Lecture 5: Amplitude Dynamics, Boundary Layers, and Harbor Resonance

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1 Amplitude Amplification at the Shore

We will now consider the consequences of ray theory for the amplification of wave amplitudes near a shoreline. In the deep ocean, tsunami waves have small amplitude and long wavelength, on the order of tens of kilometers. But when they reach the shore, as mentioned in the last lecture, they can grow to towering heights. We can gain insight into the nature of this amplitude growth using the linear methods of the previous lecture.

We will consider the evolution of a wave train incident on a sloping beach. Let the depth be a linear function of distance from the shore

$$h = x \tan\alpha \sim \alpha x, \;\; \alpha \ll 1 \tag{1}$$

Following the method of the previous lectures, factor the velocity potential into a horizontal oscillation and a vertical mode shape

$$\phi\left(x, y, z\right) = A \cosh\left(k(z+h)\right) e^{i\beta S(x,y)}. \tag{2}$$

Consequently,

$$\begin{aligned} \left(\nabla S\right)^2 &= k^2 && (3)\\ k \tanh kh &= 1. && (4) \end{aligned}$$

We now specialize to the case of one dimensional long waves. These are waves with long wavelength compared to the depth and wavefronts parallel to the shore. Mathematically, these assumptions imply

$$\begin{aligned} kh &\ll 1 && (5)\\ S &= S(x). && (6) \end{aligned}$$

In this limit, Equation (4) becomes

$$k^2 h = 1, \tag{7}$$

which implies

$$k = \frac{1}{\sqrt{h}}. \tag{8}$$

Therefore, as the waves approach shore and $h \to 0$, $k \to \infty$ and the wavelength decreases to zero. The parameter kh, however, remains small throughout:

$$kh = \frac{1}{k} \to 0$$

Incidentally, the one dimensional wave approaching parallel to the shore is the most physically relevant case. Recall from Lecture 3 the equation for the path of characteristic curves

$$\frac{d}{d\sigma}\nabla S = \frac{\lambda}{2}\nabla n^2(\mathbf{x}). \tag{9}$$

Using $n^2(\mathbf{x}) = h_0/h(\mathbf{x})$, we have

$$\frac{d}{d\sigma}\nabla S = -\frac{h_0}{h^2(\mathbf{x})}\nabla h. \tag{10}$$

Since ∇S is parallel to rays, and $-\nabla h$ points in the direction of the shore, this equation shows that rays curve in the direction of the shore. Figure 1 illustrates the intuition behind this result. As a wave approaches a beach at an angle, the section of the wavefront further from the beach is over deeper water, and therefore has a relatively faster wave speed. Accordingly, the wavefront will swing towards the shore until it is parallel with the beach, and all points on the wavefront have the same wave speed. For this reason, the one dimensional formulation is adequate to investigate the late stages of a wave's approach to the shore.

The reduction to one dimension also makes it easy to compute the phase. We have

$$\begin{aligned}\frac{\partial S}{\partial x} &= \pm k \\ &= \pm\frac{1}{\sqrt{h}} \\ &= \pm\frac{1}{\sqrt{x\tan\alpha}}.\end{aligned} \tag{11}$$

Though the derivative is singular at $x = 0$, it is integrable and we can compute the phase

$$\begin{aligned}S(x) &= \pm\int_{x_0}^{x}\frac{1}{\sqrt{h(x')}}dx' \\ &= \pm\int_{x_0}^{x}\frac{1}{\sqrt{x'\tan\alpha}}dx' \\ &= \pm\frac{2x^{1/2}}{\sqrt{\tan\alpha}} + C\end{aligned} \tag{12}$$

We choose the minus sign to be consistent with waves moving toward the shore.

We are now prepared to compute the amplitude evolution. Because the rays are straight lines in one dimension, the ray tube area $d\sigma =$ constant. We showed in Lecture 4 that

$$A_0^2\left(\sinh^2 kh + h\right)kd\sigma = \text{constant}.$$

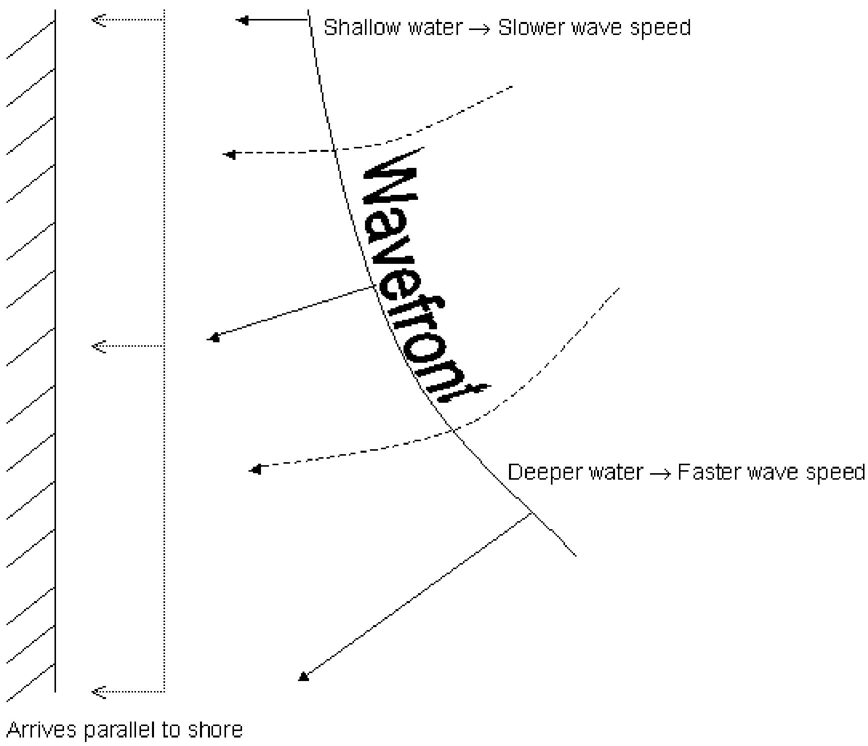

Figure 1: Waves on a sloping beach are refracted so as to approach with wavefronts parallel to the shoreline

Therefore,

$$
\begin{aligned}
A_0^2 &\sim \frac{1}{k\left(h+\sinh^2 kh\right)} \\
&\sim \frac{1}{k\left(k^2h^2+h\right)}.
\end{aligned} \tag{13}
$$

Using equation (8), we have

$$
A_0 = \frac{C'}{h^{1/4}} = \frac{C}{x^{1/4}}. \tag{14}
$$

Thus, the theory predicts wave amplitudes going to infinity at the shoreline. That's why tsunamis do so much damage.

This result seems natural considering that we have held the example of a tsunami in mind as we derived it. But one might ask what, in this theory, differentiates tsunamis from any other wave? The ocean is full of waves satisfying $kh \ll 1$ that reach shore without catastrophic consequences. Clearly, something is wrong with the theory very close to the shore.

A typical trouble with asymptotic theories is the presence of certain regions where the solutions become singular. In these regions, the asymptotic expansions fail. We need a different theory that applies in this singular region, which we call a boundary layer. If all goes well, we will be able to find a solution that applies in the boundary layer and that blends continuously into exterior solution we have just derived.

2 Boundary layer and shallow water equations

2.1 The concept of the wave boundary layer

The asymptotic analysis based on the linear wave theory discussed above, which was asymptotic in the sense that the depth and wavelength were small compared to the characteristic horizontal scale, proved to be successful for deep water waves (see the previous lecture and [3]). However, the theory fails in the proximity of the shore, yielding an infinite amplitude there. Wave propagation near the shore can be analyzed by means of the shallow water theory and the boundary layer concept.

The idea is to use the shallow water equations in the vicinity of the shore, where the depth is small, and the wavelength is large compared to the depth. It is worth noting, however, that shallow water theory is applicable over large regions of ocean basins, if we study phenomena on the synoptic scale, large ($L \approx 100$ km) compared to the mean depth of the basin ($H \approx 500$ - 1000 m), see e.g. [4].

First of all, we will ruminate for a while on the derivation of the shallow water equations, formulated for an incompressible fluid in an inertial frame - Coriolis acceleration will not be included.

Naturally, at this point anyone merely conversant with physical oceanography [1] would ask about the relevance of linear shallow water theory to the reality which occurs on the noninertial frame of the rotating Earth, where the the Coriolis acceleration should be taken

[1] and we assume that some of our readers belong to this group

into account. In fact, this effect is crucial in studies of tidal phenomena as well as for the more general model of Poincare waves (the latter problem being formulated in the shallow water approximation in the synoptic-scale ocean gives dispersive waves). However, neglecting the coriolis force may be justified in two classes of wave problems:

- waves propagating in lakes and small, shelf seas (i.e the Baltic sea) where the long (with respect to the basin depth) waves are locally generated by a strong wind yet they are short enough not to be effected by the Coriolis acceleration,
- Shoaling waves approaching normal to a shoreline. Then the waves, irrespective of whether they were nondispersive or dispersive far from the shore due to the Coriolis effect, undergo shoaling on scales on which the Coriolis effect does not contribute.

3 The structure of the boundary layer

While toying with the idea of **the horizontal wave boundary layer** $\eth$ for the problem of waves approaching the shore and affected by its presence, we could attempt to find an analogy with the **vertical** terrestial and oceanic boundary layer (table 1). By this analogy, we may expect that the solution obtained for the wave boundary layers may be matched at the borderline between the boundary layer and the outer one, just like in the case of vertical boundary layers.

3.1 Shallow water equations

We consider long waves propagating in relatively shallow water in an inertial frame of reference. We assume two-dimensional motion in the (x, y) plane. The equation of continuity for an incompressible fluid is:

$$u_x + w_z = 0, \tag{15}$$

The kinematic condition and the dynamic conditions at the free surface are:

$$(\eta_t + u\eta_x - w)|_{z=\eta} = 0, \quad p|_{z=\eta} = 0 \tag{16}$$

The kinematic condition at the bottom is:

$$(uh_x + w)|_{z=-h} = 0, \quad p|_{z=\eta} = 0 \tag{17}$$

It is convenient to formulate the problem in terms of the depth integrated horizontal velocity, namely:

$$\frac{\partial}{\partial x}\int_{-h}^{\eta} udz, \tag{18}$$

using the boundary conditions and the Leibniz rule of integration:

$$\frac{\partial}{\partial x}\int_{-h}^{\eta} udz = -\eta_t \tag{19}$$

In the shallow water theory the hydrostatic pressure approximation is used. That is, vertical acceleration is ignored. Then, the pressure at a point is determined entirely by the weight of the water column above it:

$$p = g\rho(\eta - z) \tag{20}$$

vertical (terrestial/ocean)	horizontal (wave)
ground/free water surface/ice	shoreline
Prandtl layer	shore wave boundary layer $\mathfrak{d}$ where the shallow water theory applies; for the sake of use of asymptotic methods, $\mathfrak{d}$ can be further divided into inner shore wave boundary layer $\mathfrak{d}_\mathfrak{i}$ in the immediate proximity of the shoreline and the outer shore wave boundary layer $\mathfrak{d}_\mathfrak{o}$
Ekman layer	the outer layer where the shallow water might also apply but the Coriolis term should be taken into account causing waves to be dispersive
free flow (geostrophic layer)	the outermost layer where gravity deep water wave theory applies

Table 1: Boundary layer structure and the analogy between vertical and horizontal boundary layers.

The horizontal pressure gradient is then $p_x = g\rho\eta_x$. From the equation of motion in the x direction, $u_t + uu_x = -g\eta_x$, the horizontal acceleration is independent of depth. Therefore so is u, provided that it was initially independent of z. The depth integrated u is now $\int_{-h}^{\eta} u dz = u(\eta + h)$. Using this, we obtain **the nonlinear shallow water equations**:

$$u_t + uu_x = -g\eta_x \tag{21}$$

$$\big(u(h+\eta)\big)_x = -\eta_t, \tag{22}$$

Where $\eta = \eta(x,t),\ u = u(x,t),\ h = h(x)$. If we assume that u, η and their derivatives are small, their products can be neglected compared with linear terms. Then (21) and (22) yield **the linear shallow water equations** :

$$u_t = -g\eta_x \tag{23}$$

$$(uh)_x = -\eta_t \tag{24}$$

Eliminating η from (23) and (24) gives

$$(uh)_{xx} - \frac{1}{g}u_{tt} = 0. \tag{25}$$

3.2 Linear SWE and the variable depth - asymptotic approach

Since $h = h(x)$ is independent of t, we can rewrite (25) as

$$(uh)_{xx} - \frac{1}{gh}(uh)_{tt} = 0. \tag{26}$$

This is the wave equation for a variable $U^* = uh$ with propagation velocity $c = \sqrt{gh(x)}$. For time harmonic waves $U^*(x,\ t) = U(x)\exp(-i\omega t)$, (26) becomes the Helholtz equation:

$$U_{xx} + \frac{\omega^2}{gh(x)}U = 0. \tag{27}$$

We now define $k = \omega/\sqrt{gh_0},\ n(x)^2 = gh_0/gh(x)$, in terms of a typical depth h_0. Then we can rewrite (26) as

$$U_{xx} + k^2n^2(x)U = 0. \tag{28}$$

Away from the shoreline $h(x) = 0$, the asymptotic form of $U(x,k)$ for $kh_0 >> 1$ is

$$U(x) \approx Z^{in}(x)\exp\left(ik\ S^{in}(x)\right) + Z^r(x)\exp\left(ik\ S^r(x)\right). \tag{29}$$

Here $Z^{in}(x)$, $S^{in}(x)$ and $Z^r(x)$, $S^r(x)$ are the amplitudes and phases of the incident and reflected waves, respectively. We call (29) **the outer asymptotic expansion** of U. It is not valid where $h(x) = 0$ because $Z^{in}(x)$, $Z^r(x)$ become infinite there. To determine $U(x)$ near the shore, we define $x' = kx$ and $V(x',k) = U(x,k)$. Then $h(x) = h(\frac{x'}{k}) = h(0) + h_x(0)(\frac{x'}{k}) + \mathcal{O}(k^{-2})$. Then at the shoreline $x = 0$ we get $h(0) = 0$ and we define α, the slope of the bottom, by $\tan\alpha = h_x(0)$. Then $n(x)^2 = h_0/h(x) = (kh_0)/(x'\tan\alpha) + \mathcal{O}(1)$ and (28) becomes:

$$V_{x'x'} + \big(\frac{h_0}{\tan\alpha}\frac{k}{x'} + \mathcal{O}(1)\big)V = 0. \tag{30}$$

When we neglect the $\mathcal{O}(1)$ term, (30) becomes a form of the Bessel equation. Although the coefficient of V is singular at $x' = 0$, the equation has a solution which is regular there. It is

$$V(x', k) = A\sqrt{x'}J_1\big(2\sqrt{\frac{kh_0}{\tan\alpha}}x'\big). \tag{31}$$

Here, A is an arbitrary constant. Then the solution of (31) vanishes at $x' = 0$. There is also a solution which is infinite at $x' = 0$. The asymptotic solution of (31) for x' large is:

$$V(x', k) \sim A(\sqrt{x'})e^{i\left(2\sqrt{\frac{kh_0}{\tan\alpha}}x'\right)} + B(\sqrt{x'})e^{-i\left(2\sqrt{\frac{kh_0}{\tan\alpha}}x'\right)}, \tag{32}$$

This can be matched with the outer expansion (29). The system of linear, variable depth shallow water equations is satisfactory for small amplitude waves. It does not capture effects like breaking, for which the nonlinear theory is needed.

4 Nonlinear Wave Propagation Along Rays

In these lectures, we have discussed the linear theory of waves in some detail. It would be a shame not to discuss nonlinearity a little further. For decades, models of water waves have been an interesting source of nonlinear equations. One of the most famous of these equations is that of Kordeveg and de Vries, which was derived to model the cumulative effect of nonlinearity in water waves travelling over long distances.

In this section, we will derive KdV in the context of a ray tracing theory for nonlinear long waves on a layer with spatially varying depth. In linear theories, the amplitude typically takes the form of a nearly sinusoidal wave train with amplitude and wavenumber slowly varying along a ray. Instead, we will find that the amplitude is governed by an equation of KdV form. Specifically, we will consider the equations for a disturbance on the surface of an incompressible flow of constant density without rotation. This computation is a simplified presentation of a more general analysis presented in [5], in which the effects of bottom topography, incompressibility, rotation, stratification, and a polytropic equation of state are taken into account. By including only one dimensional bottom topography, we will be examining the simplest case in which nontrivial amplitude dynamics occurs.

4.1 Scaling the Equations of Motion

We begin by introducing a carefully chosen scaling of the equations of fluid motion. The key method of asymptotic analysis is to rescale equations to introduce small parameters, thus allowing complex problems to be considered as a sequence of simpler problems. The art of this method is to tailor one's scaling to access a physically interesting limit. In this case, the scaling will pertain to waves with wavelength long compared to the depth of the layer, propagating over long distances.

Let us consider a layer of incompressible fluid in two dimensions bounded above by the free surface $z^\star = \eta^\star(x^\star, t^\star)$ and below by the rigid surface $z^\star = -h^\star(x^\star)$. Before writing the equations of motion, we introduce stretched dimensionless variables x,z,t, etc., as follows:

$$\begin{array}{cc} \epsilon = \left(\frac{H}{L}\right)^{2/3}, & (x^\star, z^\star) = H\left(\epsilon^{-3/2}x, z\right), \\ h^\star = Hh, & \eta^\star = H\eta, \\ p^\star = gH\rho_0 p, & t^\star = \epsilon^{-3/2} t \left(\frac{H}{g}\right)^{1/2} \\ (u^\star, w^\star) = \sqrt{gH}\left(u, \epsilon^{1/2} w\right). & \end{array} \tag{33}$$

and $\mathbf{v} = (u, w)$. Here, L is a typical horizontal scale of variation, so ϵ is determined by the characteristic aspect ratio of the motion.

In these stretched variables, the equations of motion take the following form:

$$\frac{\partial w}{\partial z} + \epsilon \frac{\partial u}{\partial x} = 0, \tag{34}$$

$$\epsilon \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} \right) + w\frac{\partial u}{\partial z} = 0, \tag{35}$$

$$\epsilon^2 \left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} \right) + \epsilon w \frac{\partial w}{\partial z} + \frac{\partial p}{\partial z} + 1 = 0. \tag{36}$$

The kinematic condition and the normal force balance at the free surface are, respectively,

$$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} = w, \tag{37}$$

$$p = C, \tag{38}$$

evaluated at $z = \eta(x)$. The kinematic condition at the lower boundary is

$$w = -\epsilon u \frac{\partial h}{\partial x} \tag{39}$$

evaluated at $z = -h(x)$.

When $\epsilon = 0$, equations (34) - (39) have, as a solution, the state of rest given by

$$\mathbf{v}_0 = 0, \tag{40}$$

$$p_0 = C - z \tag{41}$$

$$\eta_0 = 0. \tag{42}$$

To find approximate solutions for $\epsilon \neq 0$, we introduce a phase function $S(x,t)$ and the "fast" variable $\xi = \epsilon^{-1} S$. We then express $\mathbf{v}$, p, and η, as functions of ξ as well as of x,z,t, and ϵ. We also assume that these functions posses asymptotic expansions in ϵ of the form

$$\mathbf{v}(\xi, t, x, z, \epsilon) \sim \mathbf{v}_0(t, x, z) + \epsilon \mathbf{v}_1(\xi, t, x, z) + \epsilon^2 \mathbf{v}_2 + \cdots, \tag{43}$$

where the variables with subscript 0 are the rest state solutions given above. Under this change of variables, the derivatives transform as

$$\frac{\partial}{\partial x} \rightarrow \frac{\partial}{\partial x} + \epsilon^{-1} S_x \frac{\partial}{\partial \xi}, \tag{44}$$

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} + \epsilon^{-1} S_t \frac{\partial}{\partial \xi}. \tag{45}$$

.

Since, $\frac{\partial h}{\partial \xi} = 0$, the equations of motion become

$$\begin{aligned}
\epsilon\frac{\partial u}{\partial x} + S_x\frac{\partial u}{\partial \xi} + \frac{\partial w}{\partial z} &= 0, && (46)\\
\epsilon\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial p}{\partial x}\right) + (S_t + uS_x)\frac{\partial u}{\partial \xi} + S_x\frac{\partial p}{\partial \xi} + w\frac{\partial u}{\partial z} &= 0, && (47)\\
\epsilon^2\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x}\right) + \epsilon\left(w\frac{\partial w}{\partial z} + (S_t + uS_x)\frac{\partial w}{\partial \xi}\right) + \frac{\partial p}{\partial x} + 1 &= 0. && (48)
\end{aligned}$$

The boundary conditions become

$$\begin{aligned}
\epsilon\left(\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x}\right) + (S_t + uS_x)\frac{\partial \eta}{\partial \xi} - w &= 0,\ z = \eta, && (49)\\
w + \epsilon u\frac{\partial h}{\partial x} &= 0,\ z = -h. && (50)
\end{aligned}$$

We will now substitute the asymptotic series forms into these equations and equate coefficients of successive powers of ϵ. Additionally, we transform the boundary conditions at $z = \eta$ into boundary conditions at $z = 0$ by writing the boundary terms as a taylor expansion around $z = 0$. In this way we obtain sets of equations for the successive determination of S and of the various coefficients in the asymptotic expansion of the solution.

Equating the coefficients of order ϵ yields

$$\begin{aligned}
S_x\frac{\partial u_1}{\partial \xi} + \frac{\partial w_1}{\partial z} &= 0, && (51)\\
S_t\frac{\partial u_1}{\partial \xi} + S_x\frac{\partial p_1}{\partial \xi} &= 0, && (52)\\
\frac{\partial p_1}{\partial z} &= 0, && (53)\\
S_t\frac{\partial \eta_1}{\partial \xi} - w_1 &= 0,\ z = 0, && (54)\\
p_1 &= \eta_1,\ z = 0, && (55)\\
w_1 &= 0,\ z = -h. && (56)
\end{aligned}$$

4.2 Modes Structure and the Eiconal equation

We can solve equations (51)-(56) for the structure of wave solutions at leading order. A PDE governing the phase function S will emerge as a solvability condition for these equations, and will be seen to be equivalent to the eiconal equation of shallow water theory.

First, we eliminate η_1 and u_1 using equations (51) and (55) to write

$$\begin{aligned}
\frac{\partial \eta_1}{\partial \xi} &= \frac{\partial p_1}{\partial \xi}, && (57)\\
\frac{\partial u_1}{\partial \xi} &= -\frac{1}{S_x}\frac{\partial w_1}{\partial z}. && (58)
\end{aligned}$$

Equality in equation (57) holds for all z by equation (53). The system simplifies to

$$
\begin{aligned}
-\frac{S_t}{S_x}\frac{\partial w_1}{\partial z}+S_x\frac{\partial p_1}{\partial \xi} &= 0, && (59)\\
\frac{\partial p_1}{\partial z} &= 0, && (60)\\
\frac{\partial p_1}{\partial \xi}-\frac{w_1}{S_t} &= 0,\ z=0, && (61)\\
w_1 &= 0,\ z=-h. && (62)
\end{aligned}
$$

In order to solve these equations, we seek a product solution of the form

$$
\begin{aligned}
p_1 &= A(\xi,t)\psi(x,z), && (63)\\
w_1 &= -S_t\frac{\partial A}{\partial \xi}\phi(x,z). && (64)
\end{aligned}
$$

These forms are analogous to that used in Lecture 3, the well known WKB ansatz. In those cases the solution consists of a rapid sinusoidal oscillation with a slowly varying amplitude. In the present case, we also imagine the solution will take the form of a rapidly oscillating waveform, represented by $A(\xi,t)$, with a slow modulation and vertical structure represented by ϕ and ψ. However, because of nonlinearity, the fast waves do not take the form of sinusoids. Rather, the appropriate wave shape will emerge from the analysis.

Substituting these solution forms into equations (59)-(62) and simplifying yields

$$
\begin{aligned}
\psi &= -\theta^2(x,t)\frac{\partial \phi}{\partial z}, && (65)\\
\frac{\partial \psi}{\partial z} &= 0, && (66)\\
\psi &= -\phi,\ z=0, && (67)\\
\phi &= 0,\ z=-h. && (68)
\end{aligned}
$$

where

$$\theta^2(x,t)=\frac{S_t^2}{S_x^2}. \tag{69}$$

This is a first order system of ordinary differential equations in z in which x and t appear only as parameters. For the system to have a solution, we must have

$$\frac{S_t^2}{S_x^2}=h. \tag{70}$$

A particular solution of the system is then

$$
\begin{aligned}
w_1 &= S_t\frac{\partial A}{\partial \xi}\psi(x)\left(\frac{z}{h}+1\right), && (71)\\
u_1 &= -\frac{S_t}{S_x}\frac{\psi(x)}{h}A, && (72)\\
p_1 &= A\psi(x), && (73)\\
\eta_1 &= A\psi(x). && (74)
\end{aligned}
$$

Note that, in principle, ψ may have an arbitrary x dependence, and u_1 . This freedom corresponds physically to the fact that an arbitrary slowly varying order ϵ height field could be added to η_1, and an arbitrary order ϵ velocity field $U(x,z,t)$ could be added to u_1, that would have to be balanced only at higher order due to the form of equations (34)-(39). Since we are not interested in the interaction of waves with higher order mean flows, we will assume $U=0$ and $\frac{\partial\psi}{\partial x}=0$.

Equation (70) is the same as the eiconal equation computed when we derived shallow water theory. To solve this equation we can use the method of characteristics, as in the previous lectures. It is interesting to note that the modes and rays we have computed are the same as those determined by the linear theory of wave propagation. It is only in the determination of the amplitude $A(\xi,t)$ that nonlinearity plays a role, and to that we now turn.

4.3 Amplitudes

To determine the equations governing the amplitude function $A(\xi,t)$, we must analyze the set of equations obtained by equating the coefficients of order ϵ^2 in equations (34)-(39). Doing so, we obtain

$$S_x\frac{\partial u_2}{\partial\xi}+\frac{\partial w_2}{\partial z} = -\frac{\partial u_1}{\partial x}, \tag{75}$$

$$S_t\frac{\partial u_2}{\partial\xi}+S_x\frac{\partial p_2}{\partial\xi} = -\frac{\partial u_1}{\partial t}-S_xu_1\frac{\partial u_1}{\partial\xi}-w_1\frac{\partial u_1}{\partial z}-\frac{\partial p_1}{\partial x}, \tag{76}$$

$$\frac{\partial p_2}{\partial z} = -S_t\frac{\partial w_1}{\partial\xi}, \tag{77}$$

$$S_t\frac{\partial\eta_2}{\partial\xi}-w_2 = -\frac{\partial\eta_1}{\partial t}-u_1S_x\frac{\partial\eta_1}{\partial\xi}+\eta_1\frac{\partial w_1}{\partial z},\ z=0, \tag{78}$$

$$p_2 = \eta_2-\eta_1\frac{\partial p_1}{\partial z}=\eta_2,\ z=0, \tag{79}$$

$$w_2 = u_1\frac{\partial h}{\partial x},\ z=-h. \tag{80}$$

$$\tag{81}$$

This system is an inhomogeneous form of equations (51)-(56), with forcing given by the solutions computed at lower order. Substituting in the solutions found in equations (71)-(74) yields

$$-\frac{S_t}{S_x}\frac{\partial w_2}{\partial z}+S_x\frac{\partial p_2}{\partial\xi} = G_1(\xi,x,z,t), \tag{82}$$

$$\frac{\partial p_2}{\partial z} = G_2(\xi,x,z,t), \tag{83}$$

$$S_t\frac{\partial p_2}{\partial\xi}-w_2 = G_3(\xi,x,t),\ z=0, \tag{84}$$

$$w_2 = u_1\frac{\partial h}{\partial x},\ z=-h. \tag{85}$$

where

$$
\begin{aligned}
G_1 &= \frac{S_t}{S_x}\frac{\psi}{h}\frac{\partial A}{\partial t} - S_x\frac{\psi^2}{h}A\frac{\partial A}{\partial \xi}, && (86)\\
G_2 &= -S_t^2\frac{\partial^2 A}{\partial \xi^2}\psi\left(\frac{z}{h}+1\right), && (87)\\
G_3 &= -\frac{\partial A}{\partial t} + 2S_t\frac{\psi^2}{h}A\frac{\partial A}{\partial \xi}. && (88)
\end{aligned}
$$

As before, we seek a solvability condition for this system. This time, the condition will impose a constraint on A that will allow us to solve for the amplitude along rays.

Begin by solving (83) for p_2:,

$$
p_2 = \int_{-h} zG_2dz' + P\left(\xi, x, t\right). \tag{89}
$$

Inserting this solution into equation (82) and integrating over z gives

$$
w_2 = -\frac{S_x}{S_t}\int_{-h}^{z} G_1dz' + \frac{S_x^2}{S_t}\int_{-h}^{z}\int_{-h}^{z'}\frac{\partial G_2}{\partial \xi}dz''dz' - \frac{S_x^2}{S_t}\frac{\partial P}{\partial \xi}z + D(\xi, x, t). \tag{90}
$$

Applying the boundary condition at $z = -h$ gives

$$
S_t\frac{\partial P}{\partial \xi} = D + \frac{S_t}{hS_x}\frac{\partial h}{\partial x}\psi A. \tag{91}
$$

Finally, applying the boundary condition at $z = 0$ gives the constraint

$$
-\frac{S_x}{S_t}\int_{-h}^{0} G_1dz' + \frac{S_x^2}{S_t}\int_{-h}^{0}\int_{-h}^{z'}\frac{\partial G_2}{\partial \xi}dz''dz' = -G_3 - S_t\int_{-h}^{0}\frac{\partial G_2}{\partial \xi}dz' + \frac{S_t}{hS_x}\frac{\partial h}{\partial x}\psi A. \tag{92}
$$

By substituting for G_1, G_2, and G_3 into (92), and making extensive use of equation (70), we find an equation for A:

$$
\frac{\partial A}{\partial t} - \left[\frac{3\psi}{2h}S_t\right]A\frac{\partial A}{\partial \xi} + \left[\frac{1}{6}hS_t^3\right]\frac{\partial^3 A}{\partial \xi^3} = \mp\frac{1}{2\sqrt{h}}A. \tag{93}
$$

The sign of the right hand side is determined by the branch of the solution to the eiconal equation that is selected. It is negative for rightward travelling waves, and positive for leftward travelling waves.

Equation (93) is of KDV form, with a linear growth term reflecting the expansion and compression of ray tubes in space-time. Note that

$$
\frac{1}{2\sqrt{h}}\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\sqrt{h}
$$

is the gradient of the ray speed, and thus reflects expansion and contraction of ray tubes. For a rightward travelling wave travelling into deeper water, $\frac{\mathrm{dc}}{\mathrm{d}x} > 0$, ray tubes expand, and equation (93) predicts the decay of A along a ray.

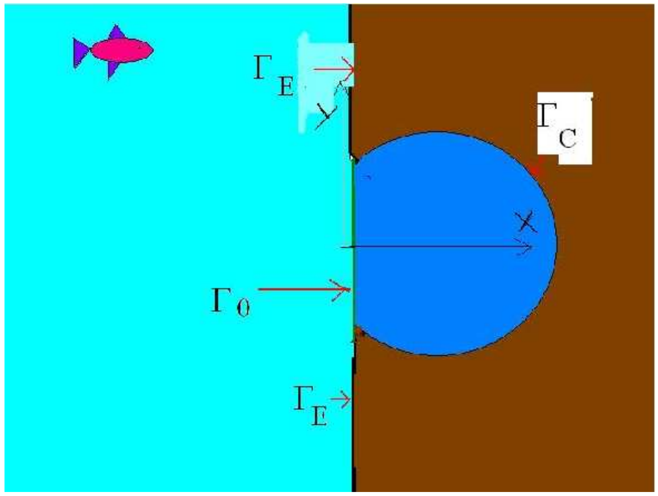

Figure 2: The configuration of the cavity resonator.

Let us now examine the case of constant depth, in which equation (93) reduces to the form

$$\frac{\partial A}{\partial t} - \lambda_1 A \frac{\partial A}{\partial \xi} + \lambda_2 \frac{\partial^3 A}{\partial \xi^3} = 0 \tag{94}$$

where λ_1 and λ_2 are constants. If we seek travelling wave solutions of the form $A(\nu)$, where $\nu = \xi - ct$, we find the ODE

$$-cA' + \lambda_1 AA' + \lambda_2 A''' = 0.$$

One periodic solution to this equation is the Jacobi function,

$$A = \sigma cn\left(\lambda^{-1}(\xi - ct)\right).$$

These are "cnoidal" waves. They resemble cosines, but with flatter troughs and sharper peaks. Furthermore, the wavespeed c is a function of the amplitude σ.

It is helpful to be quite clear about the physical picture of wave propagation that has emerged from this analysis. Due to the eiconal equation (70), surfaces of constant ξ propagate with the shallow water wave speed $\sqrt{h}$. The amplitude equation has solutions that propagate *relative* to surfaces of constant ξ. Thus, for example, the full propagation velocity of the cnoidal wave solutions above is $v = \sqrt{h} + c(\sigma)$. It is possible for these nonlinear disturbances to travel at supercritical speeds.

5 Closed and semiclosed basins

Asymptotic methods will now be applied to determine waves in semiclosed basins linked to the ocean by a small opening, such as harbors and marinas, subjected to the wave field incoming from the open ocean. The configuration considered here is a semicircular cavity, with the origin at the centre, as shown in figure 2. The boundary Γ consists of a circular arc with $r = a$ and two straight lines. The opening, of half-width s, is small compared to the radius of the cavity. The system is a two–dimensional version of an acoustic **Helmholtz resonator**, and will lead to harbour resonance. The problem may be stated as follows: given a prescribed potential at infinity ϕ_∞, corresponding to a plane wave incident at an angle α, that is $\phi_\infty = e^{ik(x\cos\alpha + y\sin\alpha)}$, find the potential $\phi(x, y)$ satisfying:

$$
\begin{aligned}
&\nabla^2\phi + k^2\,\phi = 0\\
&\frac{\partial\phi}{\partial n} \quad \text{on} \quad \Gamma\\
&\phi \to \phi_\infty(x, y) + \phi_s \quad \text{as} \quad r \to \infty
\end{aligned}
\tag{95}
$$

Here ϕ_s is an outgoing wave potential. The problem can be solved numerically, but to get an analytic insight into the behavior of the solution, we shall adopt the asymptotic approach presented in [2]. For a small opening $ks \ll 1$, the disturbance due to the cavity as seen from afar is that of a point source of some strength m. Therefore, we can write:

$$\phi = \phi_\infty - (1/4)imH_0^{(1)}(kr). \tag{96}$$

Expanding the Hankel function in (96) for $kr \to 0$:

$$\phi = \phi_\infty(0, 0) - (1/4)im[1 + (2i/\pi)\log((1/2)\tilde{\gamma}kr)] + \mathcal{O}(mk^2r^2\log(kr)), \tag{97}$$

where $\log\tilde{\gamma} = 0.577..$ is Euler constant. Note that $\phi_\infty(0, 0) = 1$ and define $\phi^0 = 1 - (1/4)im + (m/2\pi)\log((1/2)\tilde{\gamma}ks)$. Three asymptotic expansions are needed.

1. The outer expansion ϕ_{out}, valid in the infinite region away from the opening. The opening appears as a source and the solution is:

$$\phi_{out} \to \phi^0 + (m/2\pi)\log(r/s), \quad r/s << 1, \tag{98}$$

2. The potential in the cavity:

$$\phi_{in}(x, y) = -m\;\phi_C(x, y). \tag{99}$$

Here ϕ_C is a mode of the closed basin. As $r \to 0$:

$$\phi_C \to (2\pi)^{-1}\log r + const. \tag{100}$$

¿From the first two equations in (95), the solution ϕ_C is given by:

$$4\phi_C = Y_0(kr) - \frac{Y_0'(ka)}{J_0'(ka)}J_0(kr). \tag{101}$$

3. A potential ϕ_G in the neighborhood of the opening, obtained by conformal mapping (see [2]).

Matching ϕ_G to the expansion in the cavity, and to that outside of it, gives the source strength m in terms of the conductivity C. The source strength m corresponds to the flux through the opening, given by $m = C(\phi_{in} - \phi_{out})$, where C is the conductivity of the opening.

With the value of m determined, the value of the potential on the cavity wall is:

$$\phi_C = \frac{m}{2ka\pi\, J'(ka)} = \frac{2}{\pi ka}\left(\frac{1}{Y'(ka) + J'(ka)\big[i - \frac{2}{\pi}\big(1 + \log\big(\frac{1}{2}\tilde{\gamma}ks\big)\big]}\right) \tag{102}$$

An example of the potential response, given by (102) and (99) for a large cavity for three different opening widths (characterized by the values of s) is presented in figure (3a). The peaks of the response occur when the term in the denominator in (102) is small. Due to the oscillatory behavior of the Bessel functions, this coincides with zeros of either $J_1(ka) = -J_0'(ka)$ or $Y_1(ka) = -Y_0'(ka)$ [1] see also figure (3b). In case when ka is small, the Y_0' term dominates, and this corresponds with the highest peak in figure (3a), which defines the Helmholtz mode. For the larger values of ka the J_0' term takes over and condition $J_0' = 0$ determines the position of the natural eigenmodes of the closed cavity. For the cavity with small opening, the response is modified by the effect of the $-\log(\tilde{\gamma}ks)$ term, large when ks is small. We can also observe the influence of the opening size s: reduction in the size s moves the Helmholtz peak to smaller frequency and increases the amplitude of the response.

These results, obtained by asymptotic methods were tested in [2] against numerical solutions. They provide a theoretical explanation of harbor resonance, a phenomenon of practical importance in ocean engineering.

Notes be Alex Hasha and Inga Koszalka.

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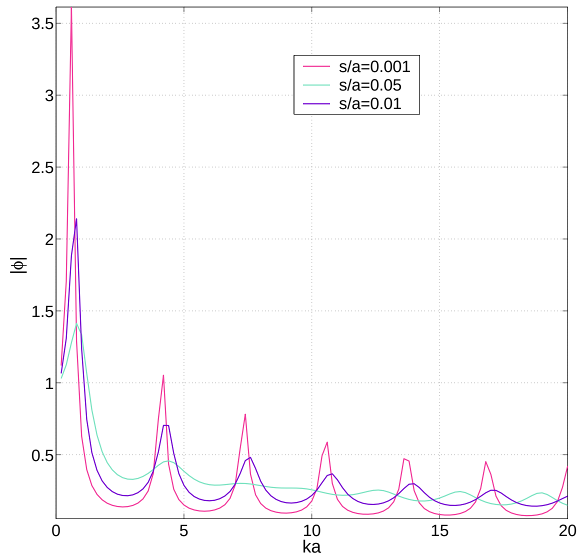

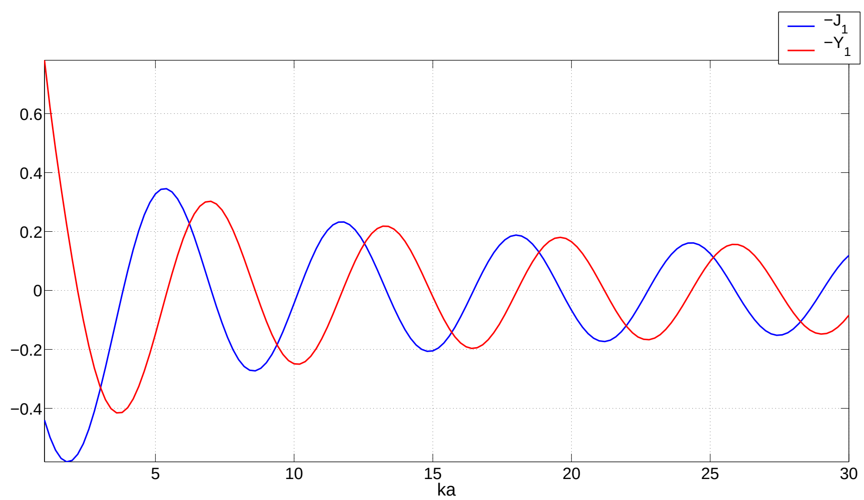

Figure 3: (a) Response, given by (102) and (99) for a large sector cavity, with unit source strength $m = 1$, for three different values of the opening width s. (b) The shape of the bessel functions $-J_1(ka) = J_0'(ka)$ and $-Y_1(ka) = Y_0'(ka)$ for different values of ka.