Lecture 10: Convection Diffusion Problems

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1 2-D Convection-Diffusion

Consider a 2-D divergence-free, periodic, steady flow field $u(t, x)$ in a domain without any boundaries. Let $\tilde{\rho}(t, x)$ be the concentration of a passive scalar, say temperature. Then the non-dimensional governing equations for the non-dimensional variables $\tilde{\rho}$ and u are:

$$\tilde{\rho}_t + u \cdot \nabla \tilde{\rho} = \epsilon \Delta \tilde{\rho} \ , \tag{1}$$

$$\nabla \cdot u = 0 \ , \tag{2}$$

together with the initial condition $\tilde{\rho}(0, x) = \tilde{\rho}^0(x)$. Note that ϵ is dimensionless parameter since $\epsilon^{-1} \sim UL/\nu = Pe$, where Pe is the *Peclet number* and L is the size of the periodic cell. By integrating (1) over $\Re^2$ and using (2), we see that if $\int_{\Re^2} \tilde{\rho}^0(x)dx = 1$, then $\int_{\Re^2} \tilde{\rho}(t, x)dx = 1$. Also if $\tilde{\rho}^0(x) \geq 0$, then $\tilde{\rho}(t, x) \geq 0$. Since $\nabla \cdot u = 0$ and the flow is 2-D, it is possible to introduce a stream function $\psi(x)$:

$$u = (-\psi_y, \ \psi_x) \ . \tag{3}$$

If $\psi(x, y) = \sin x \sin y + \delta \cos x \cos y$, then we have a cellular flow if $\delta = 0$, and a shear flow if $\delta = 1$. Since $x(t)$ is the position of a diffusing particle, the evolution equation for $x(t)$ can be written as the following SDE:

$$dx(t) = u(x(t))dt + \sqrt{2\epsilon} \, dW(t). \tag{4}$$

If there is no diffusion (i.e. there is no $\sqrt{2\epsilon}\, dW(t)$ term in (4)), a particle starting on a particular streamline remains on the streamline. If we have diffusion, there is a possibility for a particle which starts in the region (a) to move to the region (b) (See Figure 1). In that case, $\tilde{\rho}$ can be interpreted as the probability density of $x(t)$.

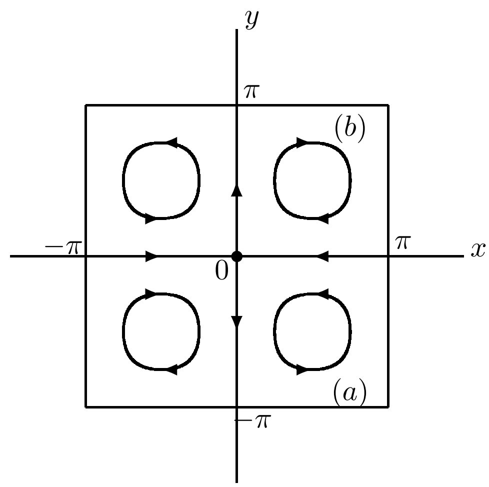

Figure 1: Rough sketch of the periodic cell

2 Effective diffusivities

Consider a diffusing particle, $\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\left[\{(x(t)-x(0))\cdot e\}^2\right]$ always exists for $u(x)$ which is periodic and satisfies $\nabla\cdot u=0$ and $\langle u(x)\rangle=0$, where $\langle\cdot\rangle$ represents the periodic cell average. We denote this limit as $\sigma^*_\epsilon(e)$, so called the *effective diffusivity*. It is a quadratic form of e.

We now take the large time, long distance limit of the PDE (1) by changing the variables $t\to n^2t,\ x\to nx$ and letting $n\to\infty$. (This process is called the *homogenization*.) $\rho_n(t,\,x)=\rho(n^2t,\,nx)$ converges to $\rho(t,\,x)$ in an appropriate sense as $n\to\infty$, where $\rho(t,\,x)$ is the solution of the homogenized equation

$$\rho_t=\nabla\cdot(\sigma^*_\epsilon\nabla\rho)\ , \tag{5}$$

with $\rho(0,\,x)=\rho^0(x)$. $\sigma^*_\epsilon(e)$ is a constant matrix, or more precisely

$$\sigma^*_\epsilon(e)=\langle(\epsilon I+\Psi)(\nabla\chi+e)\cdot e\rangle\ , \tag{6}$$

where I is the identity matrix, $\chi(x)$ is a periodic function in $\Re^2$, and

$$\Psi(x)=\begin{pmatrix}0 & -\psi(x,\,y)\\ \psi(x,\,y) & 0\end{pmatrix}. \tag{7}$$

It is found that $\sigma^*_\epsilon(e)$ satisfies the polarization relation

$$(\sigma^*_\epsilon)_{ij}=\frac{1}{4}\left[\sigma^*_\epsilon(e_i+e_j)-\sigma^*_\epsilon(e_i-e_j)\right]\ ,\quad i,\ j=1,\ 2, \tag{8}$$

where $e_1=(1,\ 0),\ e_2=(0,\ 1)$. Apart from the homogenized equation (5), the homogenization process also yields the cell problem, that is

$$\nabla\cdot[(\epsilon I+\Psi(x))(\nabla\chi+e)]=0\ . \tag{9}$$

σ_ϵ^* can be calculated by solving (9) for χ and plugging it into (6). The full derivation of (5), (6) and (9) will be shown in the next section. The physical interpretation of $\sigma_\epsilon^*(e)$ is the average flux in the direction e when there is a unit average gradient in the direction e.

3 Asymptotics for $\rho_n(t,x)$

Recall the passive scalar advection equation in the fast variables

$$\frac{\partial \rho_n}{\partial t} = \nabla \cdot ([I + \Psi_n(x)] \nabla \rho_n) \tag{10}$$

with initial condition

$$\rho_n(0,x) = \rho^0(x)$$

where I is the identity matrix, Ψ_n was defined previously, and we have set $\epsilon = 1$. First we must check that (10) solves (1)

$$\begin{aligned}
\frac{\partial \rho_n}{\partial t} &= \left(\frac{\partial \rho_n}{\partial x} + \psi_n \frac{\partial \rho_n}{\partial y}\right)_x + \left(\frac{\partial \rho_n}{\partial y} - \psi_n \frac{\partial \rho_n}{\partial x}\right)_y \\
&= (\rho_n)_{xx} + (\rho_n)_{yy} - (\psi_n)_x(\psi_n)_y + (\psi_n)_y(\psi_n)_x - \psi_n(\rho_n)_{xy} + \psi_n(\rho_n)_{yx} \\
&= \Delta \rho_n - u \cdot \nabla \rho_n.
\end{aligned}$$

Next we expand ρ_n in an asymptotic series

$$\rho_n(t,x) = \rho(t,x) + \frac{1}{n}\rho^{(1)}(t,x,nx) + \frac{1}{n^2}\rho^{(2)}(t,x,nx) + ...$$

It is clear that for this problem we have a clean separation of scales. The fast time scale does not appear because the coefficients are time homogeneous.

Let $nx = \xi$ so that $\nabla \to \nabla_x + n\nabla_\xi$. Plugging ρ_n into (10) we get

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\rho + \frac{1}{n}\rho^{(1)} + \frac{1}{n^2}\rho^{(2)} + ...\right) \quad = \\
&(\nabla_x + n\nabla_\xi) \cdot \left[(I + \Psi_n(\xi)) \cdot (\nabla_x + n\nabla_\xi)\left(\rho + \frac{1}{n}\rho^{(1)} + \frac{1}{n^2}\rho^{(2)} + ...\right)\right].
\end{aligned}$$

As is standard procedure, we equate the coefficients for powers of n. At $O(n^2)$:

$$\nabla_\xi \cdot [(I + \Psi_n(\xi))\, \nabla_\xi \rho] = 0. \tag{11}$$

Note (11) is automatically satisfied since ρ is not a function of ξ. At $O(n)$:

$$\nabla_\xi \cdot [(I + \Psi_n(\xi))\, \nabla_x \rho] + \nabla_x \cdot [(I + \Psi_n(\xi))\, \nabla_\xi \rho] + \nabla_\xi \cdot \left[(I + \Psi_n(\xi))\, \nabla_\xi \rho^{(1)}\right] = 0. \tag{12}$$

The second term in (12) is zero via (11). Upon rewriting (12) we get

$$\nabla_\xi \cdot \left[(I + \Psi_n(\xi)) \left(\nabla_\xi \rho^{(1)} + \nabla_x \rho \right) \right] = 0 \tag{13}$$

which resembles the cell problem (9). Equation (13) is a PDE for $\rho^{(1)}(\xi)$ (periodic in ξ). We can cast (13) into the cell problem by letting

$$\rho^{(1)}(t, x, \xi) = \sum_{j=1}^{d} \chi_{e_j}(\xi) \frac{\partial \rho}{\partial x_j}(t, x)$$

which separates the ξ dependence from the t, x dependence. The function $\chi_e(\xi)$ satisfies

$$\nabla_\xi \cdot \left[(I + \Psi_n(\xi)) \left(\nabla_\xi \chi_e(\xi) + e \right) \right] = 0. \tag{14}$$

At $O(1)$:

$$\begin{aligned} \frac{\partial \rho}{\partial t} = {} & \nabla_\xi \cdot \left[(I + \Psi_n(\xi)) \nabla_x \rho^{(2)} \right] + \nabla_\xi \cdot \left[(I + \Psi_n(\xi)) \nabla_x \rho^{(1)} \right] + \\ & \nabla_x \cdot \left[(I + \Psi_n(\xi)) \nabla_\xi \rho^{(1)} \right] + \nabla_x \cdot \left[(I + \Psi_n(\xi)) \nabla_x \rho \right] \end{aligned}$$

which is a PDE for $\rho^{(2)}(\xi)$ (periodic in ξ) with t, x as parameters. This can be re-written as

$$\nabla_\xi \cdot \left[(I + \Psi_n(\xi)) \nabla_\xi \rho^{(2)} \right] + S = 0 \tag{15}$$

where

$$S = \nabla_\xi \cdot \left[(I + \Psi_n(\xi)) \nabla_x \rho^{(1)} \right] + \nabla_x \cdot \left[(I + \Psi_n(\xi)) \nabla_\xi \rho^{(1)} \right] + \nabla_x \cdot \left[(I + \Psi_n(\xi)) \nabla_x \rho \right] - \frac{\partial \rho}{\partial t}.$$

Upon taking the cell average of (15), we obtain

$$\left\langle \nabla_x \cdot \left[(I + \Psi_n(\xi)) \nabla_\xi \rho^{(2)} \right] \right\rangle + \langle S \rangle = 0 \tag{16}$$

and since $\nabla_\xi \rho^{(2)}$ is a gradient of a periodic function, $\langle S \rangle = 0$ which yields

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= \left\langle \nabla_\xi \cdot \left[(I + \Psi_n(\xi)) \nabla_x \rho^{(1)} \right] \right\rangle + \left\langle \nabla_x \cdot \left[(I + \Psi_n(\xi)) \nabla_\xi \rho^{(1)} \right] \right\rangle + \\ & \quad \langle \nabla_x \cdot \left[(I + \Psi_n(\xi)) \nabla_x \rho \right] \rangle \\ &= \nabla_x \cdot \left[\left\langle (I + \Psi_n(\xi)) \left(\nabla_\xi \rho^{(1)} + \nabla_x \rho \right) \right\rangle \right] \end{aligned}$$

since $\nabla_x \rho^{(1)}$ is the gradient of a periodic function. In component form

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} &= \sum_{i,j} \frac{\partial}{\partial x_i} \left\langle A_{ij}(\xi) \left(\frac{\partial \rho^{(1)}}{\partial \xi_j} + \frac{\partial \rho}{\partial x_j} \right) \right\rangle \\
&= \sum_{i,j} \frac{\partial}{\partial x_i} \left\langle A_{ij}(\xi) \left(\frac{\partial}{\partial \xi_j} \left(\sum_k \chi_{e_k}(\xi) \frac{\partial \rho(t,x)}{\partial x_k} \right) + \frac{\partial \rho}{\partial x_j} \right) \right\rangle \\
&= \sum_{i,j} \sum_k \left\langle A_{ij} \left(\frac{\partial \chi_{e_k}}{\partial \xi_j} \frac{\partial^2 \rho}{\partial x_i \partial x_k} + \delta_{jk} \frac{\partial^2 \rho}{\partial x_i \partial x_k} \right) \right\rangle \\
&= \sum_{i,k} \left(\sum_j \left\langle A_{ij} \left(\frac{\partial \chi_{e_k}}{\partial \xi_j} + \delta_{jk} \right) \right\rangle \right) \frac{\partial^2 \rho}{\partial x_i \partial x_k}
\end{aligned}
$$

where $A_{ij} = I_{ij} + \Psi_{ij}(\xi)$. Thus, we obtain the homogenized equation

$$\frac{\partial \rho}{\partial t} = \sum_{i,k} \sigma^*_{ik} \frac{\partial^2 \rho}{\partial x_i \partial x_k} \tag{17}$$

with

$$\sigma^*_{ik} = \sum_j \left\langle A_{ij} \left(\frac{\partial \chi_{e_k}}{\partial \xi_j} + \delta_{jk} \right) \right\rangle$$

or

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (\sigma^*_\epsilon \nabla \rho)\,. \tag{18}$$

In summary, the key ideas for homogenization are:

1) Perform a multiscale expansion

$$
\begin{aligned}
t, x &\sim \text{macroscopic scales} \qquad \text{(slow)} \\
n^2 t, nx &\sim \text{microscopic scales} \qquad \text{(fast)}
\end{aligned}
$$

the resulting PDE will involve both fast and slow variables. In our case $\psi \to \psi(nx)$. In general $\psi \to \psi(n^2 t, nx, t, x)$.

2) Seek an expansion in which the principle term is slowly varying (t, x).

3) The coefficients of the slowly varying equation come from a cell problem. In this case the term of interest was ρ and we had to go to $O(1)$ to get the cell problem.

The effective diffusivity matrix, σ_ϵ^* is given by

$$\begin{aligned}\sigma_\epsilon^* &= \langle (\epsilon I + \Psi)(\nabla\chi)\cdot e\rangle \\ &= \langle (\epsilon I + \Psi)(\nabla\chi + e)\cdot(\nabla\chi + e)\rangle\end{aligned}$$

where we added $\nabla\chi$ to e because $\nabla\cdot[(\epsilon I+\Psi)(\nabla\chi+e)] = 0$ (9). Also since $(\nabla\chi+e)\cdot(\nabla\chi+e)$ is a quadratic form and Ψ is skew symmetric, we obtain

$$\begin{aligned}\sigma_\epsilon^*(e) &= \epsilon\left\langle|\nabla\chi+e|^2\right\rangle \\ &= \epsilon+\epsilon\left\langle|\nabla\chi|^2\right\rangle.\end{aligned}$$

From this it is clear that convection always enhances diffusion since $\sigma_\epsilon^*(e)\geq\epsilon$.

Finally we check convergence of the asymptotic expansion

1)

$$\max_{0\leq t\leq T, x\in\Re^2}|\rho_n(t,x)-\rho(x,t)| \leq \max_{0\leq t\leq T, x\in\Re^2}\left|\frac{1}{n}\rho^{(1)}+O\left(\frac{1}{n}\right)\right| \leq C_T\frac{1}{n}$$

provided ρ_0 decays rapidly at infinity and is smooth.

2)

$$\int_0^\infty\int_{\Re^2}(\nabla\rho_n-\nabla\rho)\,\theta(t,x)dxdt\to 0$$

where θ is a test function. This says that on average the gradient converges. Calculating $\nabla\rho_n$ we obtain

$$\nabla\rho_n = \nabla\rho+\nabla_\xi\rho^{(1)}(t,x,nx)+...$$

3)

$$\sup_{0\leq t\leq T}\int_{\Re^2}\left|\nabla\rho_n-\left(\nabla\rho+\nabla_\xi\rho^{(1)}\right)\right|^2dx\leq C_T\frac{1}{n}$$

thus $\rho^{(1)}$ closes the problem and allows us to determine $\nabla\rho_n$. Note that 3) implies 2).

Notes by Tiffany A. Shaw and Aya Tanabe.