Lecture 9: Tidal Rectification, Stratification and Mixing

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1 Additional Notes on Tidal Rectification

This lecture continues the discussion of long-wavelength tidal flow over comparatively short topography such as George's Bank. The mean flow around the bank can also be understood via the following vorticity argument.

First, consider the vertical component of the fluid's relative vorticity,

$$\xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{1}$$

ξ is defined as positive for counterclockwise flow, and negative for clockwise flow. Here, with water depth $H(x)$ and constant f, the evolution of the vorticity is given by

$$\frac{\partial \xi}{\partial t} + \mathbf{u} \cdot \nabla \xi - \frac{uf}{H^2}\frac{dH}{dx} = 0, \tag{2}$$

since the *potential vorticity* of the fluid is conserved following the flow:

$$\frac{D}{Dt}\left(\frac{\xi + f}{H + \zeta}\right) = 0. \tag{3}$$

We can see that changes in either the Coriolis parameter or the water depth will induce changes in the vorticity. If a fluid parcel is displaced north, its ambient vorticity increases ($df/dy > 0$), and it compensates by developing a negative relative vorticity, such that (3) is satisfied. Similarly, if a parcel moves into shallower water ($dH/dx < 0$) the relative vorticity has to decrease in order for potential vorticity to be conserved. Because of the analogy between planetary and topographical vorticity gradients, we can understand the mean flow due to a tidal current across the slope using the same argument that is made to explain the propagation of planetary Rossby waves.

For this analysis, we will neglect the nonlinear advective term in (2), and additionally assume that the rate of change of vorticity is a function of not just the Coriolis/depth term in (2), but that also of bottom drag. Defining λ as a friction parameter, we therefore model the rate of change of vorticity as

$$\frac{\partial \xi}{\partial t} = -\beta u - \lambda \xi, \tag{4}$$

where $\beta \equiv -\frac{f}{H^2}\frac{dH}{dx}$. If we plug in plane wave solutions

$$\xi = \xi_0 e^{-i\omega t}, \quad u = u_0 e^{-i\omega t},$$

then

$$\begin{aligned} -i\omega\xi &= -\beta u - \lambda\xi \\ \xi &= \frac{-\beta u}{\lambda - i\omega} \end{aligned} \tag{5}$$

Equation (5) helps us to interpret what happens to vorticity if there is both friction and a change of depth. In particular:

- If $\lambda = 0$, then the vorticity lags the current by a phase 90^o.
- If $\lambda \gg \omega$ then $\xi \propto -u$, so that vorticity and current are, at least in part, in phase.
- For any $\lambda \neq 0$, vorticity and current have opposite signs.

The importance of the drag term will be illustrated shortly.

Fluid that is displaced into shallow water takes on a negative relative vorticity, corresponding to clockwise rotation, which is then advected upslope. Fluid that is moving in the opposite direction takes on a positive relative vorticity, corresponding to counterclockwise rotation, which is advected downslope with that phase of the tide. Thus there is a downslope flux of positive vorticity. This flux is greatest over the steepest part of the slope, so its divergence leaves a residual vorticity, negative near the top of the slope and positive near the bottom. Since this generation process is independent of the position along the slope, the mean Eulerian current is parallel to the slope, and will flow with the shallow side to the right (when facing downstream). For a clockwise-rotating tidal current, the corresponding Stokes Drift will oppose it, which means that the net Lagrangian current is weaker than the Eulerian current.

Without friction, the current and vorticity are exactly out of phase, and the vorticity flux will therefore vanish when averaged over a tidal period, leaving zero residual current. Thus the presence of friction is necessary for a nonzero residual current.

2 Stratification and Mixing

Let us now examine stratification and mixing in a bay such as the Bay of Fundy [1]. Solar heating will, in principle, thermally stratify the water. This is what happens in a still lake, but in the oceans, where we have flow over bottom topography, the problem becomes more complex. We take three different approaches to the argument: first, since we're all geophysical fluid dynamicists here, we go through a simple dimensional argument and see what kind of solution is required. We then make a more mathematical argument by making conservation of energy considerations as in Simpson and Hunter (1974) [2]. Lastly, we use a highly simplified dynamical model to illustrate the role of mixing.

2.1 Dimensional Argument

First, we establish the dimensions of the variables of the problem, The tidal flow U has dimensions like LT^{-1}, where L is a length and T is a time. The depth of the fluid, H, has dimension L.

Since we are not so much interested in the heat flux, Q, as its effect on the water temperature, we instead consider the temperature change associated with the heat flux, or $Q(\rho c_p)^{-1}$. Actually, since temperature change also means a change in density, we will actually work with the quantity $\alpha Q(\rho c_p)^{-1}$, where $\alpha = \rho^{-1} d\rho/dT$. Then the buoyancy flux B is approximately given by

$$B = \frac{g\alpha Q}{\rho c_p}. \tag{6}$$

where B has dimensions L^2T^{-3}. The only dimensionless contribution of B, H, and U is BH/U^3, suggesting that a critical value of this separates *well-mixed* from *stratified* water.

2.2 Energy Argument

Suppose we model the warming of the bay by assuming the sun shines on the surface providing a heat flux of Q. Further suppose that only a small layer of height h is warmed by the heat giving the water in that layer a density of $\rho - \Delta\rho$. This warmer fluid lies above a larger layer of water with thickness H ($h \ll H$)and density ρ. See Figure 1 for a diagram of this model.

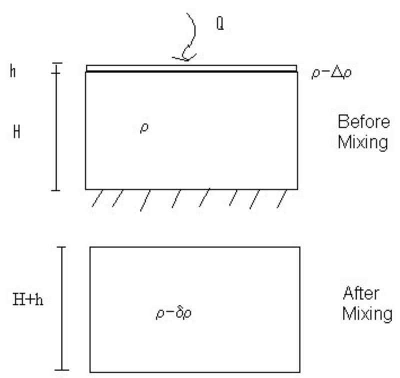

Figure 1: A Model of the Bay of Fundy

The change in temperature in the top layer can therefore be given by

$$h\frac{dT}{dt} = \frac{Q}{(\rho - \Delta\rho)\, c_p}, \tag{7}$$

where T is the temperature and c_p is the specific heat of water. Taking $\Delta\rho \ll \rho$, we can ignore the $\Delta\rho$ term in the denominator and express the rate of change of temperature in the top layer as

$$h\frac{dT}{dt} = \frac{Q}{\rho c_p}. \tag{8}$$

Using the chain rule, we can write

$$\frac{h}{\rho}\frac{d\rho}{dt} = \frac{h}{\rho}\frac{d\rho}{dT}\frac{dT}{dt} = -\frac{\alpha Q}{\rho c_p} = -\frac{B}{g}, \tag{9}$$

Now suppose that turbulent mixing homogenizes the fluid so that it all has density $\rho - \delta\rho$. Mass conservation requires

$$H\rho + h\left(\rho - \Delta\rho\right) = \left(H + h\right)\left(\rho - \delta\rho\right), \tag{10}$$

so that solving for $\rho - \delta\rho$ gives

$$\rho - \delta\rho = \frac{H\rho + h\left(\rho - \Delta\rho\right)}{H + h}. \tag{11}$$

The potential energy before mixing is given by

$$\frac{1}{2}gH^2\rho + ghH\left(\rho - \Delta\rho\right), \tag{12}$$

while after mixing the potential energy is

$$\frac{1}{2}g\left(H + h\right)^2\left(\rho - \delta\rho\right). \tag{13}$$

Therefore the change of potential energy is

$$\begin{aligned}
\Delta PE &= \frac{1}{2}g\left[\left(H + h\right)^2\left(\rho - \delta\rho\right) - 2hH\left(\rho - \Delta\rho\right) - H^2\rho\right] \\
&= \frac{1}{2}g\left[\left(H + h\right)\left(H\rho + h\left(\rho - \Delta\rho\right)\right) - 2hH\left(\rho - \Delta\rho\right) - H^2\rho\right] \\
&= \frac{1}{2}g\left[\left(H + h\right)^2\rho - \left(H + h\right)h\Delta\rho - \left(H + h\right)^2\rho + h^2\rho + 2hH\Delta\rho\right] \\
&= \frac{1}{2}g\left[hH\Delta\rho + h^2\left(\rho - \Delta\rho\right)\right].
\end{aligned} \tag{14}$$

Neglecting the quadratic terms in h gives us

$$\Delta PE = \frac{1}{2}gHh\Delta\rho. \tag{15}$$

Dividing (15) by Δt and taking the continuous limit we find that the rate of potential energy gain in the fluid is

$$\frac{dPE}{dt} = -\frac{1}{2}gHh\frac{d\rho}{dt} = \frac{1}{2}\rho BH, \tag{16}$$

where we have used (9) for the quantity $h\frac{d\rho}{dt}$.

If as in the dimensional analysis case, we assume that the tides are generating a flow in the horizontal direction of $u = U\cos\omega t$, the average rate of dissipation due to turbulence is

$$\begin{aligned}
c_d\rho < u^3 > &= c_d\rho\left(\frac{1}{\pi}\int_{-\pi/2}^{\pi/2} U^3\cos^3\theta d\theta\right) \\
&= \frac{4}{3\pi}c_d\rho U^3.
\end{aligned} \tag{17}$$

Therefore, we can expect the lake to be well mixed if the rate of dissipation is sufficiently larger than the rate of potential energy increase needed to mix the water. Thus we expect that $B/(hU^3)$ must be less than a critical value, just as predicted by dimensional analysis.

2.3 A (Poor) Model

We again consider the bay to be a layer of water of height H being warmed by the sun which provides a buoyancy flux B at the surface. Let us define the buoyancy acceleration to be

$$b = -g\left(\frac{\rho - \rho_0}{\rho_0}\right). \tag{18}$$

The buoyancy frequency is defined by

$$N^2 = \frac{db}{dz}. \tag{19}$$

If we assume that the buoyancy change diffuses vertically through the bay, we can model the evolution of the buoyancy acceleration with the diffusion equation. That is

$$\frac{\partial b}{\partial t} = \frac{\partial}{\partial z}\left(K\frac{\partial b}{\partial z}\right), \tag{20}$$

where K is the diffusion coefficient. We must solve this equation with the boundary conditions that there is no buoyancy flux in at the bottom of the bay and that the flux at the top is B. Solving this problem gives

$$b = b_0 + \frac{Bt}{H} + \frac{Bz^2}{2HK}. \tag{21}$$

It remains to represent the diffusion coefficient, K, in terms of the parameters of the problem. Suppose that

$$K = CUHF\left(R_i\right) \tag{22}$$

where C is a constant and F is an unknown function of the dimensionless Richardson number R_i which represents how much stabilizing stratification there is in the system compared with destabilizing shear:

$$R_i = \frac{N^2}{\left(\partial U/\partial z\right)^2}. \tag{23}$$

If we take $N\left(H/2\right)$ to be the characteristic buoyancy frequency of the system, we have

$$R_i = \frac{B/2K}{\left(U/2H\right)^2}. \tag{24}$$

Solving for K and setting it equal to the definition of K in (22), we have

$$K = \frac{2BH^2}{R_i U^2} = CUHF\left(R_i\right), \tag{25}$$

so that

$$\frac{BH}{U^3} = \frac{1}{2}CR_iF\left(R_i\right). \tag{26}$$

Taking

$$K = 2\times 10^{-3}UH\left(1+R_i\right)^{-7/4}, \tag{27}$$

from Bowden and Hamilton (1975) [3], so that $F(x) = (1+x)^{-7/4}$. The maximum of $xF(x)$ occurs at $x = 4/3$. Plugging this value into (26) gives

$$\frac{H}{U^3} \approx 3 \times 10^4 B^{-1}. \tag{28}$$

For values of $BH/U^3 > 3 \times 10^4$, no steady solutions are possible so that the stratification will increase with time. This value may thus separate *well-mixed* from *stratified* condition. Using $B \approx 7 \times 10^{-8} m^2/s^3$ gives

$$\frac{H}{U^3} \approx 5000. \tag{29}$$

However, experiment shows that the critical value of H/U^3 for mixing is approximately $70 m^{-2} s^3$, which means that this model does capture the physics accurately. From observation, the efficiency of tidal mixing is 2.6×10^{-3},which implies that tidal currents lose most of their energy to turbulent dissipation other than to increasing buoyancy flux.

Notes by Lisa Neef and Dave Vener.

References

[1] J. K. C. Garrett and D. Greenberg, "Tidal mixing versus thermal stratification in the bay of fundy and gulf of maine," Atm. Ocean **16**, 403 (1978).

[2] J. Simpson and J. Hunter, "Fronts in irish sea," Nature **250(5465)**, 404 (1974).

[3] K. Bowden and P.Hamilton, "Some experiments with a numerical model of circulation and mixing in a tidal estuary," Estuarine and Coastal Marine Science **3(3)**, 281 (1975).