Lecture 6: The Batchelor spectrum and tracer cascade

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1 The advection–diffusion equation

In this lecture we consider the problem of determining the spectrum of a passive tracer advected by a velocity field. The evolution of the tracer is described by the equation

$$\partial_t \Theta + \mathbf{u} \cdot \nabla \Theta = \kappa \nabla^2 \Theta, \tag{1}$$

where κ is the molecular diffusivity, $\mathbf{u}$ the advection velocity field and ∂_t denotes a partial derivative with respect to time. Depending on the particular experimental situation, the tracer Θ could be temperature or density of a passive contaminant such as ink. By Θ being passive, we mean that the dynamics of the velocity does not couple with Θ. Thus equation (1) is truly linear in Θ. We confine our attention to the case where $\mathbf{u}$ is non-divergent. Most of the mathematical details will be carried out in the two dimensional case, though many of the techniques and arguments admit ready generalizations to any number of dimensions. Throughout, we employ cartesian coordinates (x, y).

Batchelor [1] realized that the general increase of gradients of Θ accompanying the stirring action of the velocity field, which is a consequence of the quadratic term of (1), can also be thought of as a transfer between different Fourier components of the spectrum of Θ. If both $\mathbf{u}$ and Θ are written in the form of Fourier integrals, the term $\mathbf{u} \cdot \nabla \Theta$ leads to the generation of new harmonics of Θ and the growth of ever–increasing wavenumbers. The transfer of tracer variance from low wavenumbers to high wavenumbers is mathematically similar to that hypothetised by Kolomogorov for the velocity variance in a turbulent field.

The Fourier components of the tracer Θ, if the flow is spatially unbounded, are defined by

$$\Theta(x, y) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dk\, dl\, \hat{\Theta}(k, l) e^{ikx+ily}. \tag{2}$$

In the present context, it is convenient to define a spectrum function for Θ as

$$\mathcal{C}(K) = \frac{1}{V} \frac{1}{(2\pi)^2} \int_0^{2\pi} d\phi\, K |\hat{\Theta}(K, \phi)|^2, \tag{3}$$

where $\hat{\Theta}(K, \phi)$ are the Fourier components of Θ in polar coordinates $K = \sqrt{k^2 + l^2}$ and $\phi = \arctan(l/k)$. If the tracer field is isotropic the integral is trivial and $\mathcal{C}(K) = K|\hat{\Theta}(K))|^2 \,/\, 2\pi V$. The normalization factor $V = \int \int_V dx\, dy$ ensures that the spectrum has units of variance of Θ per unit wavenumber. In fact it follows from Parseval's equality that the variance of Θ is the integral of $\mathcal{C}(K)$ in wavenumber space

$$\langle \Theta^2 \rangle = \lim_{V \to \infty} \frac{1}{V} \int \int_V dx\, dy\, \Theta^2(x, y) = \int_0^{+\infty} dK\, \mathcal{C}(K). \tag{4}$$

Following the line of argument of Kolmogorov and Batchelor, we suppose that the diffusivity κ is so small as to make the effect of diffusion appreciable only at the large wavenumber end of the spectrum. The part of the equilibrium range of wavenumbers for which the Fourier components of $\mathbf{u}$ are independent of viscosity is usually termed the "inertial subrange" and an appropriate term for the part of the equilibrium range for which the Fourier components of Θ are independent of molecular diffusion is the "advection subrange". No actual destruction of tracer variance takes place at wavenumbers in, or smaller than those in the advection subrange. All the destruction takes place at high wavenumbers as a result of the action of molecular diffusion. The total rate of destruction of variance per unit volume is calculated by integrating the advection–diffusion equation over the whole domain

$$\partial_t \langle \Theta^2 \rangle = -2\kappa \langle |\nabla\Theta|^2 \rangle = -\chi. \tag{5}$$

This relation shows that the quadratic terms in (1) makes no contribution to $\partial_t \langle \Theta^2 \rangle$. Thus when one Fourier component of the spectrum of Θ is changed by the interaction between Θ and $\mathbf{u}$, other Fourier components are changed simultaneously in such a way that the sum of the contributions to $\langle \Theta^2 \rangle$ from all Fourier components remains the same. This shows that Θ variance is simply transferred from small to large wavenumbers in the advection subrange and χ is a given constant quantity.

If the velocity field $\mathbf{u}$ is characterized by a single time scale τ, it is possible to predict the spectrum of Θ on dimensional grounds. In fact the spectrum $\mathcal{C}(K)$ at a wavenumber K is determined by χ, the mean rate at which variance is cascaded, the time scale τ, and the local wavenumber K. There is only one combination of these three parameters with dimensions of $\langle \Theta^2 \rangle$ per unit wavenumer and this combination gives the functional form of the spectrum,

$$\mathcal{C}(K) \sim \chi\, \tau\, K^{-1}. \tag{6}$$

Batchelor noticed that this result is of physical interest for tracers whose diffusivity is much smaller than the viscosity of the advecting fluid. In this case the velocity field at scales shorter than the viscous cutoff is extremely smooth and the primary effect of the advection on variations of Θ is a uniform straining rate of strength τ^{-1}.

A characterization of the spectrum of Θ can be obtained also at scales larger than the viscous cutoff, if the velocity spectrum has a power law dependence on the wavenumber. The spectrum of velocity $\mathcal{E}(K)$ is the variance of velocity as a function of wavenumber and is defined in a way analogous to the spectrum of Θ such that

$$\frac{1}{2}\langle \mathbf{u} \cdot \mathbf{u} \rangle = \int_0^{+\infty} dK\, \mathcal{E}(K). \tag{7}$$

If $\mathcal{E}(K) \sim K^{-\alpha}$, a time-scale τ for the velocity field is given by the turnover time at scale K,

$$\tau \sim \frac{K^{-1}}{\sqrt{K\, \mathcal{E}(K)}} \sim K^{-(3-\alpha)/2}. \tag{8}$$

The eddy turnover time decreases with scale for $\alpha < 3$ and the picture of larger eddies feeding smaller eddies is appropriate. For $\alpha > 3$ the cascade of energy through different scales cannot be considered local in wavenumber space and the estimate of τ must be corrected to include the presence of non local effects, but this is beyond the scope of this lecture. Given an estimate for τ, the same dimensional argument used for the case of a velocity field characterized by only one time scale implies

$$\mathcal{C}(K) \sim \chi\, K^{-(5-\alpha)/2}. \tag{9}$$

According to Kolmogorov's arguments on isotropic and homogeneous turbulence, the spectrum of velocity in the inertial subrange scales as $\mathcal{E}(K) \sim K^{-5/3}$. In this case the tracer spectrum scales as $\mathcal{C}(K) \sim K^{-5/3}$ as well and is known as the Obukhov–Corrsin spectrum [2, 3].

For tracers advected by a turbulent, three-dimensional velocity field, with diffusivity κ much smaller than the viscosity of the fluid ν, we expect to see a spectrum $\mathcal{C}(K) \sim K^{-5/3}$ at scales larger

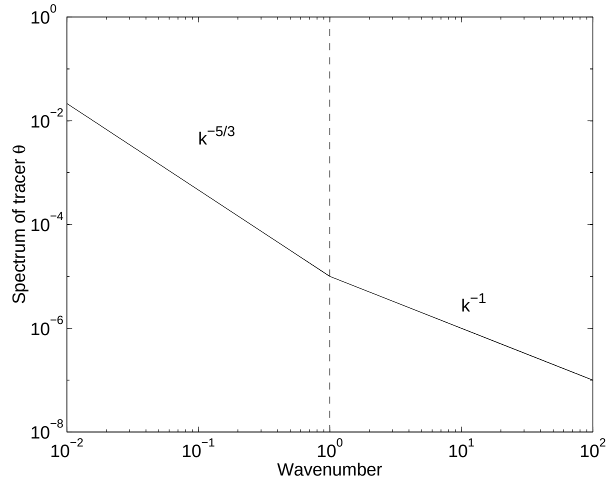

Figure 1: Spectrum of a passive tracer advected by a 3D turbulent field. Wavenumbers are normalized by the cutoff viscous scale (the scale at which momentum is dissipated).

than the viscous cutoff, and a spectrum $\mathcal{C}(K) \sim K^{-1}$ at smaller scales, as shown in Figure 1. The scaling breaks off at large wavenumbers at the dissipation cutoff (the scale at which the tracer is dissipated, here assumed smaller than the scale at which momentum is dissipated) and at small wavenumbers at the scale of the domain, if the domain is finite, or at the scale of forcing if there is some forcing feeding variance in Θ.

2 The 2-point correlation function: what can it tell us about the properties of the flow.

2.0.1 Definition and uses of the 2-point correlation function

The aim of the two point correlation function $Z_2(r)$ of a concentration field is to yield information about the typical variation of the concentration over a distance r. Let's therefore define it as

$$Z_2(r; ...) = \langle |\Theta(\mathbf{x} + r\hat{\mathbf{n}}) - \Theta(\mathbf{x})|^2 \rangle \tag{10}$$

where $\hat{\mathbf{n}}$ is a unit vector in the chosen direction, and the average can be a time, spatial, or ensemble average. Depending on the average chosen, $Z_2(r; ...)$ may also depend on the time t (for spatial or ensemble average), or the starting position $\mathbf{x}$ (for the time average if the fluid is not homogeneous). If the flow is anisotropic, the correlation function depends on $\hat{\mathbf{n}}$ as well.

The 2-point correlation function can also be rewritten as

$$\begin{aligned} Z_2(r, \hat{\mathbf{n}}) &= \langle \Theta(\mathbf{x})^2 \rangle + \langle \Theta(\mathbf{x} + r\hat{\mathbf{n}})^2 \rangle - 2\langle \Theta(\mathbf{x})\Theta(\mathbf{x} + r\hat{\mathbf{n}}) \rangle \\ &= 2\langle \Theta^2 \rangle - 2\langle \Theta(\mathbf{x})\Theta(\mathbf{x} + r\hat{\mathbf{n}}) \rangle \end{aligned} \tag{11}$$

if the flow is spatially homogeneous. More generally, one could define correlation functions of any order:

$$Z_q(r) = \langle |\Theta(\mathbf{x} + r\hat{\mathbf{n}}) - \Theta(\mathbf{x})|^q \rangle. \tag{12}$$

As q becomes larger, Z_q is increasingly dominated by the most extreme fluctuations (i.e. $\Theta(\mathbf{x} + r\hat{\mathbf{n}}) - \Theta(\mathbf{x})$ locally very large).

The 2-point correlation function yields precious information about the global spatial structure of the flow. For instance,

- if the flow is smooth at all points , then the concentration gradient exists everywhere and $|\nabla\Theta| < +\infty$. As a result, for small enough r, we get $Z_2(r) \approx \langle|\nabla\Theta|^2\rangle r^2$;
- for a flow containing well separated jumps/steps in the concentration field, then $Z_2(r) \propto r$;
- for a non-singular fractal flow, with a possible local accumulation of steps, the correlation function becomes $Z_2(r) \propto r^\nu$; where $1 < \nu < 2$. Smaller values of ν arise form singularities in the tracer field—places where the tracer value becomes infinite;
- if the flow contains integrable singularities, then $Z_2(r) \propto r^\nu$ with $0 < \nu < 1$, whereas non-integrable singularities have $\nu < 0$;
- for a flow with a white noise spectrum, there is no correlation between any two points, so that the correlation function is constant $Z_2(r) \propto r^0$.

To summarize, non-singular flows have 2-point correlation functions given by $Z_2(r) \propto r^\nu$ where $0 < \nu < 2$. There exists no interesting flows with $\nu > 2$ since this would require the gradient of the velocity to be 0 everywhere.

2.1 Relation between the 2-point correlation function and the power spectrum

We saw that the flow variance can be decomposed onto the spectral modes as

$$\langle\Theta^2\rangle = \int_0^\infty \mathcal{C}(K)\mathrm{d}K \tag{13}$$

where, say, $\mathcal{C}(K) \propto K^{-\alpha}$. In the specific case of 1-D, one can show that the correlation function can also be rewritten as [4, p. 95]

$$Z_2(r) = 2\int_0^\infty \mathcal{C}(K)(1 - \cos(Kr))\mathrm{d}K \tag{14}$$

so that

$$\begin{aligned} Z_2(r) &= 2\int_0^{1/r} \mathcal{C}(K)K^2r^2\mathrm{d}K + 2\int_{1/r}^\infty \mathcal{C}(K)(1 - \cos(KR))\mathrm{d}K \\ &\approx 2\int_0^{1/r} \mathcal{C}(K)K^2r^2\mathrm{d}K + 2\int_{1/r}^\infty \mathcal{C}(K)\mathrm{d}K \\ &\approx 2r^2\left[\frac{K^{3-\alpha}}{3-\alpha}\right]_0^{1/r} + 2\left[\frac{K^{1-\alpha}}{1-\alpha}\right]_{1/r}^\infty \end{aligned} \tag{15}$$

Note that only the tail of $\mathcal{C}(K)$ varies like $K^{-\alpha}$, so that the 0 bound of the integral does not actually pose any problems, and is a given constant, say ζ_0. Various cases can occur.

- in the case where $\alpha > 3$ then the integral is dominated by the first term, and so $Z_2(r) \approx \frac{4}{(3-\alpha)(\alpha-1)}r^{\alpha-1} + 2r^2\zeta_0$. For small r the dominant term is therefore $Z_2(r) \propto r^2$.

- in the case $1 < \alpha < 3$ then the integral is

$$Z_2(r) \approx 2r^2 \frac{r^{\alpha-3}}{3-\alpha} + 2r^2\zeta_0 + 2\frac{r^{\alpha-1}}{\alpha-1} = r^{\alpha-1}\frac{4}{(3-\alpha)(\alpha-1)} + 2r^2\zeta_0 \tag{16}$$

  For very small r, the dominant term is $Z_2(r) \propto r^{\alpha-1}$.

- in the case where $\alpha < 1$ the second integral is not convergent; $Z_2(r)$ is not defined in that case. However, in the limit $\alpha \to 1$ and $\alpha > 1$, the dominant terms are

$$Z_2(r) \approx \frac{4}{(3-\alpha)(\alpha-1)} + \frac{4}{(3-\alpha)} \ln r \propto \ln\left(\frac{r}{r_\star}\right) \tag{17}$$

  so that the length-scale $r_\star$ appears as a cutoff below which the flow is smooth on all scales. The limit $\alpha \to 1$ corresponds to the Batchelor spectrum, and $r_\star$ is the dissipation scale.

Note that in the case where α is greater than 3, the correlation function increases faster than r^2, which cannot correspond to any physical flow. We also see that provided $1 < \alpha < 3$ the following simple scaling argument applies: $Z_2(r)$ is related to the variance on a scale of r, so

$$\begin{aligned} Z_2(r) &\approx \mathcal{C}(K)\delta K \approx K^{-\alpha}K \text{ where } K = 1/r \\ &\propto r^{\alpha-1}. \end{aligned} \tag{18}$$

3 Determination of $Z_2(r)$ from the mixing properties of the flow

3.1 The non-diffusive case

In the case where the equation governing the spread of concentration contains no source terms, or diffusive terms, we have

$$\frac{\mathrm{D}\Theta}{\mathrm{D}t} = 0 \tag{19}$$

so in order to know $\Theta(\mathbf{x}, t)$ we only need to track the trajectories back in time to the initial conditions. The correlation between the concentrations of 2 points $\mathbf{x}$ and $\mathbf{x}'$ in the fluid is equal to the correlation of the concentration of $\mathbf{x}_0$ and $\mathbf{x}'_0$ in the initial conditions, where $\mathbf{x}_0$ and $\mathbf{x}'_0$ are the initial positions of $\mathbf{x}$ and $\mathbf{x}'$. However, we also know from the stretching properties of the fluid that any 2 points grow exponentially further apart as time evolves (either forwards or backwards). If $\mathbf{x}$ and $\mathbf{x}'$ are separated by r at a time t, $\mathbf{x}_0$ and $\mathbf{x}'_0$ were separated, on average, by $r_0 = r\exp(\Lambda t)$ at $t_0 = 0$, where Λ is the finite time Lyapunov exponent of the flow. If the initial conditions of the flow have a typical correlation length-scale L (for instance, the length-scale of the initial forcing in a decaying turbulence experiment), then there will be 2 regimes:

- either $r\exp(\Lambda t) < L$, then the particles have remained in the same "eddy", within the same correlation length-scale, so that we simply have

$$\Delta_r\Theta \approx |\nabla\Theta|_0 r \exp(\Lambda t) \tag{20}$$

  where $|\nabla\Theta|_0$ is the typical gradient of the eddy at $t = 0$.

- on the other hand, if $r\exp(\Lambda t) > L$ the points are uncorrelated and so $\langle|\Delta_r\Theta|^2\rangle = 2\langle\Theta^2\rangle|_0$, where $\langle\Theta^2\rangle|_0$ is the variance of the initial flow.

The resulting profile is shown in in Fig. 2; as one can see, there seems to be no logarithmic profile appearing, which would have been the sign of the Batchelor regime. This is typical of the initial value problem, where the flow keeps a memory of the initial condition for some period of time before reaching a random flow.

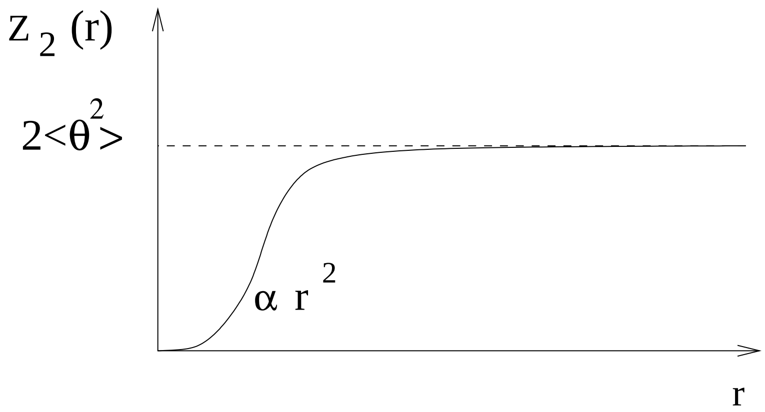

Figure 2: two-point correlation function in the unforced, non-diffusive case.

3.2 The forced case

This time we have

$$\frac{\mathrm{D}\Theta}{\mathrm{D}t} = \kappa\nabla^2\Theta + S(\mathbf{x}, t) \tag{21}$$

where $S(\mathbf{x}, t)$ is a source term, with a finite correlation length-scale, but no time correlation, i.e.

$$\langle S(\mathbf{x}, t)S(\mathbf{x}', t')\rangle = S_0^2(|\mathbf{x} - \mathbf{x}'|)\delta(t - t') \tag{22}$$

where $S_0(|\mathbf{x} - \mathbf{x}'|)$ decays on a lengthscale L. One can distinguish 2 regimes. In the initial phase the flow is mixed without dissipation; since the forcing is uncorrelated in time, the system behaves like a random walk process, so that

$$\langle\Theta^2\rangle \propto S_0^2 t \tag{23}$$

Later on, the flow reaches a steady state with a balance between the forcing and the mixing/dissipation terms. The dissipation scale $l_\star$ for which the strain balances the dissipation is $l_\star = \sqrt{\kappa/\Lambda}$ (cf. Bill's lecture). The time $t_\star$ needed for the flow to create structures on the dissipation scale is $t_\star = \frac{1}{\Lambda}\ln(L/l_\star)$. The time-scale $t_\star$ can be seen as the memory of the system. Trajectories with small Lyapunov exponent have a large memory, the flow remains correlated for longer times. The 2-point correlation function will mostly depend on scales of order of L. Indeed, if 2 points are separated by $r < L$, they must come from regions which have the same source term, so that the quantity $Z_2(r)$ is likely to be small. On the other hand, if $r > L$ then the correlation function of the concentration field will be similar to that of the source terms. As before, let's consider 2 points at $\mathbf{x}$ and $\mathbf{x}'$ at a time t and trace their trajectories backward in time. Assuming that the statistics of the forced flow are stationary, the flow builds up correlation when the separation of the 2 points is larger than L, but loses correlation for times larger than the dissipation time-scale. The 2-point correlation function will therefore depend on the difference $t_L - t_\star$, where t_L is the amount of time necessary to reach scales of order L starting from and initial separation r: $t_L = \frac{1}{\Lambda}\ln L/r$. This is illustrated in Fig. 2.

Hence

$$Z_2(r) \propto t_L - t_\star \propto \frac{1}{\Lambda}\ln\left(r/l_\star\right) \tag{24}$$

This logarithmic dependence in r of the correlation function shows that the forced case is consistent with a Batchelor regime. For more details on this subject, see Refs. [5] and [6].

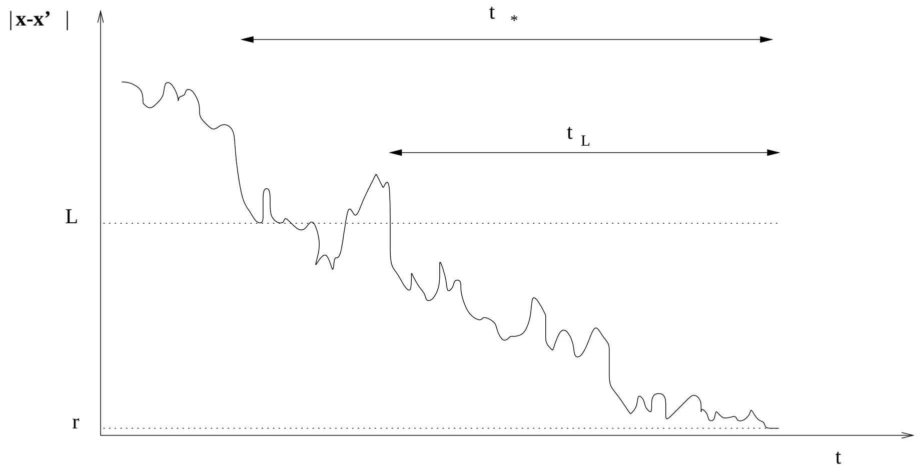

Figure 3: Evolution of the separation of 2 points, and relevant time-scales.

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