Lecture 1: Introduction to Linear and Non-Linear Waves

Lecturer: Harvey Segur. Write-up: Michael Bates

June 15, 2009

1 Introduction to Water Waves

1.1 Motivation and Basic Properties

There are many types of waves. Here, we will be paying particular attention to water waves as they provide a concrete, physical example of a dynamical system that exhibits many of the mathematical concepts that have been developed in recent years, including,

- linear stability,
- nonlinear stability,
- solitons,
- complete integrability,
- chaos,
- sensitive dependence on initial data,
- singularities,
- blow-up in finite time, and
- deterministic versus probabilistic models.

Water waves evolve on a timescale that humans can naturally relate to (as opposed to, say, optical waves). Furthermore, since we are hosted by the Woods Hole Oceanographic Institution, water waves are a fitting phenomenon to examine.

For the purposes of these notes, we shall pay particular attention to surface water waves, that is, those that you would feel if you were in a boat on the surface, or that you might observe if you were at the beach.

Surface water waves have their maximum displacement at the surface (decreasing exponentially with depth) and are approximately periodic. Furthermore, they propagate with little dissipation [1], that is, they lose very little energy to the surrounding environment as they propagate. As a result, they can propagate very long distances. Surface water waves

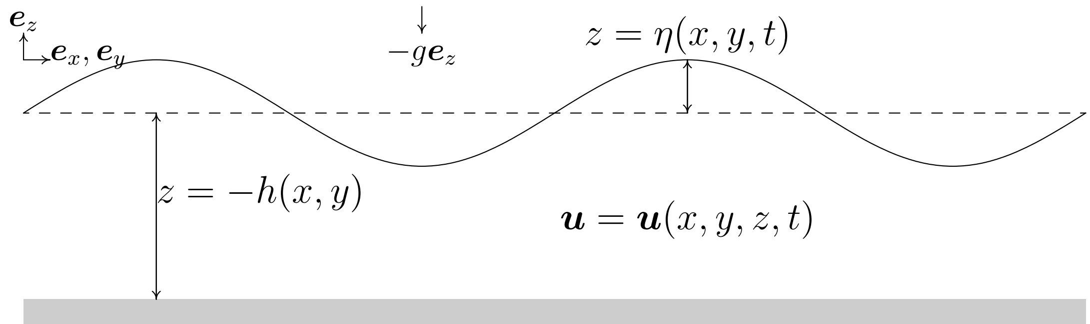

Figure 1: The shaded region indicates the solid earth boundary, the dashed line indicates mean sea level ($z = 0$) and the solid line represents the free surface. h is the distance from mean sea level to the ocean floor, and $\eta(x, y, t)$ is the deviation of the free surface from mean sea level and g is acceleration due to gravity.

are also dispersive, in that waves of different wavelengths travel at different speeds.

There are a number of other oceanic waves that will be ignored in this discussion, including,

- sound waves (or pressure waves) which arise from the compressibility of seawater. Pressure wave speeds are $O(1500\,\mathrm{ms}^{-1})$, which is much greater than, say, the 2004 Tsunami which propagated at less than $200\,\mathrm{ms}^{-1}$,
- internal waves (see Lectures 6 and 11), which arise due to variations in fluid density, have periods in the order of hours (recall surface waves have periods of order seconds),
- inertial waves (which include Rossby Waves), which are due to the rotation of the Earth and have periods greater than 12 h.

1.2 Derivation of the Governing Equations

Following Stokes[2], we derive some equations representing the evolution of the displacement of the sea surface height, $\eta = \eta(x, y, t)$, from a rest position at $z = 0$ and for the velocity throughout the fluid domain, $\boldsymbol{u} = \boldsymbol{u}(x, y, z, t)$, where $-h(x, y) < z < \eta(x, y, t)$ (h is the depth of the ocean floor), with $t > 0$ and $\eta + h > 0$.

Here (x, y, z) denote fixed, laboratory coordinates with z indicating the vertical coordinate and x and y denoting horizontal coordinates. This is illustrated in Figure 1. We can denote the location of a fixed fluid parcel as $\boldsymbol{x}(t) = \{x(t), y(t), z(t)\}$ and the velocity of a fluid parcel,

$$\boldsymbol{u}(\boldsymbol{x}, t) = (u, v, w) = \left(\frac{\mathrm{D}x}{\mathrm{D}t}, \frac{\mathrm{D}y}{\mathrm{D}t}, \frac{\mathrm{D}z}{\mathrm{D}t}\right), \tag{1}$$

where $\frac{\mathrm{D}}{\mathrm{D}t}$ is the material derivative, or in other words, the derivative that follows the fluid parcel in a Lagrangian perspective of fluid motion. The material derivative may be written in Eulerian form as,

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}. \tag{2}$$

For instance, the material derivative for the z component of velocity is,

$$\frac{\mathrm{D}w}{\mathrm{D}t} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}. \tag{3}$$

To set about deriving equations governing η and $\boldsymbol{u}$, we assume that the water (again, following Stokes [2]),

- is incompressible (which precludes sound waves),
- has uniform density (which precludes inertial waves),
- is not affected by the rotation of the Earth (which precludes inertial waves),
- is inviscid,
- and finally that gravity is uniform and is anti-parallel with $\boldsymbol{e}_z$.

Note that the theory can be expanded to include all of the neglected effects (Lectures 6 and 11 analyse the case of internal waves, while Lecture 15 discusses the effect of viscous damping on the fluid motion).

In addition, we assume that the flow is irrotational. This assumption does not change the governing equations but restricts the types of initial conditions considered, and therefore the types of solutions obtained.

We begin with mass conservation, noting that the net mass flux into an arbitrary volume V must be zero by incompressibility, so that

$$\iint_{\partial V} \rho \boldsymbol{u} \cdot \hat{\mathbf{n}} \mathrm{d}s = 0 \tag{4}$$

where ∂V is the surface bounding the volume, $\mathrm{d}s$ an infinitesimal area on that surface, and $\hat{\mathbf{n}}$ the unit normal vector to the surface. Using the divergence theorem, and the fact that we have assumed a uniform density, $\rho =$ constant, we may rewrite equation (4) as

$$\rho \iiint_V \nabla \cdot \boldsymbol{u} \, \mathrm{d}V = 0. \tag{5}$$

This expression is valid for all choices of V. We can thus say that the fluid velocity is divergence free

$$\nabla \cdot \boldsymbol{u} = 0. \tag{6}$$

We now examine the assumption that the flow is irrotational. The vorticity of the fluid is defined by

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u}. \tag{7}$$

If we assume that the vorticity is zero, then we can define a velocity potential $\phi = \phi(\boldsymbol{x}, t)$ such that

$$\boldsymbol{u} = \nabla \phi. \tag{8}$$

Recall from vector calculus that the curl of a gradient is identically zero, thus, if the vorticity of the flow is non-zero, then the velocity potential does not exist. Combining the definition for the velocity potential with the conservation (divergence) equation shows that ϕ is a solution of Laplace's equation

$$\nabla^2\phi = 0. \tag{9}$$

We have not yet discussed the validity of the irrotational assumption. To investigate this, we first state the Navier-Stokes equations,

$$\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \nabla p + \rho g \boldsymbol{e}_z = \mu\nabla^2\boldsymbol{u}. \tag{10}$$

If we neglect viscosity by setting $\mu = 0$, we regain Euler's equation,

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \frac{\nabla p}{\rho} + g\boldsymbol{e}_z = 0. \tag{11}$$

If we then take the curl of Euler's equation, we find an expression for the time evolution of vorticity,

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = \frac{\partial\boldsymbol{\omega}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{\omega} = \boldsymbol{\omega}\cdot\nabla\boldsymbol{u} \tag{12}$$

showing that if the vorticity is zero at $t = 0$, it remains so for all subsequent time.

We now examine Bernoulli's Law. We first rewrite the material derivative as

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \frac{\partial\boldsymbol{u}}{\partial t} + \frac{1}{2}\nabla|\boldsymbol{u}|^2 - \boldsymbol{u}\times\boldsymbol{\omega}\,, \tag{13}$$

where we have made use of equation (2) and of the vector identity $\boldsymbol{u}\cdot\nabla\boldsymbol{u} = \frac{1}{2}\nabla|\boldsymbol{u}|^2 - \boldsymbol{u}\times\nabla\times\boldsymbol{u}$. Since we have assumed that the initial vorticity is zero and we have shown that it shall remain so, the third term on the right is identically zero. As a result, Euler's equation becomes,

$$\nabla\left\{\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + \frac{p}{\rho} + gz\right\} = 0. \tag{14}$$

Integrating this expression gives

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + \frac{p}{\rho} + gz = F(t)\,, \tag{15}$$

where $F(t)$ is an additive constant. Note that since the velocity potential is defined up to an arbitrary additive function of time, we may absorb $F(t)$ into $\frac{\partial\phi}{\partial t}$. As such, we may set $F(t) = 0$ in (15) without loss of generality, and thus recover the well-known Bernoulli's Law.

We assume that the bottom boundary is impermeable, and thus, enforce a "no-normal flow" boundary condition,

$$\boldsymbol{u}\cdot\nabla\left\{z + h(x,y)\right\} = 0 \quad \text{at } z = -h(x,y)\,, \tag{16}$$

where $\nabla\{z+h(x,y)\}$ is the normal vector to the bottom surface. Using the definition of the velocity potential, equation (8), we obtain,

$$\frac{\partial\phi}{\partial z}\frac{\partial z}{\partial z}+\frac{\partial\phi}{\partial x}\frac{\partial h}{\partial x}+\frac{\partial\phi}{\partial y}\frac{\partial h}{\partial y}=0$$

$$\frac{\partial\phi}{\partial z}+\nabla\phi\cdot\nabla h=0\,. \tag{17}$$

On the surface, $z=\eta(x,y,t)$, we require the continuity of the pressure field p. Just above the surface, there are two contribution to pressure: a pressure due to the weight of the atmosphere and a pressure given by the surface tension, which conceptually acts like an elastic membrane stretched over the surface of the water. The surface tension is evaluated from the Young-Laplace equation. Hence,

$$p=p_{\text{air}}-\sigma\nabla\cdot\hat{\mathbf{n}}\quad\text{at } z=\eta(x,y,t)\,, \tag{18}$$

where σ is a constant, with units Nm^{-1}, and $\hat{\mathbf{n}}$ is the surface normal unit vector. From here on, we assume that $p_{\text{air}}=0$, again without loss of generality. Note that we are ignoring the effects of wind. Vector calculus tells us that

$$\hat{\mathbf{n}}=\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\,. \tag{19}$$

Using (18) in Bernoulli's Law (15) gives us the dynamic boundary condition on the free surface,

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}|\nabla\phi|^2+g\eta=\frac{\sigma}{\rho}\nabla\cdot\left\{\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right\}. \tag{20}$$

We obtain a kinematic boundary condition by assuming that a material element on the free surface stays on the free surface,

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(z(t)-\eta(x,y,t)\right)=0. \tag{21}$$

Noting that $\frac{\mathrm{D}z}{\mathrm{D}t}=w=\partial_z\phi$ and $\frac{\mathrm{D}\eta}{\mathrm{D}t}=\partial_t\eta+\boldsymbol{u}\cdot\nabla\eta$, we obtain

$$\frac{\partial\eta}{\partial t}+\nabla\phi\cdot\nabla\eta=\frac{\partial\phi}{\partial z}\,, \tag{22}$$

when evaluated at $z=\eta$.

To summarize, the governing equations are

$$\begin{aligned}
\frac{\partial\phi}{\partial t}+\frac{1}{2}|\nabla\phi|^2+g\eta&=\frac{\sigma}{\rho}\nabla\cdot\left\{\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right\} && \text{on}\quad z=\eta(x,y,t)\\
\frac{\partial\eta}{\partial t}+\nabla\phi\cdot\nabla\eta&=\frac{\partial\phi}{\partial z} && \text{on}\quad z=\eta(x,y,t)\\
\nabla^2\phi&=0 && -h(x,y)<z<\eta(x,y,t)\\
\frac{\partial\phi}{\partial z}+\nabla\phi\cdot\nabla h&=0 && \text{on}\quad z=-h(x,y).
\end{aligned}$$

The first equation relates the evolution of the velocity potential for a material element on the surface to the restoring force of gravity and the surface tension. The second equation describes the kinematic evolution of the free surface. The third equation is the continuity equation, where we have assumed that the fluid is incompressible and irrotational. The last equation is a statement that we do not allow any flow across the impermeable bottom boundary.

References

[1] F. E. Snodgrass, G. W. Groves, K. F. Hasselmann, G. R. Miller, W. H. Munk, and W. H. Powers, *Propagation of ocean swell across the pacific*, Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences., 259 (1966), pp. 431–497.

[2] G. G. Stokes, *On the theory of oscillatory waves*, Transactions of the Cambridge Philosophical Society, 8 (1847), pp. 441–473.