Lecture 2: Gravity waves and the method of stationary phase

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1 Gravity waves on a layer of uniform depth

In addition to the boundary conditions on the upper and lower interfaces, $\dot{\phi}$ must satisfy both the initial conditions and boundary conditions on lateral surfaces or at infinity. We consider free vibrations, in which neither the bottom elevation nor the pressure at the free surface are perturbed. Moreover, we specialize to the case of uniform depth $h^{(0)}$. We now summarize the governing equations derived in the last lecture:

$$\nabla^2 \tilde{\phi} = 0 \;, \qquad -h^{(0)} \leq z \leq 0 \;, \tag{1}$$

subject to the boundary conditions

$$\frac{\partial \tilde{\phi}}{\partial z} = 0 \quad \text{on} \quad z = -h^{(0)} \;, \qquad \frac{\partial \tilde{\phi}}{\partial z} = \beta \tilde{\phi} \quad \text{at} \quad z = 0. \tag{2}$$

Recall that $\tilde{\phi}$ represents the spatial variation of the velocity potential, and we have defined a wavenumber $\beta \equiv \omega^2/g$. We seek to solve this system of equations using separation of variables: $\tilde{\phi}(x, y, z) = U(x, y) f(z)$. After this substitution, (1) yields

$$\frac{U_{xx} + U_{yy}}{U} = -\frac{f''}{f} = -k^2 \;. \tag{3}$$

The boundary conditions (2) become

$$f'(-h^{(0)}) = 0 \;, \quad \text{and} \quad f'(0) = \beta f(0) \;. \tag{4}$$

In equation (3) since the left hand side is a function of only x and y and the right hand side is a function of z only, both must be constant, equal to some $-k^2$. f and U then must satisfy

$$U_{xx} + U_{yy} + k^2 U = 0 \;, \tag{5}$$

$$f'' = k^2 f \;. \tag{6}$$

(5) is called the Helmholtz or *reduced wave* equation. Solving for f and applying the first of the boundary conditions (4), we obtain

$$f(z) = A \cosh[k(z + h)] \;, \tag{7}$$

where A is an arbitrary constant, and we have suppressed the superscript upon h. In order to satisfy the free surface boundary condition, we need that $k\sinh(kh) = \beta\cosh(kh)$, or equivalently

$$\omega^2 = gk\tanh(kh)\ . \tag{8}$$

This is the *dispersion relation*, k is the *wavenumber*, and ω is the *angular frequency*. It may seem surprising that (8) only has one positive and one negative real solution. There are infinitely many pure imaginary roots $k = i\kappa$ to this equation, but these represent evanescent modes.

There are two physically interesting special limits

- Deep water:
$$kh \to \infty \qquad \omega^2 \sim gk. \tag{9}$$

- Shallow water
$$kh \to 0 \qquad \omega^2 \sim ghk^2. \tag{10}$$

Another useful concept is the *phase velocity* defined by

$$c \equiv \frac{\omega}{k} \sim \begin{cases} \sqrt{g/k} & \text{as} \quad kh \to \infty \\ \sqrt{gh} & \text{as} \quad kh \to 0 \end{cases} \tag{11}$$

Notice that to leading order the phase velocity is independent of the wavenumber in the shallow water limit. Media with this property are said to be *non-dispersive*.

We can now solve for the x and y variations of the velocity field. Plane wave solutions of the Helmholtz equation have the form

$$U(x,y) = e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \quad \text{where} \quad |\boldsymbol{k}| = k\ . \tag{12}$$

Therefore

$$\phi(x,y,z,t,k) = Ae^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}\cosh[k(z+h)], \tag{13}$$

$$\eta(x,y,t,k) = -\left.\frac{1}{g}\phi_t\right|_{z=0} = \frac{i\omega}{g}Ae^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}\cosh(kh). \tag{14}$$

Finding the wavenumber for a prescribed frequency ω requires solving a transcendental equation (8). It is easier to give k and then determine ω directly from (8).

2 Trajectories of fluid particles

By further integration we can obtain expressions for the motion of the water particles. We choose axes so that $\boldsymbol{k}$ points in the x-direction, and consider the small excursions of a fluid particle $(x+\delta x(t),\ z+\delta z(t))$, where δx and δz evolve according to the linearized equations:

$$\frac{d\delta x}{dt} = ikAe^{i(kx-\omega t)}\cosh[k(z+h)] \tag{15}$$

$$\frac{d\delta z}{dt} = kAe^{i(kx-\omega t)}\sinh[k(z+h)]\ . \tag{16}$$

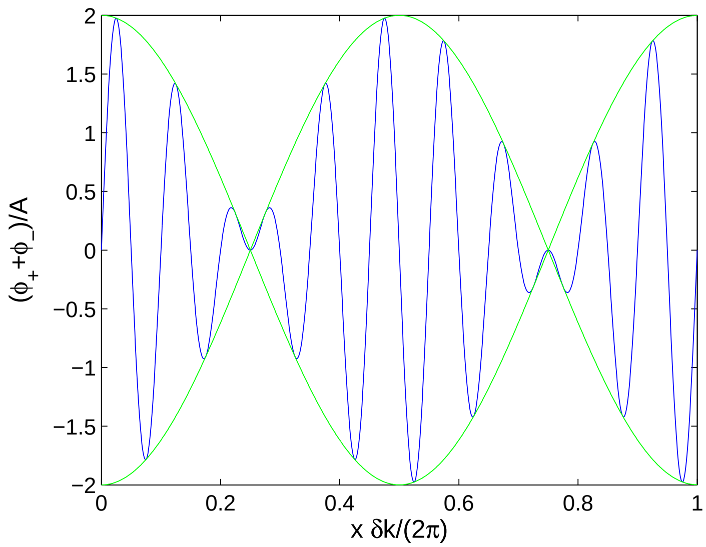

Figure 1: (Blue curve) Superposition of two waves of narrowly separated wavenumber and frequency. (Green curve) The predicted curve envelope. Here $\delta k = k/10$.

On integrating up, and explicitly writing out the real part (assuming that A is real for convenience) we arrive at:

$$\delta x(t) = -\frac{Ak}{\omega}\cos(kx - \omega t)\cosh[k(z+h)] \tag{17}$$

$$\delta z(t) = -\frac{Ak}{\omega}\sin(kx - \omega t)\sinh[k(z+h)] \ . \tag{18}$$

The particles therefore trace out an ellipse in the $(x,\ z)$ plane, with horizontal axis $(Ak/\omega)\cosh[k(z+h)]$ and vertical axis $(Ak/\omega)\sinh[k(z+h)]$.

3 Group velocity

First consider the superposition of disturbances having slightly different wavenumbers, and identical amplitudes:

$$\phi_\pm = A\sin[(k \pm \delta k)x - (\omega \pm \delta\omega)t] \ . \tag{19}$$

The sum of the two velocity potentials may be simply written:

$$\phi_+ + \phi_- = 2A\cos(\delta k x - \delta\omega t)\sin(kx - \omega t) \ . \tag{20}$$

The sin and cos factors describe respectively the wave, traveling at the phase velocity $c = \omega/k$, and the much slower oscillations in the amplitude envelope, which propagates at the *group velocity* $c_g \equiv d\omega/dk \approx \delta\omega/\delta k$. This 'beat' structure persists for wave packets with any small spread of wavenumbers. The wave packet travels at the group velocity, while individual wave components move through the wave packet at their phase velocities. For gravity waves on a uniform fluid layer we have:

$$c_g = \frac{c}{2}\left(1 + \frac{2kh}{\sinh(2kh)}\right). \tag{21}$$

Again we consider two limits for (21):

$$c_g \to \begin{cases} \frac{1}{2}c & \text{as} \quad kh \to \infty \\ c & \text{as} \quad kh \to 0 \end{cases} . \tag{22}$$

Now we calculate the period-integrated energy flux F through a surface x=const, with unit width in the y-direction. This is equal to the rate of working of the pressure force across the surface, or since in the linear approximation $p = -\rho\partial\phi/\partial t$, to:

$$F = -\rho \int_t^{t+T} \left[\int_S \frac{\partial\phi}{\partial t}\frac{\partial\phi}{\partial n} dS\right] dt, \tag{23}$$

where $T = \frac{2\pi}{\omega}$. Note that it is only necessary to evaluate the z integral over the interval $-h < z < 0$; the contribution from the sliver $0 < z < \eta$ is of lower order. After some calculations, we discover

$$F_{av} = \frac{F}{T} = \frac{A^2\rho\omega^2}{2g}\cosh^2(kh)c_g. \tag{24}$$

Next let us calculate the total energy (associated with the flow) in a box R: $\{x_0 < x < x_0+\lambda$, $y_0 < y < y_0 + 1$, $-h < z < \eta\}$.

$$E = \rho \int\int\int_R [\tfrac{1}{2}(\nabla\phi)^2 + gz]dV + \tfrac{1}{2}\lambda g h^2 \ . \tag{25}$$

The integral represents the total (kinetic plus potential) energy of the fluid contained within the box, and the additional summand is simply the hydrostatic potential energy which must be subtracted off. With a little algebra, we find that

$$E_{av} = \frac{E}{\lambda} = \frac{A^2\rho\omega^2}{2g}\cosh^2(kh). \tag{26}$$

From (21), (24) and (26) we get

$$F_{av} = E_{av}c_g \ , \tag{27}$$

so that the energy of the wave propagates with the group velocity c_g.

4 Superposition of plane waves

A general solution of the wave equation can be written as a linear combination of plane wave solutions of the type considered:

$$\eta(x, y, t) = \int \mathrm{d}^2\boldsymbol{k}\,\frac{i\omega(\boldsymbol{k})}{g}\left(A(\boldsymbol{k})\cosh kh\ e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)} - B(\boldsymbol{k})\cosh kh\ e^{i(\boldsymbol{k}\cdot\boldsymbol{x}+\omega t)}\right) \tag{28}$$

The amplitude functions $A(\boldsymbol{k})$ and $B(\boldsymbol{k})$ can be determined to make η and η_t satisfy initial conditions. The fluid elevation at time $t = 0$ is

$$\eta(x, y, 0) = \int \mathrm{d}^2\boldsymbol{k}\, \frac{i\omega(\boldsymbol{k})}{g} \cosh kh\, (A(\boldsymbol{k}) - B(\boldsymbol{k}))\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \ . \tag{29}$$

Thus the Fourier transform $\tilde{\eta}$ of the initial fluid height is related to A and B by

$$\tilde{\eta}(\boldsymbol{k}, 0) = \frac{i\omega(\boldsymbol{k})}{g} \cosh kh(A(\boldsymbol{k}) - B(\boldsymbol{k})) \ . \tag{30}$$

Similarly for the Fourier transform of the initial velocity of the free surface,

$$\tilde{\eta}_t(\boldsymbol{k}, 0) = \frac{\omega(\boldsymbol{k})^2}{g} \cosh kh(A(\boldsymbol{k}) + B(\boldsymbol{k})) \ . \tag{31}$$

Thus from (30) and (31) the amplitude functions are determined by:

$$\tilde{\eta}_t - i\omega\tilde{\eta}|_{t=0} = \frac{2\omega^2}{g} \cosh kh\, A(\boldsymbol{k}) \ , \tag{32}$$

$$\tilde{\eta}_t + i\omega\tilde{\eta}|_{t=0} = \frac{2\omega^2}{g} \cosh kh\, B(\boldsymbol{k}) \ . \tag{33}$$

However convoluted the initial conditions upon the surface elevation, we can obtain the complete evolution of the fluid surface height if only we can evaluate these integrals. In the days when computation was expensive, this was a knotty problem, since the range of integration in wave number space extends out to infinity, and the amplitude functions may decay only weakly as the wavenumber is made vanishingly small. In the next section, we describe a powerful asymptotic method for evaluation of the integral far from any disturbance sources.

5 Asymptotic evaluation of integrals

We consider the simplest case of a one-dimensional disturbance, and seek to evaluate the following integral:

$$\eta(x, t) = \int_{-\infty}^{\infty} A(k) e^{i(kx - \omega(k)t)} dk \ . \tag{34}$$

When x and t are large, the integrand of (34) oscillates very rapidly and everything is canceled out. So the dominant contribution to $\eta(x, t)$ comes from the neighborhood of $k = k_s$, a stationary point at which the derivative of the phase function $S(k, x, t) \equiv kx - \omega(k)t$ with respect to k vanishes. We approximate S about $k = k_s$ by a Taylor series expansion:

$$S(k) \approx S(k_s) + S_k(k_s)(k - k_s) + \tfrac{1}{2} S_{kk}(k)(k - k_s)^2 + \dots \ . \tag{35}$$

Noting that $S_k(k_s) = 0$, we find that (34) can be approximated by the Gaussian integral

$$\begin{aligned} \eta(x, t) &\sim A(k_s) e^{iS(k_s)} \int_{-\infty}^{\infty} e^{\frac{i}{2} S_{kk}(k_s)(k - k_s)^2} dk \\ &\sim A(k_s) e^{iS(k_s)} \frac{\sqrt{2\pi} e^{\pm i\frac{\pi}{4}}}{\sqrt{|S_{kk}(k_s)|}} \ , \end{aligned} \tag{36}$$

where $\pm = sgn(S_{kk}(k_s))$. Therefore, for $x,\ t \gg 1$,

$$\eta(x,t) \sim \sum_{\pm k_s\left(\frac{x}{t}\right)} A\left(k_s\left(\frac{x}{t}\right)\right) e^{i(k_s x - \omega(k_s)t)} \frac{\sqrt{2\pi} e^{\pm i\frac{\pi}{4}}}{\sqrt{|\omega''(k_s)|t}}\ , \tag{37}$$

since $S_{kk} = -\omega''(k)t$. We must sum over the contributions from each of the wavenumbers k_s giving group velocities $c_g(k_s) = x/t$. For the given dispersion relation (8), there is precisely one such wavenumber. This method is called the *method of stationary phase.*

There is a singularity in (37) if $\omega''(k_s) = 0$, i.e. at the maximum attainable group velocity. From the form of the dispersion relation (8) it may be determined that this occurs iff $k = 0$ (the 'if' statement is an obvious corollary of c_g being an even function of k, but the converse statement requires a little more work). Much of the energy introduced into the medium at the source piles up in the fastest-traveling part of the wave packet. It follows that the formula is not valid in the neighborhood of values $x,\ t$ where $x/t = c_g(0) = \sqrt{gh}$. To correct it, it is necessary to extend the Taylor series for x/t close to $\sqrt{gh}$:

$$\begin{aligned} \eta(x,t) &\sim A(k_s)e^{iS(k_s)} \int_{-\infty}^{\infty} \mathrm{d}k\, e^{i(k-k_s)^3 S_{kkk}(k_s)/6} \\ &\sim A(k_s)e^{i(k_s x - \omega t)} \int_{-\infty}^{\infty} \mathrm{d}\sigma\, \frac{e^{\pm i\sigma^3/6}}{|S_{kkk}(k_s)|^{1/3}} = \left(\frac{16\pi^3}{|S_{kkk}(k_s)|}\right)^{1/3} e^{ik_s x - i\omega t} Ai(0) \end{aligned} \tag{38}$$

Since $S_{kkk}(k_s)$ is proportional to t at $k_s = 0$, η decays as $t^{-1/3}$. This decay is significantly weaker than the $t^{-1/2}$ decay at points behind the front. In order to obtain a uniform approximation which combines (37) for $x/t \neq \sqrt{gh}$ and (38) for $x/t \approx \sqrt{gh}$, we can retain both the cubic and quadratic terms in the expansion of $S(k)$. The resulting integral can be written as an Airy function for η. This integral was first used by George Airy in his analysis of light-scattering by spherical raindrops, the first quantitative model for the distribution of colors in a rainbow.

This transitional expression is valid near the front $x/t = \sqrt{gh}$ but it becomes inaccurate far from the front. A result valid over the whole range of x and t can be obtained by writing $S(k)$ as a cubic polynomial in a new variable in a suitable way. This method, developed by Chester, Friedman and Ursell, yields an Airy function of a more complicated argument. It was applied recently to tsunami waves by Berry.

Notes by Marcus Roper and Aya Tanabe.