

On One-Fluid MHD Models with Electron Inertia

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Abstract

We investigate the limitations of various MHD models that include electron inertia by considering expansion in terms of non-dimensional parameters, and by classifying those that have energy conservation. It is revealed that a correction term in the momentum equation, which is usually neglected, is needed to conserve the total energy. In order to investigate the effect of this correction term, a modified Grad-Shafranov equation is obtained in the “straight torus,” i.e., where toroidal curvature is neglected, for the cases when the plasma has constant density or is barotropic. In the case with toroidal curvature, some conditions on the magnetic field in a torus are needed even if the plasma has constant density.

1 Introduction

The usual ideal magnetohydrodynamic (MHD) model is often used in various fields, for instance, in astrophysics, nuclear fusion and geophysics. Of course, as with all models, ideal MHD has some limitations, such as in a specific regions where magnetic reconnection takes place, which is considered to have an important role in energy transfer. Other effects can also be important and the MHD model may breakdown for other reasons. Consequently, many researchers have considered other MHD models (usually called extended MHD) and investigated their properties both analytically and numerically. However different researchers have used different extended MHD models [1, 3, 4, 10], and the relation and limitation between these models seems to be unknown. For all of these models the difference only exists in the generalized Ohm’s law, with the momentum equation being the same as the usual MHD model.

Here we first classify models with electron inertial in terms of energy conservation; we find which models have an energy conservation law. Next, to investigate the effect of electron inertia, we try to generalize the Grad-Shafranov equation, which describes the equilibrium state with no flow and axisymmetry in a torus [5, 6]. Finally, we introduce some equilibrium states with flow in an incompressible plasma.

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This report is organized as follows. In Section 2 we discuss the limitations of some inertial MHD models and investigate their energy conservation. In Section 3, we classify some inertial MHD models in terms of energy conservation. In Section 4 we try to modify the Grad-Shafranov equation to include the effect of electron inertia term and in Section 5 we introduce some equilibrium states with constant density and with flow. Finally, Conclusions and discussions are given in Section 6.

2 Limitation and Energy Conservation of Inertial MHD Model

In this section we derive the asymptotically consistent inertial MHD (IMHD) model from a complete one-fluid model. We consider an electron-ion plasma that is completely ionized. The one-fluid model has been derived from kinetic theory (see e.g. [3, 2]) of charged particles, that is, a plasma. We begin with the model of Lüst [8] who derived the continuity equation, the momentum equation, and the generalized Ohm's law for the one-fluid model.

Using the fact that the electron mass is much lighter than the ion mass and assuming the plasma is quasi-neutral, we obtain the continuity equation, the momentum equation, and the generalized Ohm's law as follows:

$$\begin{aligned}
0 &= \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}), \\
\rho \left(\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} \right) &= -\nabla p + \mathbf{j} \times \mathbf{B} - \frac{m_e}{e} (\mathbf{j} \cdot \nabla) \frac{\mathbf{j}}{en}, \\
\mathbf{E} + \mathbf{V} \times \mathbf{B} &= \frac{1}{\sigma} \mathbf{j} + \frac{1}{en} (\mathbf{j} \times \mathbf{B} - \nabla p_e) \\
&\quad + \frac{m_e}{e^2 n} \left[\frac{\partial \mathbf{j}}{\partial t} + \nabla \cdot (\mathbf{V} \mathbf{j} + \mathbf{j} \mathbf{V}) \right] \\
&\quad - \frac{m_e}{e^2 n} (\mathbf{j} \cdot \nabla) \frac{\mathbf{j}}{en},
\end{aligned}$$

where ρ is the density of plasma, \mathbf{V} the bulk velocity, p the pressure, \mathbf{j} the current density, \mathbf{B} the magnetic field, m_e the electron mass, e the elementary charge, n the number density of each species of charged particles, \mathbf{E} the electric field, σ the conductivity, and p_e the electron pressure. Note that the momentum equation is equivalent to the summation of the momentum equations of the electron and ion species, and the generalized Ohm's law is the difference. Ohm's law can be viewed as representing the momentum equation of electron.

The last term on the right-hand-side of the momentum equation exists due to the electron inertia. We call the second term on the left-hand-side of the generalized Ohm's law the "nonlinear term," the first term on the right-hand-side the "collision term," the second term the "Hall term," and the third and fourth terms will be called the "electron inertia terms." To compare the size of these terms, we use the following characteristic numbers:

$$\begin{aligned}
Re_m &\equiv \frac{\text{Nonlinear term}}{\text{Collision term}} = \sigma \mu_0 U L, \\
C_H &\equiv \frac{\text{Hall term}}{\text{Collision term}} = \frac{\sigma B}{en}, \\
C_I &\equiv \frac{\text{Electron inertia term}}{\text{Collision term}} = \frac{\sigma m_e}{e^2 n \tau},
\end{aligned}$$

where U , L , B , and τ are the characteristic velocity scale, length scale, magnitude of magnetic field, and time scale of current change, respectively. The number Re_m is called the magnetic Reynolds number.

In this report we focus on the situation where the electron inertia term is much larger than the collision term and the Hall term; however, the nonlinear term is still considered to be large enough to be comparable with the electron inertia term, that is,

$$Re_m \gg 1, C_I \gg 1, \frac{C_I}{C_H} \gg 1.$$

Since the last relation is equivalent to

$$\frac{C_I}{C_H} = \frac{m_e}{eB} \frac{1}{\tau} = \frac{1}{\Omega_{Ge}\tau},$$

where Ω_{Ge} is the electron gyro-frequency, this relation can be interpreted as saying that the characteristic time scale of the current change is much shorter than the gyro-period of the electron.

Finally, we obtain the IMHD model given by the following equations:

$$0 = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}), \quad (1)$$

$$\rho \left(\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} \right) = -\nabla p + \mathbf{j} \times \mathbf{B} - \epsilon \frac{m_e}{e} (\mathbf{j} \cdot \nabla) \frac{\mathbf{j}}{en}, \quad (2)$$

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = \epsilon \frac{m_e}{e^2 n} \left[\frac{\partial \mathbf{j}}{\partial t} + \nabla \cdot (\mathbf{V} \mathbf{j} + \mathbf{j} \mathbf{V}) \right] - \delta \frac{m_e}{e^2 n} (\mathbf{j} \cdot \nabla) \frac{\mathbf{j}}{en}, \quad (3)$$

$$0 = \frac{\partial S}{\partial t} + (\mathbf{V} \cdot \nabla) S, \quad (4)$$

where S is the entropy of the plasma and the last equation means the plasma is adiabatic. We have inserted book keeping parameters ϵ and δ to label terms, yet both of these have value unity. The above equations are to be solved with the pre-Maxwell's equations,

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0, \\ \nabla \cdot \mathbf{B} &= 0, \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\ \nabla \times \mathbf{B} &= \mu_0 \mathbf{j}. \end{aligned}$$

Note that because of quasi-neutrality, the current density is solenoidal, that is,

$$\nabla \cdot \mathbf{j} = 0.$$

Next, let us consider the energy of this IMHD model. Taking the dot product of \mathbf{V} and the momentum equation, the dot product of \mathbf{j} and the generalized Ohm's law, and from

the pre-Maxwell equations, we obtain the following energy relation:

$$\begin{aligned}
0 = & \frac{\partial}{\partial t} \left(\frac{1}{2} \rho |\mathbf{V}|^2 + \rho U + \epsilon \frac{m_e}{e^2 n} \frac{|\mathbf{j}|^2}{2} + \frac{|\mathbf{B}|^2}{2\mu_0} \right) \\
& + \nabla \cdot \left[\left(\frac{1}{2} \rho |\mathbf{V}|^2 + p + \rho U + \epsilon \frac{m_e}{e^2 n} \frac{|\mathbf{j}|^2}{2} \right) \mathbf{V} \right. \\
& \left. + \epsilon \frac{m_e}{e^2 n} (\mathbf{V} \cdot \mathbf{j}) \mathbf{j} - \delta \frac{m_e}{2e^3 n^2} |\mathbf{j}|^2 \mathbf{j} + \frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \right].
\end{aligned}$$

Because of the electron inertia the term $\frac{m_e}{e^2 n} \frac{|\mathbf{j}|^2}{2}$ is included in the energy density term.

Note that, from the generalized Ohm's law, \mathbf{E} includes the time derivative term $\epsilon \frac{m_e}{e^2 n} \frac{\partial \mathbf{j}}{\partial t}$, so the above formulation is not in the usual conservation form.¹ However, upon integrating the above energy relation over the whole domain V with appropriate boundary conditions, it is revealed that the total energy H , which is defined as

$$H \equiv \int_V \left(\frac{1}{2} \rho |\mathbf{V}|^2 + \rho U + \epsilon \frac{m_e}{e^2 n} \frac{|\mathbf{j}|^2}{2} + \frac{|\mathbf{B}|^2}{2\mu_0} \right) d\mathbf{r},$$

is conserved.

3 Classification of Inertial MHD Model in terms of Energy Conservation

In this section we consider the classification of some IMHD models in terms of the energy conservation. In order to classify these models, we label the different epsilon term the different epsilon as is shown in the following arguments.

First, we consider the compressible IMHD model which is governed by pre-Maxwell's

¹When we solve the governing equations (1)–(4) with the full Maxwell's equations, we obtain the following energy relation:

$$\begin{aligned}
0 = & \frac{\partial}{\partial t} \left(\frac{1}{2} \rho |\mathbf{V}|^2 + \rho U + \epsilon \frac{m_e}{e^2 n} \frac{|\mathbf{j}|^2}{2} + \frac{|\mathbf{B}|^2}{2\mu_0} + \frac{1}{2} \epsilon_0 |\mathbf{E}|^2 \right) \\
& + \nabla \cdot \left[\left(\frac{1}{2} \rho |\mathbf{V}|^2 + p + \rho U + \epsilon \frac{m_e}{e^2 n} \frac{|\mathbf{j}|^2}{2} \right) \mathbf{V} \right. \\
& \left. + \epsilon \frac{m_e}{e^2 n} (\mathbf{V} \cdot \mathbf{j}) \mathbf{j} - \delta \frac{m_e}{2e^3 n^2} |\mathbf{j}|^2 \mathbf{j} + \frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \right],
\end{aligned}$$

and this relation is of the usual conservation form.

equations and by the following equations:

$$\begin{aligned}
0 &= \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}), \\
\rho \left(\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} \right) &= -\nabla p + \mathbf{j} \times \mathbf{B} - \epsilon_{\text{EOM}} \frac{m_e}{e} (\mathbf{j} \cdot \nabla) \frac{\mathbf{j}}{en}, \\
\mathbf{E} + \mathbf{V} \times \mathbf{B} &= \epsilon_t \frac{m_e}{e^2 n} \frac{\partial \mathbf{j}}{\partial t} + \epsilon_{\text{ad}} \frac{m_e}{e^2 n} (\mathbf{V} \cdot \nabla) \mathbf{j} + \epsilon_{\text{cp}} \frac{m_e}{e^2 n} \mathbf{j} (\nabla \cdot \mathbf{V}) \\
&\quad + \epsilon_{\text{M}} \frac{m_e}{e^2 n} (\mathbf{j} \cdot \nabla) \mathbf{V} - \delta \frac{m_e}{e^2 n} (\mathbf{j} \cdot \nabla) \frac{\mathbf{j}}{en}, \\
0 &= \frac{\partial S}{\partial t} + (\mathbf{V} \cdot \nabla) S,
\end{aligned}$$

where S is the entropy and δ , ϵ_t (time derivative), ϵ_{ad} (advection), ϵ_{cp} (compressible), ϵ_{M} (related to EOM), ϵ_{EOM} (in EOM) are labels of these terms. In this system, the energy conservation is as follows:

$$\begin{aligned}
&\frac{\partial}{\partial t} \left(\frac{1}{2} \rho |\mathbf{V}|^2 + \rho U + \epsilon_t \frac{m_e}{e^2 n} \frac{|\mathbf{j}|^2}{2} + \frac{|\mathbf{B}|^2}{2\mu_0} \right) \\
&\quad + \nabla \cdot \left[\left(\frac{1}{2} \rho |\mathbf{V}|^2 + p + \rho U + \epsilon_{\text{ad}} \frac{m_e}{e^2 n} \frac{|\mathbf{j}|^2}{2} \right) \mathbf{V} \right. \\
&\quad \quad \left. + \epsilon_{\text{M}} \frac{m_e}{e^2 n} (\mathbf{V} \cdot \mathbf{j}) \mathbf{j} - \delta \frac{m_e}{e^3 n^2} \frac{|\mathbf{j}|^2}{2} \mathbf{j} + \frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \right] \\
&= (\epsilon_t - \epsilon_{\text{ad}}) \frac{m_e}{e^2 n} \frac{|\mathbf{j}|^2}{2} \frac{\nabla \cdot (n \mathbf{V})}{n} + (\epsilon_{\text{ad}} - \epsilon_{\text{cp}}) \frac{m_e}{e^2 n} |\mathbf{j}|^2 (\nabla \cdot \mathbf{V}) \\
&\quad + (\epsilon_{\text{M}} - \epsilon_{\text{EOM}}) \frac{m_e}{e} \mathbf{V} \cdot \left\{ (\mathbf{j} \cdot \nabla) \frac{\mathbf{j}}{en} \right\}.
\end{aligned}$$

Then, since the ϵ_t term have to exist in this model because when deriving the IMHD model we evaluate the electron inertia term using the characteristic time scale of current change, we find that the total energy is conserved only when all the epsilon terms exist or ϵ_t , ϵ_{ad} and ϵ_{cp} exist. Therefore we conclude that the epsilon term in the momentum equation is very important to conserve the total energy in IMHD model.

Secondly, we consider the incompressible IMHD model which is governed by pre-Maxwell's equations and by the following equations:

$$\begin{aligned}
\rho &= \rho_0 = \text{const.} \quad \Leftrightarrow \quad n = n_0 = \text{const.}, \\
\rho_0 \left(\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} \right) &= -\nabla p + \mathbf{j} \times \mathbf{B} - \epsilon_{\text{EOM}} \frac{m_e}{e^2 n_0} (\mathbf{j} \cdot \nabla) \mathbf{j}, \\
\mathbf{E} + \mathbf{V} \times \mathbf{B} &= \epsilon_t \frac{m_e}{e^2 n_0} \frac{\partial \mathbf{j}}{\partial t} + \epsilon_{\text{ad}} \frac{m_e}{e^2 n_0} (\mathbf{V} \cdot \nabla) \mathbf{j} \\
&\quad + \epsilon_{\text{M}} \frac{m_e}{e^2 n_0} (\mathbf{j} \cdot \nabla) \mathbf{V} - \delta \frac{m_e}{e^3 n_0^2} (\mathbf{j} \cdot \nabla) \mathbf{j}.
\end{aligned}$$

Note that ϵ_{cp} do not exist in the above equations because of the incompressibility. In this

system, the energy conservation is as follows;

$$\begin{aligned}
& \frac{\partial}{\partial t} \left(\frac{1}{2} \rho_0 |\mathbf{V}|^2 + \epsilon_t \frac{m_e}{e^2 n_0} \frac{|\mathbf{j}|^2}{2} + \frac{|\mathbf{B}|^2}{2\mu_0} \right) \\
& + \nabla \cdot \left[\left(\frac{1}{2} \rho_0 |\mathbf{V}|^2 + p + \epsilon_{\text{ad}} \frac{m_e}{e^2 n_0} \frac{|\mathbf{j}|^2}{2} \right) \mathbf{V} \right. \\
& \quad \left. + \epsilon_M \frac{m_e}{e^2 n_0} (\mathbf{V} \cdot \mathbf{j}) \mathbf{j} - \delta \frac{m_e}{e^3 n_0^2} \frac{|\mathbf{j}|^2}{2} \mathbf{j} + \frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \right] \\
& = (\epsilon_M - \epsilon_{\text{EOM}}) \frac{m_e}{e^2 n_0} \mathbf{V} \cdot \{(\mathbf{j} \cdot \nabla) \mathbf{j}\}.
\end{aligned}$$

Then it is revealed that total energy is conserved when $\epsilon_M = \epsilon_{\text{EOM}} = 0$ or 1 , and there are no other constraints against ϵ_t and ϵ_{ad} .

ϵ_t	ϵ_{ad}	ϵ_{cp}	ϵ_M	Ohm's law $\mathbf{E} + \mathbf{V} \times \mathbf{B} =$	ϵ_{EOM}	Conserved?
Compressible plasma						
1	1	1	1	$\frac{m_e}{e^2 n} \left(\frac{\partial \mathbf{j}}{\partial t} + \nabla \cdot (\mathbf{V} \mathbf{j} + \mathbf{j} \mathbf{V}) \right)$	1	OK!
1	1	1		$\frac{m_e}{e^2 n} \left(\frac{\partial \mathbf{j}}{\partial t} + \nabla \cdot (\mathbf{V} \mathbf{j}) \right)$		OK!
1				$\frac{m_e}{e^2 n} \frac{\partial \mathbf{j}}{\partial t}$		$\frac{m_e}{e^2 n} \frac{ \mathbf{j} ^2}{2} \frac{\nabla \cdot (n \mathbf{V})}{n}$
1	1			$\frac{m_e}{e^2 n} \left(\frac{\partial \mathbf{j}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{j} \right)$		$\frac{m_e}{e^2 n} \mathbf{j} ^2 (\nabla \cdot \mathbf{V})$
1	1	1	1	$\frac{m_e}{e^2 n} \left(\frac{\partial \mathbf{j}}{\partial t} + \nabla \cdot (\mathbf{V} \mathbf{j} + \mathbf{j} \mathbf{V}) \right)$		$\frac{m_e}{e} \mathbf{V} \cdot \left\{ (\mathbf{j} \cdot \nabla) \frac{\mathbf{j}}{en} \right\}$
Incompressible plasma						
1		—		$\frac{m_e}{e^2 n_0} \frac{\partial \mathbf{j}}{\partial t}$		OK!
1	1	—		$\frac{m_e}{e^2 n_0} \left(\frac{\partial \mathbf{j}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{j} \right)$		OK!
1	1	—	1	$\frac{m_e}{e^2 n_0} \left(\frac{\partial \mathbf{j}}{\partial t} + \nabla \cdot (\mathbf{V} \mathbf{j} + \mathbf{j} \mathbf{V}) \right)$	1	OK!
1	1	—	1	$\frac{m_e}{e^2 n_0} \left(\frac{\partial \mathbf{j}}{\partial t} + \nabla \cdot (\mathbf{V} \mathbf{j} + \mathbf{j} \mathbf{V}) \right)$		$\frac{m_e}{e^2 n_0} \mathbf{V} \cdot \{(\mathbf{j} \cdot \nabla) \mathbf{j}\}$

Table 1: Classification of some IMHD models in terms of energy conservation. The epsilons in the generalized Ohm's law are listed from the first to fourth columns, and the generalized Ohm's law is described in the fifth column. The epsilon in the momentum equation are listed in the sixth column. When the total energy is conserved "OK!" is written in the last column, otherwise the remaining terms are written in the last column. Note that in a incompressible plasma, there is no ϵ_{cp} term in the generalized Ohm's law, then we write — in the third column in a incompressible plasma.

We summarize the classification of some IMHD models in terms of the energy conservation in Table 1.

4 Equilibrium States with No Flow – Grad-Shafranov Equation

In the previous section we find that ϵ_{EOM} term is very important to conserve the energy in the MHD model. Then in this section, to investigate the effect of this term, we try to modify the Grad-Shafranov (GS) equation ([5],[6]) which can be derived from the (ideal) MHD model and describe the equilibrium state with no flow in a torus and with toroidal axisymmetry, for example, the axisymmetric toroidal plasma in a tokamak.

4.1 Modified Grad-Shafranov Equation in a “Straight Torus” with Constant Density

Let us consider the Grad-Shafranov equation of plasma with constant density in a “straight torus,” which means a torus with no curvature, that is, a cylinder. We assume that the plasma is steady and z -independent, where z means the axis of the cylinder. Then, using the cylindrical coordinate (r, θ, z) , the magnetic field can be described as

$$\mathbf{B} = B_z \mathbf{e}_z + \mathbf{e}_z \times \nabla \psi,$$

and the current density is

$$\mathbf{j} = \frac{1}{\mu_0} \nabla \times \mathbf{B} = \frac{1}{\mu_0} [\mathbf{e}_z (\nabla_{\perp}^2 \psi) - \mathbf{e}_z \times (\nabla_{\perp} B_z)],$$

where B_z and ψ are both z -independent and

$$\begin{aligned} \nabla_{\perp} &\equiv \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_{\theta} \frac{1}{r} \frac{\partial}{\partial \theta}, \\ \nabla_{\perp}^2 &\equiv \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}, \end{aligned}$$

and $\mathbf{e}_r, \mathbf{e}_{\theta}$ are the unit vector toward the radial and polar angle direction, respectively. Then the governing equations can be simplified to the following equations:

$$\mathbf{0} = -\nabla p + \mathbf{j} \times \mathbf{B} - \epsilon \frac{m_e}{e^2 n_0} (\mathbf{j} \cdot \nabla) \mathbf{j}, \quad (5)$$

$$\mathbf{0} = \nabla \times \mathbf{E} = -\delta \frac{m_e}{e^3 n_0^2} \nabla \times [(\mathbf{j} \cdot \nabla) \mathbf{j}], \quad (6)$$

where p is also z -independent.

First we focus on the momentum equation (5). Considering the z component of the momentum equation, we obtain

$$0 = \{ \psi - \tilde{\epsilon} \nabla_{\perp}^2 \psi, B_z \},$$

where $\tilde{\epsilon} \equiv \epsilon m_e / (\mu_0 e^2 n_0)$ and

$$\{f, g\} \equiv \mathbf{e}_z \cdot (\nabla_{\perp} f \times \nabla_{\perp} g),$$

where f and g are both z -independent ². Then we obtain the first constraint, that is,

$$\psi - \tilde{\epsilon} \nabla_{\perp}^2 \psi \equiv L(B_z), \quad (7)$$

where L is an arbitrary function of B_z .

Taking the dot product of \mathbf{j} and the momentum equation, we obtain

$$0 = \left\{ \mu_0 p + \tilde{\epsilon} \frac{(\nabla_{\perp}^2 \psi)^2}{2} + \tilde{\epsilon} \frac{|\nabla_{\perp} B_z|^2}{2}, B_z \right\},$$

$$\Leftrightarrow M(B_z) \equiv \mu_0 p + \tilde{\epsilon} \frac{(\nabla_{\perp}^2 \psi)^2}{2} + \tilde{\epsilon} \frac{|\nabla_{\perp} B_z|^2}{2}, \quad (8)$$

where $M(B_z)$ is an arbitrary function of B_z , and this is the second constraint.

Taking the dot product of $\nabla_{\perp} \psi$ and the momentum equation and substituting the above two constraints (7) and (8), we obtain the third constraint described as

$$0 = \frac{dM}{dB_z} + \frac{dL}{dB_z} (\nabla_{\perp}^2 \psi) + B_z - \tilde{\epsilon} (\nabla_{\perp}^2 B_z). \quad (9)$$

Considering $\epsilon \rightarrow 0$, the first constraint (7) means B_z is a function of ψ , that is, $B_z \equiv F(\psi)$, then the second constraint means p is also a function of ψ , and the third constraint becomes

$$\nabla_{\perp}^2 \psi = -\mu_0 \frac{dp}{d\psi} - F \frac{dF}{d\psi}, \quad (10)$$

which is the incompressible Grad-Shafranov equation in a ‘‘straight torus.’’

Next we consider the constraint from the δ term (6) with the above three constraints with finite ϵ . After some calculations, we obtain the following equation from (6):

$$\mathbf{0} = -\frac{dL}{dB_z} \nabla_{\perp} \psi \times \nabla_{\perp} B_z + \nabla \times (\nabla_{\perp} \psi \times \nabla_{\perp} B_z),$$

then we find that

$$\nabla_{\perp} \psi \times \nabla_{\perp} B_z \equiv \mathbf{0}, \quad \Leftrightarrow \quad B_z \equiv F(\psi), \quad (11)$$

where F is an arbitrary function of ψ , and this is the fourth constraint.

From the first constraint (7) and the fourth one (11), $\nabla_{\perp}^2 \psi$ is also a function of ψ . Then $|\nabla_{\perp} \psi|^2$ is also a function of ψ because of the relation that $\nabla_{\perp}^2 B_z = F'' |\nabla_{\perp} \psi|^2 + F' \nabla_{\perp}^2 \psi$ with the third constraint (9) and the fact that $\nabla_{\perp}^2 \psi$ is a function of ψ . So we define

$$|\nabla_{\perp} \psi|^2 \equiv K(\psi).$$

Then, considering the second constraint (8) with the relation that $|\nabla_{\perp} B_z|^2 = F'^2 |\nabla_{\perp} \psi|^2$, it is revealed that p is also a function of ψ .

²If $\{f, g\} = 0$, f is an arbitrary function of g , or g is an arbitrary function of f .

Finally, after some calculations, we obtain the following equation from the third constraint (9):

$$\nabla_{\perp}^2 \psi = -\frac{1}{1 - \tilde{\epsilon} F'^2} (\mu_0 p' + F F') - \frac{\tilde{\epsilon} F'^2}{1 - \tilde{\epsilon} F'^2} \frac{K'}{2}, \quad (12)$$

where prime means the derivative of ψ , and this is the modified incompressible Grad-Shafranov equation in a “straight torus.” Note that when $\epsilon \rightarrow 0$, the above equation becomes the (usual) incompressible Grad-Shafranov equation in a “straight torus” already shown in (10).

4.2 Modified Grad-Shafranov Equation in a “Straight Torus”

Next, let us consider the Grad-Shafranov equation of compressible plasma in a “straight torus.” Then the governing equations are

$$\mathbf{0} = -\frac{1}{m_e n} \nabla p + \frac{1}{m_e n} \mathbf{j} \times \mathbf{B} - \epsilon \left(\frac{\mathbf{j}}{en} \cdot \nabla \right) \frac{\mathbf{j}}{en}, \quad (13)$$

$$\mathbf{0} = \nabla \times \mathbf{E} = -\delta \frac{m_e}{e} \nabla \times \left[\left(\frac{\mathbf{j}}{en} \cdot \nabla \right) \frac{\mathbf{j}}{en} \right]. \quad (14)$$

We assume p , n , B_z and ψ are all z -independent. Comparing the z component of momentum equation, we obtain the first constraint:

$$\left\{ B_z, \psi - \epsilon \frac{m_e}{\mu_0 e^2} \frac{\nabla_{\perp}^2 \psi}{n} \right\} = 0, \quad \Leftrightarrow \quad \psi - \epsilon \frac{m_e}{\mu_0 e^2} \frac{\nabla_{\perp}^2 \psi}{n} \equiv L(B_z). \quad (15)$$

Taking the dot product of \mathbf{j} and the momentum equation (13), we obtain

$$0 = \frac{1}{n} [B_z, \mu_0 p] + \left[B_z, \epsilon \frac{m_e}{\mu_0 e^2} \frac{(\nabla_{\perp}^2 \psi)^2}{2n^2} + \epsilon \frac{m_e}{\mu_0 e^2} \frac{|\nabla_{\perp} B_z|^2}{2n^2} \right], \quad (16)$$

and taking the dot product of $\nabla_{\perp} \psi$ and the momentum equation (13), we obtain

$$\begin{aligned} 0 = & -\nabla_{\perp} \psi \cdot \nabla_{\perp} (\mu_0 p) - (\nabla_{\perp}^2 \psi) |\nabla_{\perp} \psi|^2 - \nabla_{\perp} \psi \cdot \nabla_{\perp} \left(\frac{B_z^2}{2} \right) \\ & - \epsilon \frac{m_e}{\mu_0 e^2} \frac{\nabla_{\perp} \psi}{n} \cdot \nabla_{\perp} \left(\frac{|\nabla_{\perp} B_z|^2}{2} \right) + \epsilon \frac{m_e}{\mu_0 e^2} \frac{(\nabla_{\perp}^2 B_z)}{n} \nabla_{\perp} \psi \cdot \nabla_{\perp} B_z \\ & + \epsilon \frac{m_e}{\mu_0 e^2} \frac{1}{n^2} [B_z, n] \cdot [B_z, \psi], \end{aligned} \quad (17)$$

where we use the first constraint (15).

Next we focus on the constraint of δ term (14), and we can define the potential Φ as the following relation:

$$\begin{aligned} \left(\frac{\mathbf{j}}{en} \cdot \nabla \right) \frac{\mathbf{j}}{en} = & \frac{1}{\mu_0^2} \frac{1}{e^2 n^2} \left[\nabla_{\perp} \left(\frac{|\nabla_{\perp} B_z|^2}{2} \right) - (\nabla_{\perp}^2 B_z) \nabla_{\perp} B_z \right. \\ & \left. - \frac{1}{n} \{B_z, n\} \mathbf{e}_z \times (\nabla_{\perp} B_z) \right] + \frac{1}{\mu_0^2} \frac{1}{e^2 n} \left\{ \frac{\nabla_{\perp}^2 \psi}{n}, B_z \right\} \mathbf{e}_z, \\ \equiv \nabla \Phi = & \nabla_{\perp} \Phi + \frac{\partial \Phi}{\partial z} \mathbf{e}_z. \end{aligned}$$

Then, since n , B_z and ψ are all z -independent, it is revealed that

$$\frac{\partial}{\partial z} (\nabla_{\perp} \Phi) = \mathbf{0}, \quad \text{and} \quad \frac{\partial}{\partial z} \left(\frac{\partial \Phi}{\partial z} \right) = 0,$$

therefore we obtain

$$C = \frac{\partial \Phi}{\partial z} = \frac{1}{\mu_0^2} \frac{1}{e^2 n} \left\{ \frac{\nabla_{\perp}^2 \psi}{n}, B_z \right\},$$

where C is a constant. Using the constraint (15), the above equation becomes

$$C = \frac{1}{\epsilon \mu_0} \frac{1}{n} \{\psi, B_z\}.$$

Multiplying nB_z^2 by the above equation and integrating in the whole area S perpendicular to the z -axis,

$$\begin{aligned} C \int_S n B_z^2 dS &= \frac{1}{\epsilon \mu_0} \int_S B_z^2 \{\psi, B_z\} dS \\ &= \frac{1}{\epsilon \mu_0} \int_S \psi \{B_z, B_z^2\} dS = 0, \end{aligned}$$

where we use $B_z = 0$ on the boundary ∂S . Since n is positive and $B_z^2 \neq 0$ in some region in the area S , the constant C must be 0. This fact means that the potential Φ is also z -independent, that is,

$$\frac{\partial \Phi}{\partial z} = 0, \quad \Leftrightarrow \quad \left\{ \frac{\nabla_{\perp}^2 \psi}{n}, B_z \right\} = 0,$$

then it is revealed that $\nabla_{\perp}^2 \psi/n$ is an arbitrary function of B_z . Considering this fact with the first constraint (15), we find that ψ is a function of B_z , that is,

$$B_z \equiv F(\psi). \tag{18}$$

In this case, (16) becomes

$$0 = \frac{1}{n} [\psi, \mu_0 p] + [\psi, \epsilon \mu_0 m_e \Phi] = \frac{1}{n} [\psi, \mu_0 p] + \left[\psi, \epsilon \frac{m_e}{\mu_0 e^2} \frac{F'^2}{2n^2} |\nabla_{\perp} \psi|^2 \right], \tag{19}$$

and (17) becomes

$$\begin{aligned} 0 &= -\nabla_{\perp} \psi \cdot \frac{\nabla_{\perp}(\mu_0 p)}{n} - F F' \frac{|\nabla_{\perp} \psi|^2}{n} \\ &\quad - \left(1 - \epsilon \frac{m_e}{\mu_0 e^2} \frac{F'^2}{2n} \right) \frac{\nabla_{\perp}^2 \psi}{n} |\nabla_{\perp} \psi|^2 - \epsilon \frac{m_e}{\mu_0 e^2} \frac{F'^2}{2n^2} \nabla_{\perp} \psi \cdot \nabla_{\perp} (|\nabla_{\perp} \psi|^2) \\ &= -\nabla_{\perp} \psi \cdot \frac{\nabla_{\perp}(\mu_0 p)}{n} - F F' \frac{|\nabla_{\perp} \psi|^2}{n} - \frac{\nabla_{\perp}^2 \psi}{n} |\nabla_{\perp} \psi|^2 - \epsilon \mu_0 m_e \nabla_{\perp} \psi \cdot \nabla_{\perp} \Phi. \end{aligned} \tag{20}$$

Let us consider the situation that the plasma is barotropic, that is, $p = p(n)$. We define the function P as

$$\nabla_{\perp} P \equiv \frac{1}{n(p)} \nabla_{\perp} p.$$

From (19), we obtain the following constraint:

$$P + \epsilon m_e \Phi \equiv \tilde{P}(\psi),$$

where \tilde{P} is an arbitrary function of ψ . Since $\nabla_{\perp}^2 \psi/n$ is a function of B_z , and B_z is a function of ψ , that is, $\nabla_{\perp}^2 \psi/n \equiv G(\psi)$ where G is a function of ψ . And so (20) becomes

$$0 = -\mu_0 \tilde{P}' - \frac{FF'}{n} - G,$$

where prime means the derivative of ψ . Therefore we find that n is also a function of ψ . From (19), it is revealed that $|\nabla_{\perp} \psi|^2$ is a function of ψ , that is, $|\nabla_{\perp} \psi|^2 \equiv K(\psi)$, where K is a function of ψ . Finally, after some calculations for (20), we obtain the following equation:

$$\nabla_{\perp}^2 \psi = -\frac{1}{1 - \epsilon \frac{m_e}{\mu_0 e^2 n} F'^2} (\mu_0 p' + FF') - \frac{\epsilon \frac{m_e}{\mu_0 e^2 n} F'^2}{1 - \epsilon \frac{m_e}{\mu_0 e^2 n} F'^2} \frac{K'}{2}, \quad (21)$$

where prime means the derivative of ψ , and this is the modified Grad-Shafranov equation in a “straight torus.” This includes the modified Grad-Shafranov equation with constant density in a “straight torus” which is described as (12).

4.3 Modified Grad-Shafranov Equation in a Torus with Constant Density

Thirdly, let us consider the Grad-Shafranov equation of plasma with constant density in a torus. We assume that the plasma is steady and ϕ -independent, where we use the cylindrical coordinate (R, ϕ, Z) and ϕ means the azimuth. In this case the magnetic field can be described as

$$\mathbf{B} = B_{\phi} \mathbf{e}_{\phi} + \frac{1}{R} \mathbf{e}_{\phi} \times \nabla_{\perp} \psi,$$

where B_{ϕ} and ψ are both ϕ -independent. The current density can be described as

$$\mathbf{j} = \frac{1}{\mu_0} \left[\frac{1}{R} (\Delta^* \psi) \mathbf{e}_{\phi} + \frac{1}{R} \nabla_{\perp} (RB_{\phi}) \times \mathbf{e}_{\phi} \right],$$

where ∇_{\perp} and Δ^* are defined as

$$\begin{aligned} \nabla_{\perp} &\equiv \mathbf{e}_R \frac{\partial}{\partial R} + \mathbf{e}_Z \frac{\partial}{\partial Z}, \\ \Delta^* &\equiv R \frac{\partial}{\partial R} \left(\frac{1}{R} \frac{\partial}{\partial R} \right) + \frac{\partial^2}{\partial Z^2}, \end{aligned}$$

and \mathbf{e}_R , \mathbf{e}_Z are the unit vector toward the radial and height direction, respectively. The governing equations are as follows:

$$\mathbf{0} = -\nabla p + \mathbf{j} \times \mathbf{B} - \epsilon \frac{m_e}{e^2 n_0} (\mathbf{j} \cdot \nabla) \mathbf{j}, \quad (22)$$

$$\mathbf{0} = \nabla \times \mathbf{E} = -\delta \frac{m_e}{e^3 n_0^2} \nabla \times [(\mathbf{j} \cdot \nabla) \mathbf{j}]. \quad (23)$$

First we focus on the momentum equation (22). Comparing the ϕ component of this equation, we obtain the first constraint as follows:

$$\psi - \tilde{\epsilon} \Delta^* \psi \equiv L(RB_\phi), \quad (24)$$

where L is an arbitrary function of RB_ϕ and

$$\tilde{\epsilon} \equiv \epsilon \frac{m_e}{\mu_0 e^2 n_0}.$$

Taking the dot product of \mathbf{j} and the momentum equation, we also obtain the second constraint, that is,

$$M(RB_\phi) \equiv \mu_0 p + \tilde{\epsilon} \frac{(\Delta^* \psi)^2}{2R^2} + \tilde{\epsilon} \frac{|\nabla_\perp(RB_\phi)|^2}{2R^2}, \quad (25)$$

where M is an arbitrary function of RB_ϕ . Finally, taking the dot product of $\nabla_\perp \psi$ and the momentum equation and using the first and second constraints, we obtain the third constraint which is described as

$$0 = R^2 \frac{dM}{d(RB_\phi)} + \frac{dL}{d(RB_\phi)} \Delta^* \psi - \tilde{\epsilon} \Delta^*(RB_\phi) + RB_\phi. \quad (26)$$

When $\epsilon \rightarrow 0$, using (24)–(26), we can obtain the (usual) Grad-Shafranov equation which is described as

$$\Delta^* \psi = -\mu_0 R^2 p' - FF',$$

where $p \equiv p(\psi)$ and $RB_\phi \equiv F(\psi)$, and prime means the derivative of ψ .

Next we consider the constraint of (23). This equation means

$$\begin{aligned} \nabla \Phi &\equiv (\mathbf{j} \cdot \nabla) \mathbf{j} \\ &= \nabla_\perp \left[\frac{|\nabla_\perp(RB_\phi)|^2}{2\mu_0^2 R^2} + \frac{(\Delta^* \psi)^2}{2\mu_0^2 R^2} - \frac{M}{\mu_0^2 \tilde{\epsilon}} \right] \\ &\quad + \frac{1}{\mu_0^2 \tilde{\epsilon}} \frac{1}{R^2} [-(\Delta^* \psi) \nabla_\perp \psi - RB_\phi \nabla_\perp(RB_\phi) + \nabla_\perp \psi \times \nabla_\perp(RB_\phi)]. \end{aligned}$$

Therefore, we find

$$\frac{\partial}{\partial \phi} \nabla_\perp \Phi = \frac{\partial}{\phi} \frac{\partial \Phi}{\partial \phi} = 0, \quad \Leftrightarrow \quad \frac{\partial \Phi}{\partial \phi} = C,$$

where C is a constant.

If $C = 0$, that is, Φ is also ϕ -independent, then we find $RB_\phi \equiv F(\psi)$ where F is an arbitrary function of ψ . Then, from the first constraint (24) $\Delta^*\psi \equiv N(\psi)$ where N is an arbitrary function of ψ .

$$\mathbf{0} = \nabla \times \nabla \Phi = \frac{1}{\mu_0^2 \tilde{\epsilon}} \frac{2}{R^3} (N + FF') \mathbf{e}_R \times \nabla_\perp \psi,$$

therefore we find that

$$N + FF' = 0, \quad \text{or} \quad \psi = \psi(R). \quad (27)$$

Especially the second situation that ψ is only a function of R and independent of Z may be unrealistic. Further analyses are needed.

If $C \neq 0$, the potential Φ becomes a multivalued function, so this means the domain must not be simply connected space. This situation is satisfied in a tokamak. Further analyses are also needed.

4.4 Summary of Modified Grad-Shafranov equation in IMHD model

	Straight torus	Torus
Plasma with constant density	3 constraints (7)– (9) are obtained.	3 constraints (24)– (26) are obtained.
	Modified GS (12) is obtained.	Only the condition (27) is obtained if $\partial\Phi/\partial\phi = 0$.
Compressible plasma	1 constraints (15) and 2 (weak) constraints (16), (17) are obtained.	??
	If barotropic, modified GS (21) is obtained.	???

Table 2: Summary of the constraints of the equilibrium states with no flow in a “straight torus” and a torus and the modified Grad-Shafranov (GS) equations. The upper row in each situation shows the result with considering only ϵ term in the momentum equation, while the lower row shows the result with considering both ϵ and δ terms. At compressible plasma in a “straight torus,” “weak constraints” mean that these constraints are not the form that some values are purely the functions of another values but are simply the written formulation. We could not obtain any efficient conditions for compressible plasma in a torus and so “??” is written in the above table.

In Table 2, we summarize the constrains of the equilibrium states with no flow in a “straight torus” or a torus and the modified Grad-Shafranov equation obtained in the previous sections.

In “straight tori,” the modified Grad-Shafranov equations are obtained with the assumption that the plasma is barotropic. In torus, however, because of its curvature, useful equations could not be obtained and only some conditions are obtained even if the plasma has constant density. In compressible plasma in torus, there were no efficient results obtained. The physical interpretation of this effect of its curvature is a future work and further analyses are needed.

5 Incompressible Equilibrium States with Flow

In this section we introduce some equilibrium states in an incompressible plasma with flow, but further analyses are future works.

The governing equations are as follows:

$$\rho = \rho_0 = \text{const.} \Leftrightarrow n = n_0 = \text{const.}, \quad (28)$$

$$\mathbf{0} = -\nabla p + \mathbf{j} \times \mathbf{B} - \epsilon \frac{m_e}{e^2 n_0} (\mathbf{j} \cdot \nabla) \mathbf{j} - \rho_0 (\mathbf{V} \cdot \nabla) \mathbf{V}, \quad (29)$$

$$\nabla \Phi = \mathbf{E} = -\mathbf{V} \times \mathbf{B} + \epsilon \frac{m_e}{e^2 n_0} [(\mathbf{V} \cdot \nabla) \mathbf{j} + (\mathbf{j} \cdot \nabla) \mathbf{V}] - \delta \frac{m_e}{e^3 n_0^2} (\mathbf{j} \cdot \nabla) \mathbf{j}, \quad (30)$$

because the plasma is incompressible and

$$\mathbf{0} = \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \Leftrightarrow \mathbf{E} \equiv \nabla \Phi.$$

5.1 The Situation $\mathbf{V} \propto \mathbf{j}$

We assume

$$\mathbf{V} = C \frac{\mathbf{j}}{en},$$

then the governing equations (28)–(30) become as follows;

$$\begin{aligned} 0 &= -\nabla p + \mathbf{j} \times \mathbf{B} - \left(C^2 \frac{\rho_0}{e^2 n_0^2} + \epsilon \frac{m_e}{e^2 n_0} \right) (\mathbf{j} \cdot \nabla) \mathbf{j}, \\ 0 &= \nabla \Phi + \frac{C}{en} \mathbf{j} \times \mathbf{B} - \left(2C \epsilon \frac{m_e}{e^3 n_0^2} - \delta \frac{m_e}{e^3 n_0^2} \right) (\mathbf{j} \cdot \nabla) \mathbf{j}. \end{aligned}$$

If

$$2C \epsilon \frac{m_e}{e^3 n_0^2} - \delta \frac{m_e}{e^3 n_0^2} = \frac{C}{en} \left(C^2 \frac{\rho_0}{e^2 n_0^2} + \epsilon \frac{m_e}{e^2 n_0} \right)$$

and we define Φ as $\Phi \equiv -Cp/(en_0)$, the two equations are equivalent. Indeed, if $\delta = 0$,

$$C = \pm \sqrt{\epsilon \frac{m_e}{e^2 n_0 \rho_0}} = \pm \sqrt{\epsilon \frac{m_e}{m_i + m_e}},$$

and the governing equation becomes

$$0 = -\nabla p + \mathbf{j} \times \mathbf{B} - 2\epsilon \frac{m_e}{m_i + m_e} (\mathbf{j} \cdot \nabla) \mathbf{j},$$

where $\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$. Studying the physical meaning of the value of C is a future work.

5.2 Beltrami – “Jeltrami” Flow

In this section we consider the Beltrami – “Jeltrami” flow, which means the velocity and current density satisfy the following equations:

$$\begin{aligned}\nabla \times \mathbf{V} &= \lambda \mathbf{V}, \\ \nabla \times \mathbf{j} &= \mu \mathbf{j},\end{aligned}$$

where λ and μ are some functions in general. In this case the magnetic field satisfies the following equation:

$$\mathbf{B} = \frac{\mu_0}{\mu} \mathbf{j} + \nabla \chi,$$

where χ is an arbitrary harmonic function because \mathbf{B} is solenoidal. Then the governing equations (28)– (30) become as follows:

$$\begin{aligned}\nabla \tilde{p} &= \mathbf{j} \times \nabla \chi, \\ \nabla \tilde{\Phi} &= -\mathbf{V} \times \nabla \chi + \frac{\mu_0}{\mu} \left[1 + \left(\frac{m_e}{\mu_0 e^2 n_0} \right) \mu (\mu - \lambda) \right] \mathbf{j} \times \mathbf{V},\end{aligned}$$

where \tilde{p} and $\tilde{\Phi}$ are defined as

$$\begin{aligned}\tilde{p} &\equiv p + \rho_0 \frac{|\mathbf{V}|^2}{2} + \epsilon \frac{m_e}{e^2 n_0} \frac{|\mathbf{j}|^2}{2}, \\ \tilde{\Phi} &\equiv \Phi + \epsilon \frac{m_e}{e^2 n_0} (\mathbf{V} \cdot \mathbf{j}) + \delta \frac{m_e}{e^3 n_0^2} \frac{|\mathbf{j}|^2}{2}.\end{aligned}$$

For simplicity, if μ , λ and χ satisfy the following relations,

$$\mu = \lambda, \quad \text{and} \quad \chi \equiv 0,$$

the governing equations becomes

$$\begin{aligned}\nabla \tilde{p} &= 0, \\ \nabla \tilde{\Phi} &= \frac{\mu_0}{\mu} \mathbf{j} \times \mathbf{V}.\end{aligned}$$

6 Conclusions and Discussions

We have investigated limitations of IMHD model by introducing some non-dimensional parameters and classify IMHD models according to whether or not they conserve energy.

It is revealed that the correction term $\frac{m_e}{e} (\mathbf{j} \cdot \nabla) \frac{\mathbf{j}}{en}$ in the momentum equation, which is usually neglected, is needed to conserve the total energy.

In order to investigate the effect of this correction term, we attempted to modify the Grad-Shafranov equation, which describes the equilibrium state in a torus with no flow, and obtain a modified Grad-Shafranov equation in a “straight torus” when the plasma is barotropic (the density being a function of the pressure). However, we were only able to

do this with some conditions on the magnetic field in a torus due to its curvature. The physical interpretation of this effect of curvature will be addressed in future work and further analyses are needed.

We also introduced some equilibrium states with constant density and with flow. One has \mathbf{V} proportional to \mathbf{j} and another was the Beltrami–"Jeltrami" flow. Concrete analyses of these will also be considered in future work.

The dispersion relation of the Alfvén wave in usual MHD model is $\omega/k = V_A \cos \theta$, where ω is the frequency, k the wavenumber, $V_A \equiv \sqrt{B_0^2/(\mu_0 \rho_0)}$ the speed of Alfvén wave (B_0 is the magnitude of magnetic field and ρ_0 the density), and θ is the angle between the uniform magnetic field and the wavevector. On the other hand, the dispersion relation of the IMHD model is $\omega/k = V_A \cos \theta / \sqrt{1 + d_e^2 k^2}$, where $d_e^2 \equiv m_e / (\mu_0 e^2 n_0)$ (m_e is the electron mass, e the elementary charge and n_0 the number density) [9]. Therefore, higher wavenumber waves propagate more slowly in IMHD model than in usual MHD model. The same tendency can be found in the fast and slow magnetosonic waves. These facts suggest that the stability of some states can be changed by the effect of electron inertia, and this is this will also be considered in future work.

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