Lecture 5: The Spectrum of Free Waves Possible along Coasts

Myrl Hendershott

1 Introduction

In this lecture we discuss the spectrum of free waves that are possible along the straight coast in shallow water approximation.

2 Linearized shallow water equations

The linearized shallow water equations(LSW) are

$$u_t - fv = -g\zeta_x, \tag{1}$$

$$v_t + fu = -g\zeta_y, \tag{2}$$

$$\zeta_t + (uD)_x + (vD)_y = 0. \tag{3}$$

where D, η are depth and free surface elevations respectively.

3 Kelvin Waves

Kelvin waves require the support of a lateral boundary and it occurs in the ocean where it can travel along coastlines.

Consider the case when $u = 0$ in the linearized shallow water equations with constant depth. In this case we have that the Coriolis force in the offshore direction is balanced by the pressure gradient towards the coast and the acceleration in the Longshore direction is gravitational. Substituting $u = 0$ in equation (1) through (3) (assuming constant depth D) gives

$$fv = g\zeta_x, \tag{4}$$

$$v_t = -g\zeta_y, \tag{5}$$

$$\zeta_t + Dv_y = 0, \tag{6}$$

eliminating ζ between equation (4) through equation (6) gives

$$v_{tt} = (gD)v_{yy}. \tag{7}$$

The general wave solution can be written as

$$v = V_1(x, y + ct) + V_2(x, y - ct), \tag{8}$$

where $c^2 = gD$ and V_1, V_2 are arbitrary functions. Now note that ζ satisfies

$$\zeta_{tt} = (gD)\zeta_{yy}. \tag{9}$$

Hence we can try $\zeta = AV_1(x, y+ct) + BV_2(x, y-ct)$. From equation (4) or equation (5) we get $A = -\sqrt{H/g}$, $b = \sqrt{H/g}$. V_1 and V_2 can be determined as follows. From equation (4) we have

$$\frac{\partial v}{\partial x} = -\frac{f}{\sqrt{gD}} V_1, \tag{10}$$

$$\frac{\partial v}{\partial x} = \frac{f}{\sqrt{gD}} V_2. \tag{11}$$

Solving the two equations we get

$$V_1 = V_{10}(0, y+ct)e^{-\frac{x}{R}}, \tag{12}$$

$$V_2 = V_{20}(0, y-ct)e^{\frac{x}{R}}, \tag{13}$$

where $R = \sqrt{gD/f}$. Since the second solution increases exponentially from the boundary it is deemed physically unfit and so the general solution can be written as

$$u = 0, \tag{14}$$

$$v = F(y+ct)e^{-\frac{x}{R}}, \tag{15}$$

$$\zeta = -\sqrt{\frac{D}{g}} F(y+ct)e^{-\frac{x}{R}}, \tag{16}$$

where F is some arbitrary function. Since we have exponential decay away from the boundary the kelvin wave is said to be trapped.

4 Poincare Continuum

Let us relax the condition $u = 0$ and keep the equations (1) through (2). With constant f and a flat bottom we can seek a Fourier solution,with u, v, ζ taken as constant factor times the function $e^{(-i\sigma t + ilx + iky)}$. The system of equations (1),(2),(3) becomes

$$-i\sigma u - fv = -igl\zeta, \tag{17}$$

$$-i\sigma v + fu = -igk\zeta, \tag{18}$$

$$-i\sigma\zeta + D(ilu + ikv) = 0. \tag{19}$$

Eliminating u, v from equations (17) through (19) we get the following dispersion relation,

$$\sigma^2 = f^2 + gD(l^2 + k^2). \tag{20}$$

This implies $\sigma^2 > f^2 + gDk^2$. Hence inside the hyperbola $\sigma^2 = f^2 + gDk^2$ there is a continuum of waves with a given frequency σ and this region is called as Poincare's continuum.

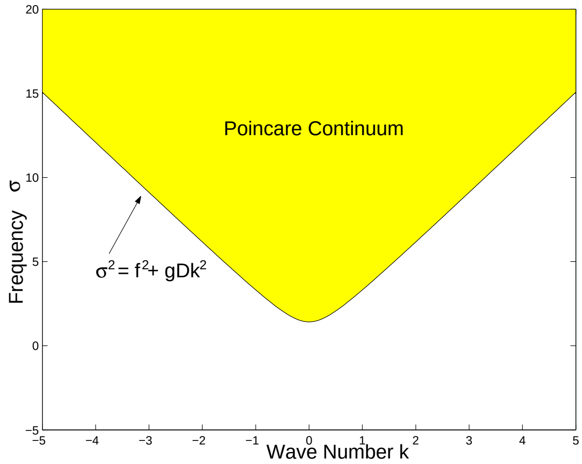

Figure 1: Poincare continuum.

5 Reflection along a straight coast

Consider the reflection of a wave $ae^{(-i\sigma t - ilx + iky)}$ at a straight coast(figure 2). Assuming constant depth we now derive the equation for the reflected wave. At the coast we have the condition that the horizontal velocity $u = 0$. The reflected wave will have the form $ae^{(-i\sigma t + ilx + iky)} + ae^{(-i\sigma t - ilx + iky)}$. Substituting in equations (1),(2) we get,

$$-i\sigma u - fv = -\zeta_x, \tag{21}$$

$$-i\sigma v + fu = -\zeta_y. \tag{22}$$

Solving for u we get,

$$u = \frac{-i\sigma g\zeta_x + fg\zeta_y}{\sigma^2 - f^2}. \tag{23}$$

Hence the boundary condition $u = 0$ implies that

$$-i\sigma g\zeta_x + fg\zeta_y = 0. \tag{24}$$

Using (24) we get,

$$b = a\frac{-l\sigma + ifk}{l\sigma + ifk}. \tag{25}$$

Hence the reflected waves can be represented as

$$a\left[e^{(-i\sigma t + ilx + iky)} + \frac{-l\sigma + ifk}{l\sigma + ifk}e^{(-i\sigma t + ilx + iky)}\right]. \tag{26}$$

6 Waves on a beach with a sloping bottom

We now consider the spectrum of free waves possible in a beach with a sloping bottom where the depth is given by $D = ax$. Let $\zeta = \eta(x)e^{(-i\sigma t + iky)}$ and letting $u, v \propto e^{(-i\sigma t + iky)}$

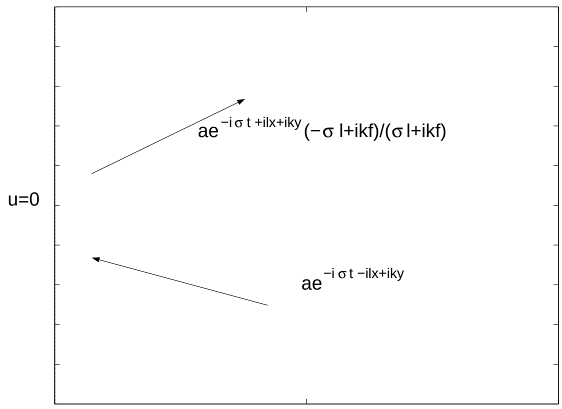

Figure 2: Reflection along a straight coast.

in equations (1),(2),(3) we get

$$i\sigma u - fv = -g\zeta_x, \tag{27}$$

$$i\sigma v + fu = -ikg\zeta, \tag{28}$$

$$i\sigma\eta + (uax)_x + ik(vax) = 0. \tag{29}$$

Eliminating u, v from equations (27),(28) and (29), we get

$$\eta_{xx} + \frac{\eta_x}{x} + \left[\frac{\sigma^2 - f^2}{gax} - \frac{fk}{\sigma x} - k^2\right]\eta = 0. \tag{30}$$

If $\nabla.UD = 0$, then we have

$$\eta_{xx} + \frac{\eta_x}{x} - [\frac{fk}{\sigma x} + k^2]\eta = 0. \tag{31}$$

By making use of the substitution $x = \frac{z}{2k}$ equation (30) can be reduced to the standard Lagurre's Differential Equation

$$\eta_{zz} + \frac{\eta_z}{z} + \left[\frac{\lambda}{z} - \frac{1}{4}\right]\eta = 0, \tag{32}$$

where

$$\lambda = \frac{\sigma^2 - f^2}{2gak} - \frac{f}{2\sigma}. \tag{33}$$

The solution to equation (32) can be written in terms of the eigenfunctions $\eta_n = e^{-z/2}L_n(z), n = 0, 1, 2, ...,$ where $L_0(z) = 1, L_1(z) = 1 - 2z, L_2(z) = 1 - 2z + z^2...$ are the standard Lagurre's polynomials. The condition that the eigensolutions are finite as $z \to 0, \infty$ gives us

$$\lambda = n + \frac{1}{2}, \qquad n = 0, 1, 2, \ldots. \tag{34}$$

Substituting for lambda from equation (33) gives us the following dispersion relation,

$$(\sigma^2 - f^2) - \frac{fgak}{\sigma} = (2n+1)gak, \qquad n = 0, 1, 2, \ldots. \tag{35}$$

Notice that if $f = 0$ we have

$$\sigma^2 = (2n+1)gak. \tag{36}$$

If $\nabla.UD = 0$ then

$$\sigma = \frac{f}{2n+1}. \tag{37}$$

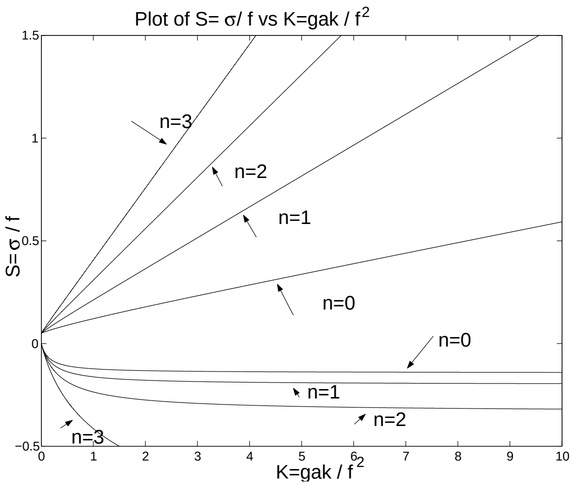

Figure 3: Dispersion relation for waves on a beach with sloping bottom

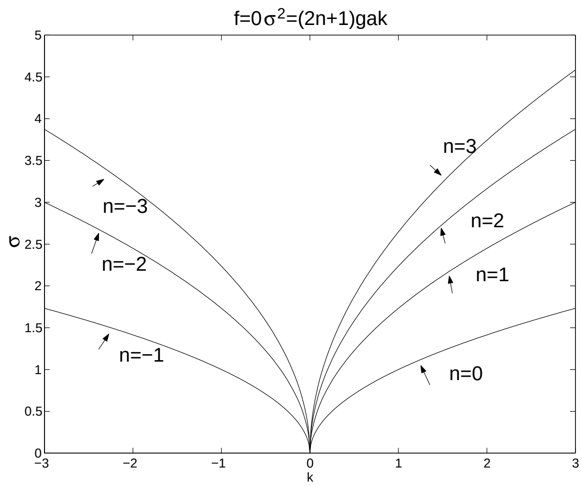

Figure 4: Dispersion relation for waves on a beach with sloping bottom with f=0

7 Refraction of gravity waves over a shelf

A shelf will cause gravity waves to refract. We begin with a shelf of arbitrary topography H(x) (see Figure 5). The local dispersion relation for gravity waves without rotation is

$$\sigma^2 = gH(x)\left[l(x)^2 + k^2\right], \tag{38}$$

where H and l depend on x. Taking the derivative of σ with respect to l and k gives the group velocity in the x and y directions respectively:

$$c_g = (l, k)\sqrt{\frac{gH}{l^2 + k^2}}. \tag{39}$$

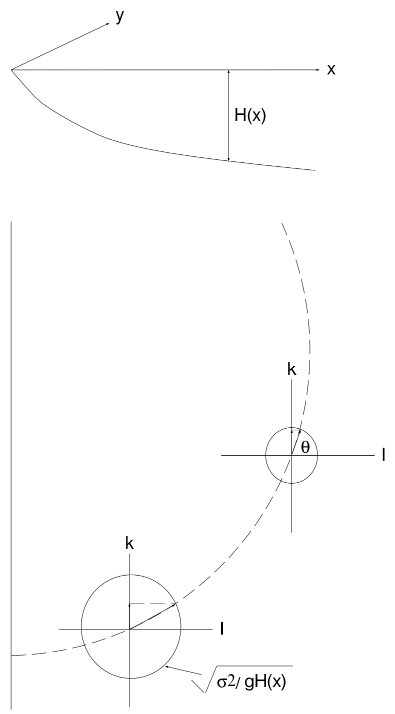

Figure 5: Path of a reflected gravity wave over a sloping shelf.

Equation (39) shows that the group velocity vector is parallel to the wave vector, hence the wave group moves in the direction of the wave vector. As the wave group moves, σ and k don't change, only l does. So if the topography $H(x)$ is known, then l can be computed in each position and is seen to decrease with distance x for a constant direction slope. Consequently, the direction of the wave vector changes with x. The angle θ that the wave vector makes with the horizontal can be computed using the dispersion relation as

$$\sin\theta = \frac{k}{\sigma}\sqrt{gH(x)}. \tag{40}$$

As H increases, so does θ, leading to the reflection pattern seen in Figure 5. The initial angle with which the wave hits the coastline dictates how far back into the ocean it reflects before returning to the coast again.

The solution for the free surface elevation also shows that all gravity waves are trapped along the coastline with this topography. Using the previous results for short waves with no Coriolis force and an arbitrary depth profile, we get the following expression for η:

$$\eta_{xx} + \left(\frac{\sigma^2}{gH} - k^2\right)\eta = 0. \tag{41}$$

The coefficient of η indicates a turning point in the solution for the free surface. While the coefficient is positive $\left(\frac{\sigma^2}{gH(x)} - k^2 > 0\right)$, a wave solution results. When the coefficient is negative $\left(\frac{\sigma^2}{gH(x)} - k^2 < 0\right)$, an Airy solution results, which decays in the positive x direction.

When the shelf takes the form of a step (see Figure 6), some of the gravity waves will become refractively trapped, but there will be a continuum of waves that will be reflected back into the open ocean. Whether a wave becomes trapped or reflects back is determined by its initial angle of entry onto the shelf.

Munk et al. (1964) investigated refractively trapped gravity waves (or edge waves) along the California coast. Using the actual shelf topography with the assumption of a straight coastline for Southern California, they computed the dispersion relation for the gravest edge wave modes. Their computations matched very well with measurements.[1]

8 Refraction of topographic Rossby waves over a shelf

When a patch of fluid moves on or offshore, potential vorticity must be conserved:

$$\frac{D}{Dt}\left(\frac{\xi + f_0}{H + \eta}\right) = 0. \tag{42}$$

Assuming that $H >> \eta$ and neglecting the nonlinear terms, this implies that

$$\xi_t = u\frac{f_0 H_x}{H}. \tag{43}$$

If there is an initial patch of positive potential vorticity, a positive velocity is induced on one side of the patch, while a negative velocity is induced on the other side. For a shelf sloping in a constant direction ($dH/dx > 0$), Equation (43) states that the vorticity will decrease with the negative velocity and increase with the positive velocity (Figure 7). This propagates the positive vorticity patch in the direction of Kelvin waves.

The dispersion relation for topographic Rossby waves can be derived from analogy with planetary Rossby waves. For planetary Rossby waves, the dispersion relation is

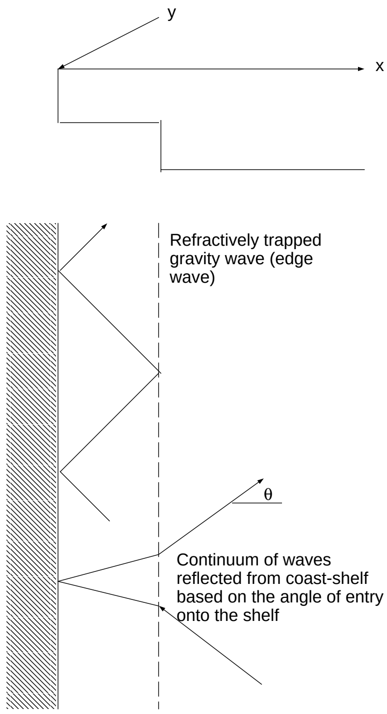

Figure 6: Path of a reflected gravity wave over a step shelf.

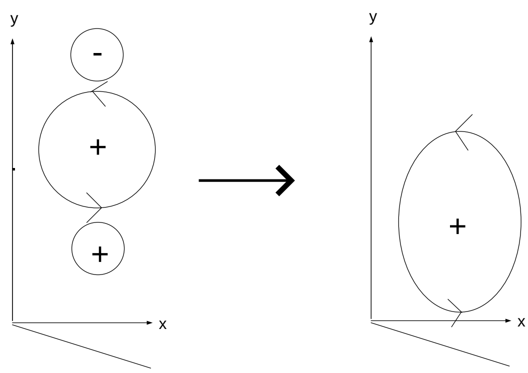

Figure 7: The patches of vorticity generated by an initial patch of positive vorticity.

$$\sigma = -\frac{\beta l}{l^2 + k^2 + f_0^2/gH}. \tag{44}$$

Replacing β with $f_0 H(x)/H$ for topographic Rossby waves gives

$$\sigma = -\frac{f_0 H(x)/H}{l^2 + k^2 + f_0^2/gH}. \tag{45}$$

Similarly to the gravity waves discussed above, the alongshore wavenumber k and the frequency σ do not change as the wave moves in the x direction, so l responds to the change in depth, resulting in the same arcing pattern seen for gravity waves in Figure 5.

9 Wave excitation by tides

Munk et al. (1970) measured tidal pressures and currents off the coast of Southern California and analyzed the data for the tidal components. The dispersion relations for the possible trapped waves against a straight coast with a shelf can be computed. The observations can be fit by superposing all the possible trapped waves of tidal frequencies. Using a plot of the dispersion relations for the various wave types described above (see Figure 8 for a composite illustration of the various dispersion relations), the types of waves that can be excited by a tide is determined by selecting the tidal frequency of interest and seeing which types of waves are possible at that frequency. For example, in Figure 8, if the frequency of interest is f_0, only the Kelvin wave would be excited. If the frequency is very low, the Kelvin and the Rossby waves would be excited. If the frequency is higher than f_0, waves in the Poincaré continuum would be excited in addition to certain edge wave modes and the Kelvin wave. The model of Munk et al. (1970), based on wave excitation, reproduces the observed tidal amplitudes of the M_2 tide, although the K_1 tide is not well modeled. The model was successful in predicting the tidal amplitude very well over a range of model parameters. Additionally the model was also able to predict the amphidrome of the M_2 tide off the coast of Southern California. Comparison of their modeled values for the current over the course of a month shows that the model agrees well with the data, but there are large differences at some points. This is attributed to the inability of the model to resolve baroclinic modes and their currents. [2]

Cartwright (1969) describes measurements which show strong diurnal tidal currents near the St. Kilda islands. The tidal amplitude of the diurnal K_1 tide are much less than the amplitude of the semi-diurnal M_2 tide, but measurements show that the currents due to these tides are of the same order. He attributes this to excitation of a Rossby wave by the diurnal tides.[3]

10 Waveguide modes

In an infinitely long channel in the x direction on an f plane with sides at $y = 0$ and $y = a$ with $v = 0$ at boundaries (Figure 9a), waves will travel and have structure along the waveguide, similar to Kelvin waves propagating along a coast. The kinds of waves which can propagate can be determined by solving

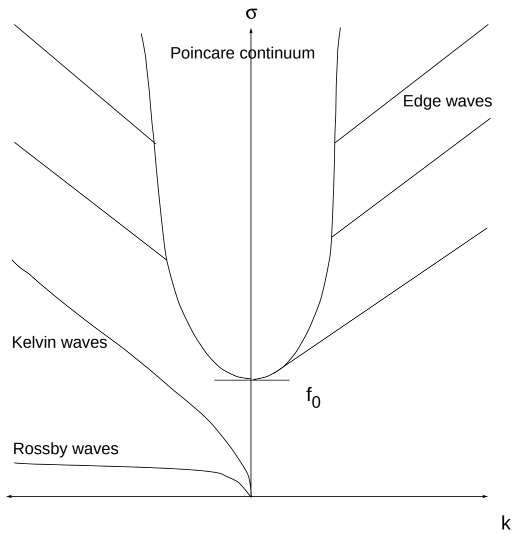

Figure 8: Composite illustration of the dispersion relations for the various wave types discussed above.

$$\nabla^2\eta + \frac{\sigma^2 - f^2}{gD}\eta = 0, \tag{46}$$

where D is the water depth. The boundary condition $v = 0$ can be entered into the governing equations and assuming that v takes a solution form of $e^{i\sigma t + kx + ly}$ the following condition at the boundary results:

$$\sigma\eta_y + f\eta_x = 0 \tag{47}$$

at $y = 0$ and $y = a$. The solution to Equation (46) is assumed to take the form:

$$\eta \sim e^{ikx}\left(\cos\frac{m\pi y}{a} + \alpha_m \sin\frac{m\pi y}{a}\right).$$

This solution form is valid if

$$k^2 = \frac{\sigma^2 - f^2}{gD} - \left(\frac{m\pi}{a}\right)^2. \tag{48}$$

If $k^2 > 0$, the waves will propagate. If $k^2 < 0$ the waves are evanescent and will decay. The boundary condition can be restated as

$$i\sigma\frac{m\pi}{a}\left(-\sin\frac{m\pi y}{a} + \alpha_m \cos\frac{m\pi y}{a}\right) + ifk\left(\cos\frac{m\pi y}{a} + \alpha_m \sin\frac{m\pi y}{a}\right) = 0. \tag{49}$$

at $y = 0, a$. To satisfy the boundary conditions,

$$\sigma_m = -\frac{f}{\sigma}\frac{ka}{m\pi} \qquad\qquad m = 1, 2, \ldots. \tag{50}$$

Note that the $m = 0$ mode does not satisfy the boundary conditions t $y = a$.

As m increases, k decreases and will become imaginary, making evanescent waves. These waves are unimportant in the infinite channel case, but are important in the closed channel case discussed in the next section. For the waves to propagate

$$m < \left(\frac{\sigma^2 - f^2}{gD}\frac{a^2}{\pi^2}\right)^{1/2} \qquad \text{and} \qquad \sigma^2 > f^2.$$

Kelvin waves are also possible and can propagate along both boundaries, but in opposite directions. The two Kelvin waves must be superimposed to describe the behavior of η. This superposition of the two waves leads to amphidromes in the channel, separated by a half wavelength, where there is no change in η. The wave crests rotate around these amphidromes with a period of $2\pi/\sigma$. Figure 9 shows examples of the location of the amphidromes, depending on the magnitude of the two Kelvin waves.

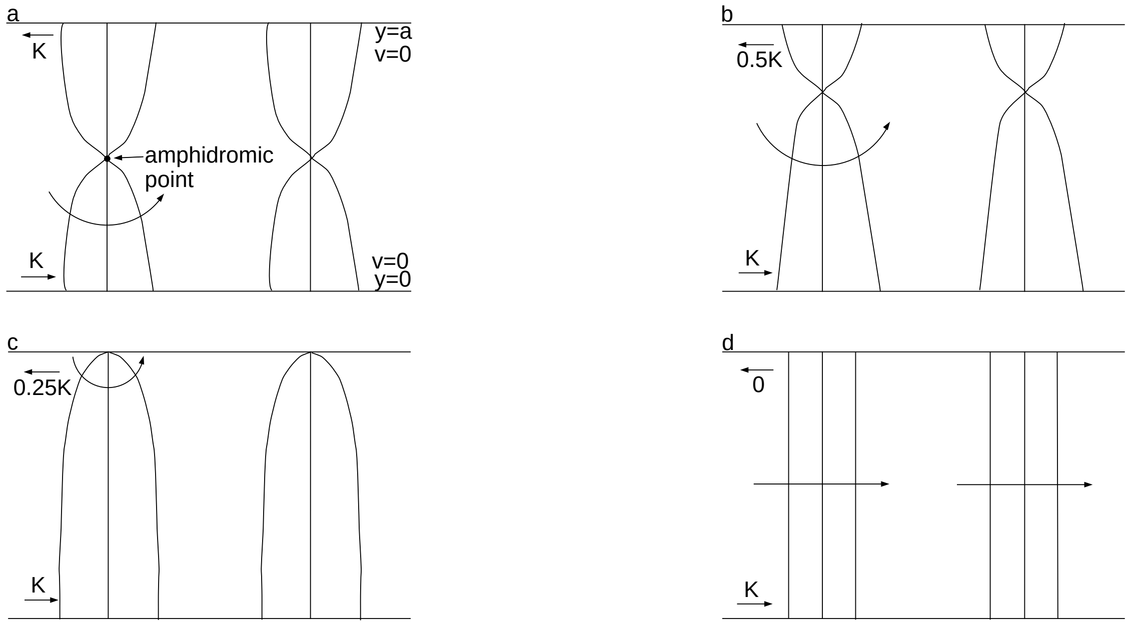

Figure 9: Kelvin waves propagating in a channel. Panel (a) shows the variables. a. The Kelvin waves are of equal magnitude. b. One Kelvin wave is twice the magnitude of the other. c. One Kelvin wave is four times the magnitude of the other. d. The Kelvin wave only exists in one direction resulting in cotidal lines that progress along the channel.

11 Kelvin wave reflection

In a channel with a closed end as in Figure 10, an incident Kelvin wave will be reflected out of the channel. While $v = 0$ at the boundaries for these waves, $u \neq 0$. For the boundary condition of $u = 0$ to be met at the end of the channel, an infinite series of evanescent Poincaré waves (as discussed above) is needed so that the velocity imposed by the Kelvin wave on the end of the channel is countered. These waves decay exponentially so at a sufficient distance from the end of the channel, the channel is seen as infinite and the solution looks like that discussed above. In order to have these evanescent waves and consequently perfect reflection of the incident Kelvin wave,

$$\left(\sigma^2 - f^2\right) \frac{a^2}{\pi^2 g D} < 1.$$

This can be achieved in two manners: (1) the waves are sub-inertial ($\sigma < f$) or the channel is sufficiently deep or narrow (D or a small, respectively). If neither of these criteria are met, the reflection is not a simple Kelvin wave.

Figure 10 shows the solution for perfect reflection of a Kelvin wave in a flat-bottomed perpendicular wall basin. Modes are obtained for each integral number of Kelvin wavelengths around the circumference of the amphidrome closest to the end of the channel. Bottom topography and shelves make other modes possible as well. Hendershott and Speranza (1971) show examples of this phenomenon. They show that when there is dissipation, the amphidromes move toward $y = a$ in Figure 10. In the Gulf of California, which is shallow near the end of the gulf and hence there is large dissipation, the co-tidal lines show only the incident wave - the reflecting wave has been damped out by the dissipation. In the Adriatic Sea, where the shelf is still deep, the amphidrome is closer to the center of the Sea, implying that the incident and reflected Kelvin waves are of the same amplitude.[4]

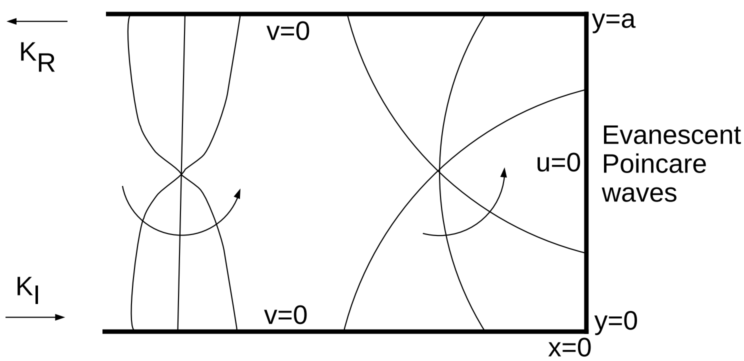

Figure 10: A series of evanescent Poincaré waves is required for reflection of the Kelvin wave out of the closed channel.

12 Tides in gulfs

The conventional treatment of tides in gulfs considers the solution to be the sum of an independent tidal solution (where there is tidal forcing and the tidal amplitude at the mouth of the gulf is zero) and a co-oscillating tidal solution (where there is no forcing and the amplitude at the mouth is observed). Figure 11 shows schematically how this solution is obtained. This method can work only if the tidal amplitude at the mouth of the gulf is observed.

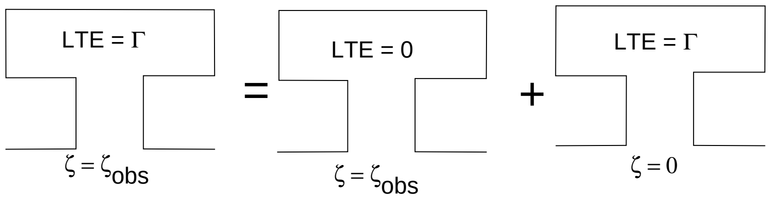

Figure 11: Treatment of tides in a gulf as the sum of an independent and co-oscillating tidal solutions.

Garrett (1975) proposes another methodology to solve this case. Figure 12 shows the two solutions that are combined to achieve the true solution. The boundary condition is that the normal velocity is zero at the coastlines of the gulf and the ocean. Garrett also represents the true solution as the sum of two simpler solutions. In one of these, the zero normal velocity boundary condition is also imposed at the mouth of the gulf. The second solution has a normal mass flux boundary condition at the mouth that is equal to a function $F(s)$ to be determined, where s is the distance across the mouth:

$$\vec{u}D \cdot \hat{n} = F(s), \tag{51}$$

To determine $F(s)$, Garrett first supposes that $\vec{u}D \cdot \hat{n} = \delta(s-\sigma)$ produces elevations $\zeta_G(s) = K_G(s,\sigma)$ and $\zeta_O(s) = -K_O(s,\sigma)$ on the gulf and ocean sides of the mouth respectively, where δ is the Dirac delta function. To satisfy continuity at the mouth, the sum of the two solutions for the gulf and the ocean need to be equal:

$$\zeta(s) = \zeta_G^1(s) + \int_{mouth} K_G(s,\sigma)F(\sigma)d\sigma = \zeta_O^1 - \int_{mouth} K_O(s,\sigma)F(\sigma)d\sigma. \tag{52}$$

To solve this problem, first ζ_G^1,ζ_O^1,K_G,K_O are found and then the total solution,

$$\zeta_G(r) = \zeta_G^1(r) + \int_{mouth} K_G(r,\sigma)F(\sigma)d\sigma, \tag{53}$$

can be solved to give the amplitude of the gulf tide.[5]

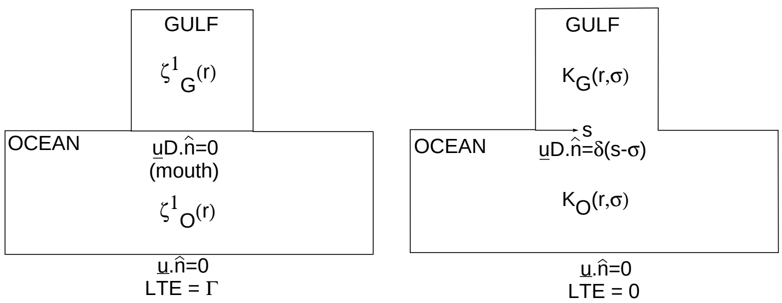

Figure 12: Illustration of the solutions used by Garrett (1975) for the gulf tides.

Notes by Vishwesh and Danielle

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