Lecture 10: Whitham Modulation Theory

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1 Introduction

The Whitham modulation theory provides an asymptotic method for studying *slowly varying periodic waves*, and is essentially a nonlinear WKB theory. Equations are derived which describe the slow evolution of the governing parameters for these nonlinear periodic waves (such as the amplitude, wavelength, frequency, etc.), and are called the *modulation (or Whitham) equations*. The Whitham equations have a remarkably rich mathematical structure, and are at the same time a powerful analytic tool for the description of nonlinear waves in a wide variety of physical contexts. One of the most important aspects of the Whitham theory is the analytic description of the formation and evolution of dispersive shock waves, or undular bores. These are coherent nonlinear wave-structures which resolve a wave-breaking singularity when it is dominated by dispersion rather than by dissipation. There are also a number of important connections between the Whitham theory, the inverse scattering transform (IST), and the general theory of integrable hydrodynamic systems.

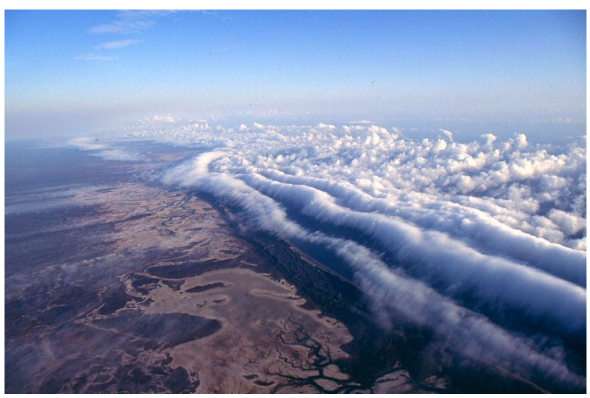

Figure 1: Satellite view of a Morning Glory wave

2 KdV Cnoidal waves

Let us consider the Korteweg de Vries (KdV) equation as an example, namely,

$$u_t + 6uu_x + u_{xxx} = 0\,. \tag{1}$$

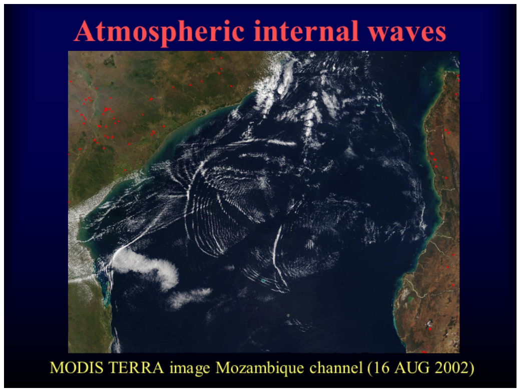

Figure 2: Atmospheric solitary waves near Mozambique

Figure 3: Undular Bore on the Dordogne river

As seen in many of the previous lectures, this equation has a one-phase periodic traveling wave solution, which is also called the *cnoidal wave* of the KdV equation (1):

$$u(x,t) = r_2 - r_1 - r_3 + 2(r_3 - r_2)\mathrm{cn}^2(\sqrt{r_3 - r_1}\,\theta; m)\,, \tag{2}$$

where $\mathrm{cn}(y;m)$ is the Jacobi elliptic cosine function defined as

$$\mathrm{cn}(y;m) = \cos\phi,$$

with ϕ satisfying

$$y = \int_0^\phi \frac{dt}{\sqrt{1 - m^2\sin^2 t}}.$$

This form of the wave solution is slightly different from the one introduced in previous lectures. Here the three parameters are $r_1 \le r_2 \le r_3$, and the phase variable θ and the modulus $m \in (0,1)$ are related to these parameters in the following way:

$$\theta = x - Vt\,, \quad V = -2(r_1 + r_2 + r_3)\,, \tag{3}$$

$$m = \frac{r_3 - r_2}{r_3 - r_1}\,, \quad L = \oint d\theta = \frac{2K(m)}{\sqrt{r_3 - r_1}}, \tag{4}$$

where $K(m) = \int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{1-m^2\sin^2 t}}$ is the complete elliptic integral of the first kind, and L is the "wavelength" along the x-axis. When $m \to 1$, $\mathrm{cn}(y;m) \to \mathrm{sech}(y)$ and (2) becomes a solitary wave; when $m \to 0$, $\mathrm{cn}(y;m) \to \cos(y)$ reducing to a sinusoidal wave (see [6]).

It is sometimes advantageous to use the parameters r_1, r_2 and r_3 instead of more physical parameters (such as the amplitude, speed, wavelength etc.), as they arise directly from the basic ordinary differential equation for the KdV traveling wave solution (2). Indeed, if we substitute (2) into the KdV equation (1), and integrate it once we get

$$u_\theta^2 = -2u^3 + Vu^2 + Cu + D\,, \tag{5}$$

where C, D are constants. This can be transformed to

$$w_\theta^2 = -4\mathbb{P}(w), \tag{6}$$

where

$$w = \frac{u}{2} - \frac{V}{4}\,, \quad \text{and} \quad \mathbb{P}(w) = \prod_{i=1}^{3}(w - r_i)\,.$$

This means that the cnoidal wave (2) is parameterized by the zeros r_1, r_2, r_3 of the cubic polynomial $\mathbb{P}(w)$. In a modulated periodic wave, the parameters r_1, r_2, r_3 are slowly varying functions of x and t, described by the Whitham modulation equations. These can be obtained either by a multi-scale asymptotic expansion, or more conveniently by averaging conservation laws, as described below.

3 Averaged conservation laws

Let us introduce an average over the period of the cnoidal wave (2) as

$$\langle \mathfrak{F} \rangle = \frac{1}{L}\oint \mathfrak{F} d\theta = \frac{1}{L}\int_{r_2}^{r_3} \frac{\mathfrak{F} d\mu}{\sqrt{-\mathbb{P}(\mu)}}\,, \tag{7}$$

for any function $\mathfrak{F}(x,t)$ such that (7) is finite. It can then be shown that

$$\langle u \rangle = 2(r_3 - r_1)\frac{E(m)}{K(m)} + r_1 - r_2 - r_3\,, \tag{8}$$

$$\langle u^2 \rangle = \frac{2}{3}[V(r_3 - r_1)\frac{E(m)}{K(m)} + 2Vr_1 + 2(r_1^2 - r_2 r_3)] + \frac{V^2}{4}\,, \tag{9}$$

where $E(m) = \int_0^{\pi/2} \sqrt{1 - m^2\sin^2\theta} d\theta$ is the complete elliptic integral of the second kind, and $K(m)$ was described earlier.

Next, recall that the KdV equation has a set of conservation laws (see Lecture 4):

$$(P_j)_t + (Q_j)_x = 0\,, \quad j = 1, 2, 3\,, \tag{10}$$

The averaging (7) procedure, when applied to these laws, yields equations of the kind:

$$\langle P_j \rangle_t + \langle Q_j \rangle_x = 0, \quad j = 1, 2, 3\,. \tag{11}$$

When combined with (8) and (9), the system (11) then describes the slow evolution of the parameters r_j for the cnoidal wave (2).

The first two conservation laws of the KdV equation (1) are

$$u_t + (3u^2 + u_{xx})_x = 0\,, \tag{12}$$

$$(u^2)_t + (4u^3 + 2uu_{xx} - u_x^2)_x = 0\,. \tag{13}$$

These respectively describe "mass" and "momentum" conservation. The next conservation law would be that of "energy", although here only two are needed, since after averaging the third equation can be replaced by the law for the conservation of waves:

$$k_t + \omega_x = 0\,, \quad \text{where} \quad k = \frac{2\pi}{L}\,, \quad \omega = kV\,. \tag{14}$$

This must be consistent with the modulation system (11), and can be introduced instead of any of three averaged conservation laws (11). In fact, any three independent conservation laws can be used, and will lead to equivalent modulation systems.

4 Whitham modulation equations

In general the Whitham modulation equations have the structure

$$\mathbf{b}_t + \mathbf{A}(\mathbf{b})\mathbf{b}_x = 0\,. \tag{15}$$

Here $\mathbf{b} = (r_1, r_2, r_3)^t$, and the coefficient matrix $\mathbf{A}(\mathbf{b}) = \mathbf{P}^{-1}\mathbf{Q}$ where the matrices $\mathbf{P}, \mathbf{Q}$ have the entries $P_{ij} = \langle P_i \rangle_{r_j}$ and $Q_{ij} = \langle Q_i \rangle_{r_j}$ for $i, j = 1, 2, 3$. The eigenvalues of the coefficient matrix $\mathbf{A}$ are called the characteristic velocities. If all the eigenvalues $v_j(\mathbf{b})$ of $\mathbf{A}(\mathbf{b})$ are real-valued, then the system is nonlinear hyperbolic and the underlying traveling wave is modulationally stable. Otherwise the traveling wave is modulationally unstable.

For this KdV case all the eigenvalues are real so the cnoidal wave is modulationally stable. It can be shown that

$$v_j = -2\sum r_j + \frac{2L}{\partial L/\partial r_j}\,, \quad j = 1, 2, 3. \tag{16}$$

The parameters r_j have been chosen because they are the Riemann invariants of the system (15) for the present case of the KdV equation. Thus this system has the diagonal form

$$(r_j)_t + v_j(r_j)_x = 0\,, \quad j = 1, 2, 3, \tag{17}$$

where we recall that $v_j(r_1, r_2, r_3)$ are the characteristic velocities (16)

$$\begin{aligned} v_1 &= -2\sum r_j + 4(r_3 - r_1)(1-m)K/E\,, \\ v_2 &= -2\sum r_j - 4(r_3 - r_2)(1-m)K/(E - (1-m)K)\,, \\ v_3 &= -2\sum r_j + 4(r_3 - r_2)K/(E - K)\,. \end{aligned}$$

5 Limiting cases of Whitham modulation equations

In the sinusoidal wave limit ($m \to 0$), a solution of (17) is $r_2 = r_3, m = 0$, $v_1 = -6r_1, v_2 = v_3 = 6r_1 - 12r_3$ so that the system collapses to

$$\begin{aligned} r_{1t} - 6r_1 r_{1x} = 0\,, \quad r_{3t} + (6r_1 - 12r_3)r_{3x} = 0\,, \\ \text{or} \quad d_t + 6dd_x = 0\,, \quad k_t + \omega_x = 0\,. \end{aligned} \tag{18}$$

Here $-r_1 = d$ is the mean level, $r_3 - r_1 = k^2/4$ is the wavenumber, and the dispersion relation is $\omega = 6dk - k^3$. An expansion for small m is needed to recover the wave action equation.

In the solitary wave limit ($m \to 1$), a solution of (17) is $r_1 = r_2, m = 1$, $v_1 = v_2 = -4r_1 - 2r_3, v_3 = -6r_3$, so that the system collapses to

$$\begin{aligned} r_{1t} + (-4r_1 - 2r_3)r_{1x} = 0\,, \quad r_{3t} - 6r_3 r_{3x} = 0\,. \\ \text{or} \quad d_t + 6dd_x = 0\,, \quad a_t + Va_x = 0\,. \end{aligned} \tag{19}$$

Now $-r_3 = d$ is the background level, $2(r_3 - r_1) = a$ is the solitary wave amplitude, and $-4r_1 - 2r_3 = 6d + 2a = V$ is its speed.

6 Shocks and undular bores

Let's now consider the similarity solution of the modulation system (17) which describes an undular bore developing from an initial discontinuity:

$$u(x,0) = \Delta \quad \text{for} \quad x < 0\,, \quad \text{and} \quad u(x,0) \quad \text{for} \quad x > 0\,, \tag{20}$$

where $\Delta > 0$ is a constant.

When the dispersive term in the KdV equation (1) is omitted, the KdV becomes the Hopf equation:

$$u_t + 6uu_x = 0\,. \tag{21}$$

This is readily solved by the method of characteristics and the solution is multivalued, $u = \Delta$ for $-\infty < x < 6\Delta t$ and $u = 0$ for $0 < x < \infty$. A shock is needed with speed 3Δ. See Fig. 6 for a solution of this equation with $\Delta = 1$, at $t = 3$.

When the dispersive term is retained, the shock is replaced by a modulated wave train (dispersive shock wave or *undular bore*). Behind the undular bore $u = \Delta$ (or in terms of the Riemann invariants $r_1 = -\Delta, r_2 = r_3$), while ahead of it $u = 0$ ($r_1 = r_2, r_3 = 0$). Because of the absence of a length scale in this problem, the corresponding solution of the Whitham modulation system must depend on the self-similar variable $\tau = x/t$ alone, which reduces the system (17) to

$$(v_j - \tau)\frac{dr_j}{d\tau} = 0\,, \quad i = 1, 2, 3\,. \tag{22}$$

Hence two Riemann invariants must be constant, namely $r_1 = -\Delta$, $r_3 = 0$ while r_2 varies in the range $-\Delta < r_2 < 0$, given by $v_2 = \tau$.

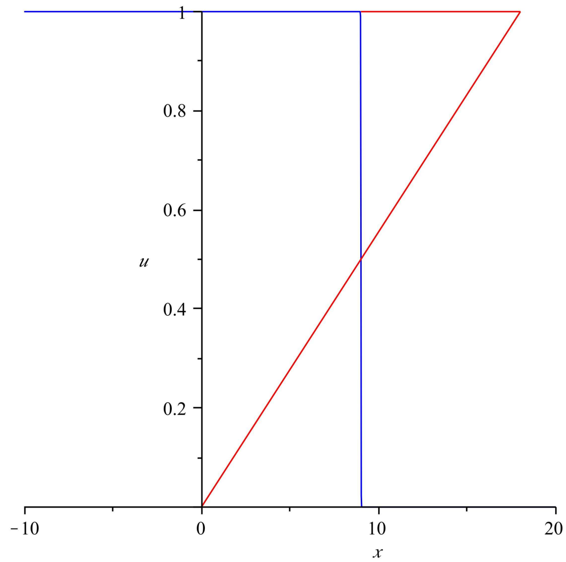

Figure 4: A shock solution to Hopf equation (21) equation, for $\Delta = 1$ (blue) and $\Delta = 3$ (red).

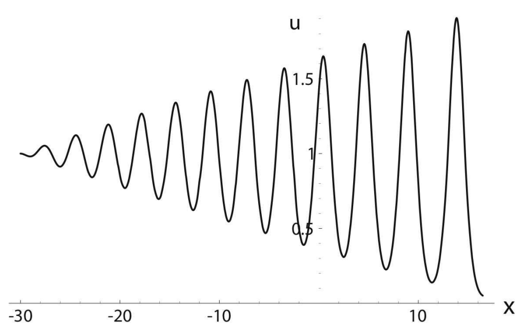

Figure 5: An undular bore

Finally, using the expressions (2, 3, 4) for the cnoidal wave, we get the solution for the undular bore expressed in terms of the modulus m,

$$\frac{x}{t\Delta} = 2(1+m) - \frac{4m(1-m)K(m)}{E(m)-(1-m)K(m)}\,, \tag{23}$$

$$\frac{u}{\Delta} = 1 - m + 2m\,\mathrm{cn}^2(\Delta^{1/2}(x-Vt);m)\,, \quad \frac{V}{\Delta} = -2(1+m)\,. \tag{24}$$

The leading and trailing edges of the undular bore are determined from (23) by putting $m = 1$ and $m = 0$: the undular bore is thus found to exist in the zone

$$-6 < \frac{x}{\Delta t} < 4\,. \tag{25}$$

Note that this solution is an unsteady undular bore which spreads out with time – a steady undular bore would require some friction. The leading solitary wave amplitude is 2Δ, exactly twice the height of the initial jump. Also the wavenumber is constant. For each wave in the wave train, $m \to 1$ as $t \to \infty$, so each wave tends to a solitary wave.

7 Rarefraction wave

When $\Delta < 0$, the initial discontinuity (20) creates a rarefraction wave

$$\begin{aligned} &u = 0\,, \quad \text{for} \quad x > 0\,, \\ &u = \frac{x}{6t}\,, \quad \text{for} \quad 6\Delta t < x < 0\,, \\ &u = \Delta\,, \quad \text{for} \quad x < 6\Delta t\,. \end{aligned} \tag{26}$$

This is a solution of the full KdV equation (1), but needs smoothing at the corners with a weak modulated periodic wave (see Fig. 6).

8 Further developments

1. The Whitham theory can be applied to any nonlinear wave equation which has a (known) periodic travelling wave solution. These include the NLS equations, Boussinesq equations, Su-Gardner equations.

2. For a broad class of integrable nonlinear wave equations, a simple universal method has been developed by Kamchatnov (2000), enabling the construction of periodic solutions and the Whitham modulation equations directly in terms of Riemann invariants.

3. The "undular bore" solution can be extended to the long-time evolution of a system from arbitrary localized initial conditions, described by Gurevich and Pitaevskii (1974) (and many subsequent works), and by Lax and Levermore (1983).

4. There have been applications in many physical areas, including surface and internal undular bores, collisionless shocks in rarefied plasmas (e. g. Earth's magnetosphere bow shock), nonlinear diffraction patterns in laser optics, and in Bose-Einstein condensates.

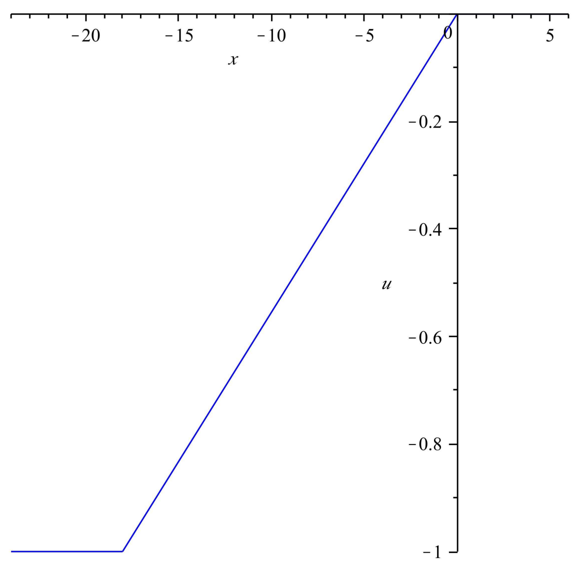

Figure 6: A rarefaction wave, with $\Delta = -1$, at instant $t = 3$.

References

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