across the edge. Figures 7 shows the evolution in phase space of the rescaled and original decaying trajectories, from an initial condition above and on the outside of the spiral structure. This measure of "turbulence" against λ is plotted in figure 8. The original trajectory decays quickly, as do those rescaled further from the laminar state. However for $\lambda < 0.97$ a large increase in the time average of the L2 norm is observed, this is caused by stepping across the edge meaning another spiral must be completed before relaminarization. We will compare these results to the carrying out the same analysis at later point along the original trajectory, with results in figures 9 & 10. The original point lies below the edge, so the reverse λ scaling is observed. For $\lambda < 1.08$ trajectories relaminization quickly, but larger values lie the other side of the edge. The results from these two points highlight the transition that is made as a trajectory in this model passes around the edge on route to the laminar point. By using a relatively short time average of the L2 norm for our projection we restrict the edge crossings that we can observe. Crossing the outer-most loop of the spiral is captured as a transition, but the subsequent crossings of inner parts of the edge are not caught by this relatively short time average. By using this metric we observe a behaviour which is has a parallel in the full dynamics, which will be discussed later.

In the case of the low dimensional model the outcome of the test could be predicted from the phase portrait alone. Having studied this simple system the same analysis can be performed on the full dynamics and allow some interpretation of the complex

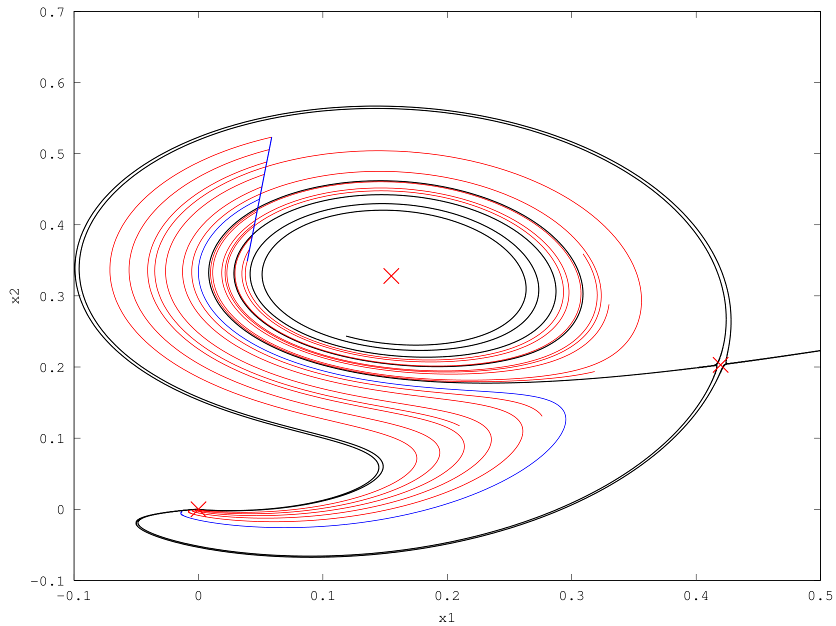

Figure 7: Phase space of the 2D model. Blue curving line denotes the trajectory decaying from "turbulence". Blue straight line shows range of rescaled initial conditions, the evolution of which are plotted in red. Original decaying point lies above the edge. Rescaled initial conditions span the edge.

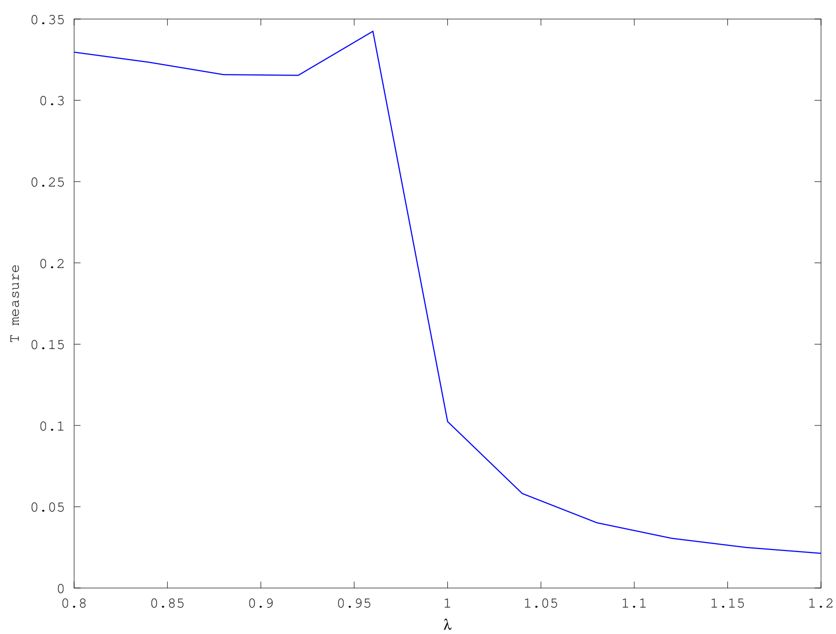

Figure 8: T measure against λ. Measures whether each initial condition maintains "turbulent" evolution during measure time. Values above 0.2 indicate "turbulent" evolution. Values below indicate decay to the laminar state.

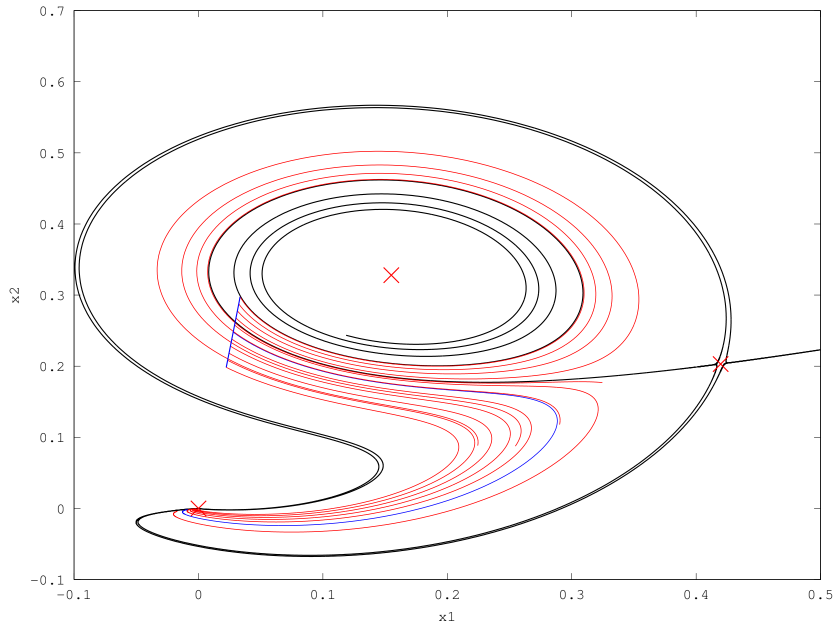

Figure 9: Phase space of the 2D model. Test this time carried out at later time on the original decaying trajectory. Initial condition now lies below the edge. Rescaled initial conditions still span the edge.

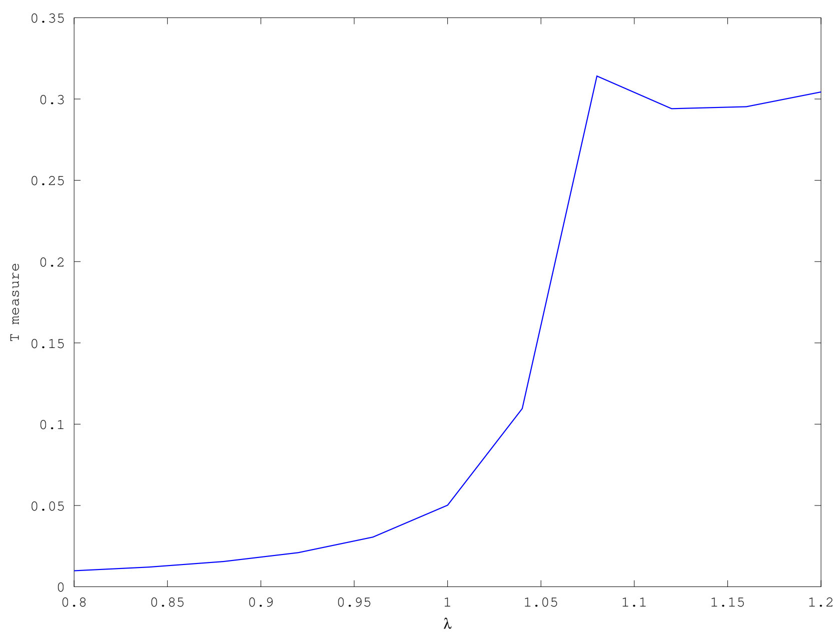

Figure 10: T measure against λ for second point on the trajectory. Graph confirms no edge beneath trajectory (smaller λ) but edge exists above (larger λ).

behaviour of the edge. Before we can begin this work we should consider the methods previously used to study the edge. These methods have revolved around a bisection technique between an original turbulent state and another which smoothly relaminarizes. The criterion as to the turbulent or laminar evolution of the new flow field are usually energy or E_{3D} thresholds, with the turbulent threshold set just below the turbulent value of the metric. These techniques work well for finding the attracting objects embedded within the edge, but using this definition the edge will not be found near the turbulent part of phase space. This can be compared to using the T measure (with a short time average) to define the edge in the low dimensional model. If this was used, only the outer spiral of the edge would be detected. In the two dimensional model the edge is defined to be the stable manifold of the edge state, which removes the issue. In the full system, in geometries which have a single attracting edge state, we can use the same definition. While the edge state may be located using the bisection technique, we suggest that the edge is defined as the stable manifold of the edge state. It should be noted that this definition does not automatically solve the problem of tracking the edge near turbulent, which will be discussed later in this work. With this new definition of the edge, we can move forward to comparing the full dynamics with the low dimensional model.

The test begins with a single relaminarizing trajectory, along which two test points are selected. The flow fields at these points are rescaled using the same methodology as before,

$$\mathbf{u}_{in} = \lambda \mathbf{u} \tag{7}$$

where $\lambda = 1$ recovers the original dynamics. For each value of λ the initial condition is generated, evolved for a set time, and the time average of the L2 norm of $\mathbf{u}$ recorded (again called the T measure). Figure 11 shows the evolution of the L2 norm of $\mathbf{u}$ for the original trajectory and rescaled conditions for $\lambda \in [0.8, 1.3]$, and in figure 12 is plotted the T measure against λ. By choice the original trajectory decays quickly, as do its neighbours in λ. However for both larger and smaller values of λ the T measure increases, indicating a crossing of the edge to turbulent dynamics on both sides of the point. A second region where trajectories decay exists at $\lambda \simeq 0.9$. The main point to receive is that there exists an edge (in this case several pieces of the edge) below the relaminarizing trajectory. The next step is to carry the process out at a later time, the results of which are presented in figures 13 & 14. A transition has occurred since the previous analysis, as there now exists no edge beneath the relaminarizing trajectory. Turbulent dynamics are only to be found by choosing $\lambda > 1.2$. While the results of conducting this test on the full dynamics are far more complicated than those of the low dimensional model, certain characteristics are maintained in both situations. Early in the decay, part of the edge exists beneath the trajectory in phase space. Yet later, this edge beneath is no longer present. This result is unsurprising given that by the definition of the edge this transition from one "side" to another has to occur. However the time and manner of the transition can provide evidence as to the topology of the edge.

Early in the decay the test shows the existence of an edge below the decaying trajectory. Using the bisection techniques briefly discussed earlier we can track the dynamics on this edge. Later in the decay of the original trajectory there exists only an edge above the decaying trajectory. Again bisection can be used to track the dynamics of this part of the edge. Figure 15 shows the evolution of these two pieces of edge, the edge beneath in green and the edge above in red. The dynamics on these two pieces of edge initially move in different directions, however 200 time units later they become involved in the same dynamics which is maintained for the rest of the time the edge is accurately tracked[2]. This result suggests that the two pieces of edge above and below

[2] i.e. when the coloured pairs diverge. There is no evidence that if the edge was tracked longer these

the trajectory are dynamically connected. Building on the previous result we now have a more complete picture of the route past the edge. The transition from the trajectory lying above the edge to below it, added to the dynamical connection of those two pieces of edge fits with the low dimensional picture built by the 2D model of Lebovitz. The complete geometry of the edge in the full dynamics is much more complicated than a 2D model could hope to model, as shown by the multiple layers of edge observed in figure 12. Two further details from this work also suggest agreement with the low dimensional model, that the edge goes up into the turbulent part of phase space. If we examine the location of the transition in relative edge position of a trajectory (from above the edge to below the edge) under the E_{3D} projection of the dynamics we note that this transition occurs at values of E_{3D} associated with turbulent dynamics. This metric is commonly used to evaluation if a flow field is turbulent and therefore suggests the edge also exists near turbulence in phase space. This claim could be proved by developing a method to track the edge into turbulence, however a method to perform this is not clear. The second observation to be made concerns the evolution of the dynamics after the transition in edge location has occurred. As can be seen in figure 15, the relaminarizing trajectory tracks the dynamics of the edge for approximately 50 time units before diverging. This picture is consistently observed and helps our understanding of the manner in which trajectories pass by the edge. The trajectories running approximately parallel to the edge in phase space fits the picture produced from the low dimensional model. None of the results found in this work are individually convincing arguments for the spiral topology of the edge. However together they begin to form a body of evidence which supports a more complicated version of this cartoon of the edge geometry. The manner of the transition around the edge appears to agree with the picture constructed. The E_{3D} projection suggests this transition happens near turbulence, and the dynamics of the edge near this transition appear to be connected. The evolution of the trajectory as it leaves turbulence remains close to the edge for a significant amount of time (> 50 time units) both before and after transition, which again is in agreement with the low dimensional model.

5 Conclusion

The focus of this work has been understanding the role that the edge plays during decay from turbulence. Beginning from a hypothesis that turbulence decayed through a point in the edge, we examined the statistics and physical processes involved in decay. We finished by comparing the dynamics of the edge in a low dimensional system to the full dynamics of plane Couette flow. There was no evidence to support the hypothesis of a unique decay point, instead what was observed was a wide variety of decay rates and routes back to the laminar state. We statistically confirmed ideas about the physical processes involved during the decay. Statistically all parts of the flow field begin to decay at the same point in time, but the downstream rolls and x-dependent part of the flow field decay at a greater relative rate. The last 200 time units of decay involve only downstream fast and slow streaks. The rate that these streaks decay is set by the horizontal length-scales involved. The streak structure during decay is simple, involving predominately the first and second horizontal Fourier modes. The relative energy in these two modes sets the decay rate. To understand the geometry of the edge in the full dynamics we ran tests to compare this to a 2D model. This 2D model had been selected as the simplest model containing edge structure seen in several low dimensional models of shear flows. In these models the edge extended to infinity in one direction, but in another wrapped up infinitely many times around a structure (in the case of

edge dynamics would separate.

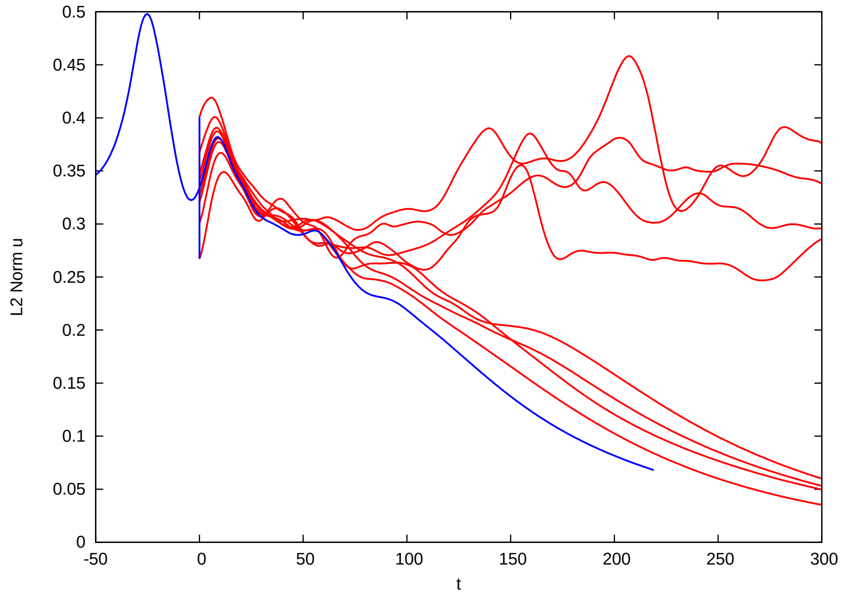

Figure 11: L2 norm of **u** against time. Blue curve shows decaying trajectory in full dynamics. Blue straight line shows range of initial condition from λ rescaling. Red curves show selected evolution trajectories from these new initial condition.

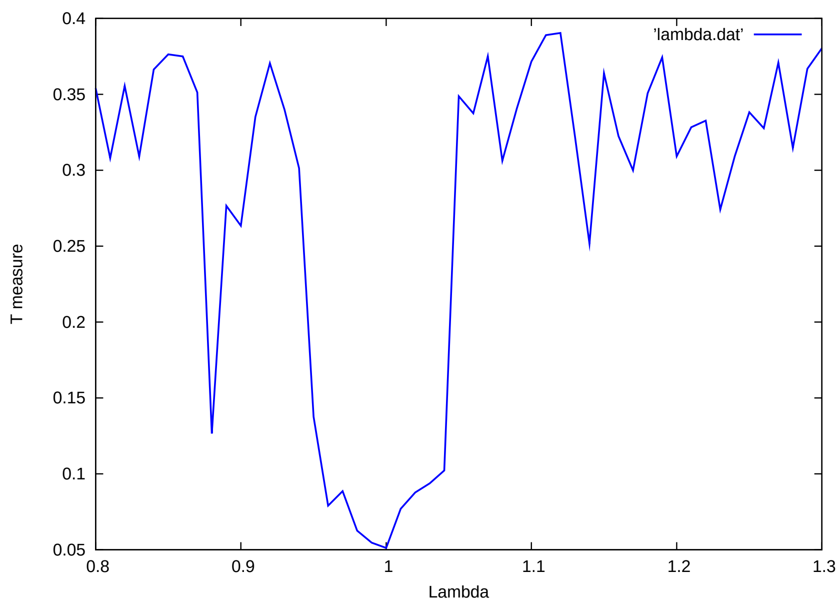

Figure 12: T measure against λ for first point in full dynamics. T measure values larger than 0.2 indicate turbulent evolution. Highly complex edge structure evident, with 3 regions of turbulent dynamics and 2 regions of laminar dynamics. Points to 3 individual edge pieces.

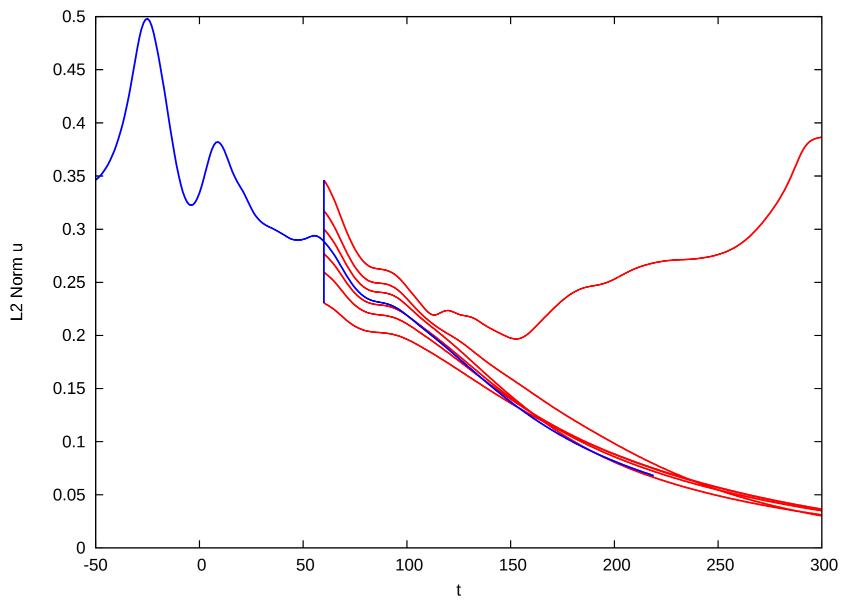

Figure 13: L2 norm of **u** against time for second point 60 time units later. Blue curve shows decaying trajectory in full dynamics. Blue straight line shows range of initial condition from λ rescaling. Red curves show selected evolution trajectories from these new initial condition.

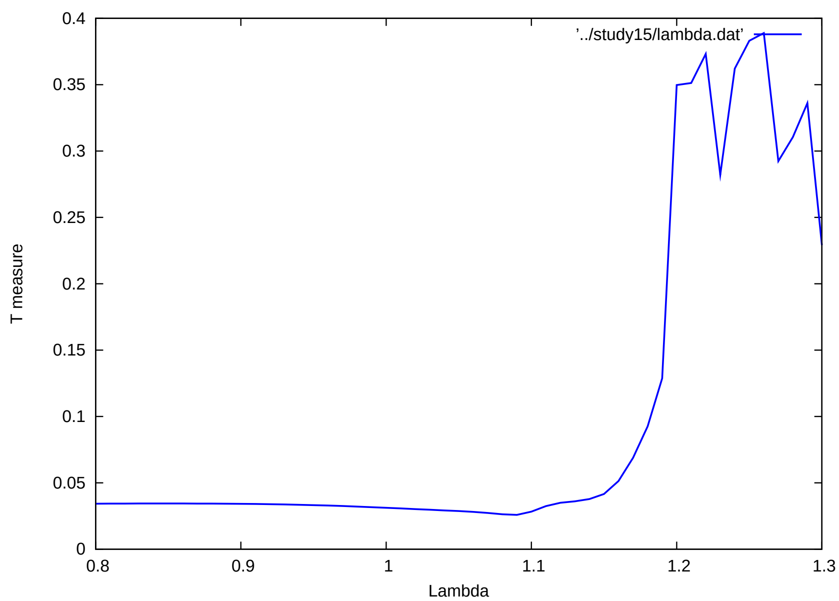

Figure 14: T measure against λ for second point. Simpler structure, evidence for single edge piece intersecting the rescaling line. Edge lies above the relaminarizing trajectory.

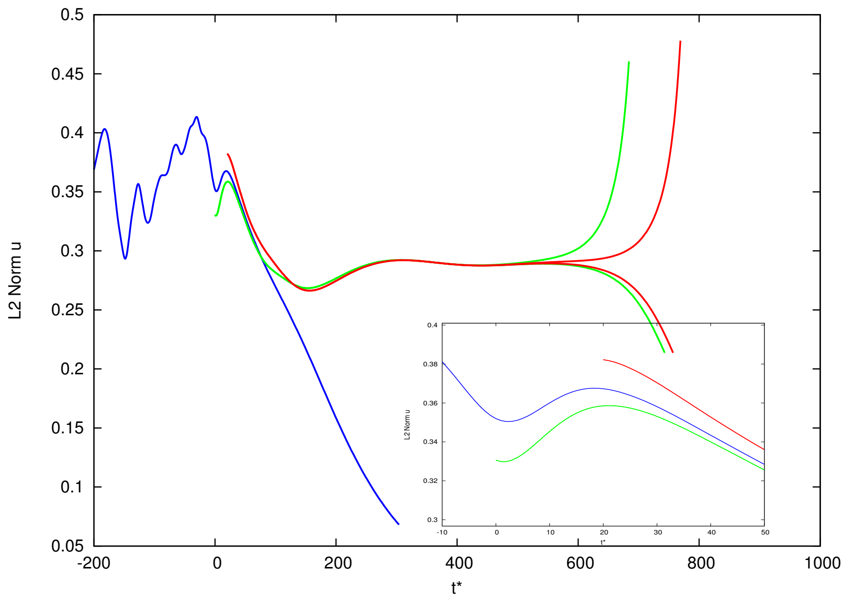

Figure 15: L2 norm of $\mathbf{u}$ against t (relabeled to locate initial edge bisection at zero). Blue curve - relaminarizing trajectory which transitions from having edge below to edge above only. Green curves - bounding curves for edge bisection tracking the dynamics of the edge below the blue curve at $t = 0$. Red curves - bounding curves for edge bisection tracking the dynamics of the edge above the blue curve at $t = 20$. Both pieces of edge become involved in same dynamics, indicating dynamical connection. Inset: close-up around $t = 0$.

the 2D model a fixed point). Comparing to this model we saw some evidence in the full dynamics for similar, if more complicated structure. Decaying trajectories passed around the edge, and the parts it passed around were found to be dynamically connected in some cases. The value of E_{3D} at this transition suggests that the edge passes up into phase space. However we do not claim that this evidence proves this model to be accurate. What we have succeeded in doing, is test the validity of this model with a series of tests, and further work is required before stronger conclusions can be made. There are several areas of this work where interesting extensions are clear. Conducting the statistical analysis for a larger domain size would allow study of the conjectures on streak structure. Are six-streak structures observed in larger domains during decay, and at what Reynolds numbers? Another question opened up by this work is into the significance of the location where a decaying trajectory passes around the edge. Does this occur at a specific time before decay, and are there particular characteristics of the flow field as the edge is rounded?

Despite having held attention of great minds since 1883, turbulence in shear flows is an area where real progress is currently being made around the globe. Modern ideas and powerful computers have enabled the discovery of new and complex solutions, which bear resemblance to experimental work. Progress has been made furthering our understanding of the edge, and the role that it plays in the dynamics of turbulent shear flows. The edge itself changes strongly with Reynolds number, with different solutions playing the role as the edge state. The idea that the edge is wrapped and folded around the turbulent dynamics, which a trajectory must negotiate in order to return to the laminar state, is an idea which requires further study before conclusions can be drawn.

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