Lecture 11: Internal solitary waves in the ocean

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1 Introduction

In Lecture 6, we sketched a derivation of the KdV equation applicable to internal waves, and discussed the extended KdV equation for critical cases where the quadratic nonlinear term is small.horizontal direction. In this lecture, we describe internal solitary waves in the ocean, where the bottom topography may vary from the deep ocean to the shallow seas of the coastal oceans, and the background hydrography can also vary along the path of the wave. Hence the asymptotic models must incorporate a variable background state. On the assumption that this is slowly varying relative to the waves, the outcome is a KdV-type equation, but with variable coefficients, namely the variable-coefficient extended KdV (veKdV) equation. When references to Lecture 6 are made, Eq. (n) in Lecture 6 will be referred to as $(6.n)$ here. The presentation of the properties and results for the veKdV equation closely follows that of the vKdV equation discussed in Lecture 9.

2 Variable-coefficient extended KdV equation

The ocean has variable depth, as well as variations in the basic state hydrology and background currents. As seen in the previous lecture, these effects can be formally incorporated into the theory by supposing that the basic state is a function of the slow spatial variable $\chi = \epsilon^3 x$. Thus here we assume a depth $h(\chi)$, a horizontal shear flow $u_0(\chi, z)$ with a corresponding vertical velocity field $\epsilon^3 w_0(\chi, z)$, a density field $\rho_0(\chi, z)$, a corresponding pressure field $p_0(\chi, z)$ and a free surface displacement $\eta_0(\chi)$. With this scaling, the slow background variability enters the asymptotic analysis at the same order as the weakly nonlinear and weakly dispersive effects, and an asymptotic analysis produces a variable coefficient extended KdV equation. The modal system is again defined by (6.12-13) (N is the buoyancy frequency)

$$\{\rho_0(c-u_0)^2\phi_z\}_z + \rho_0 N^2\phi = 0\,, \quad \text{for} \quad -h < z < 0\,, \tag{1}$$

$$\phi = 0 \quad \text{at} \quad z = -h\,, \quad (c-u_0)^2\phi_z = g\phi \quad \text{at} \quad z = 0\,, \tag{2}$$

but now the linear long wave speed $c = c(\chi)$ and the modal functions $\phi = \phi(\chi, z)$, where the χ-dependence is parametric.

With all small parameters removed, the governing equation is

$$A_\tau + \alpha A A_\xi + \alpha_1 A^2 A_\xi + \lambda A_{\xi\xi\xi} = 0\,. \tag{3}$$

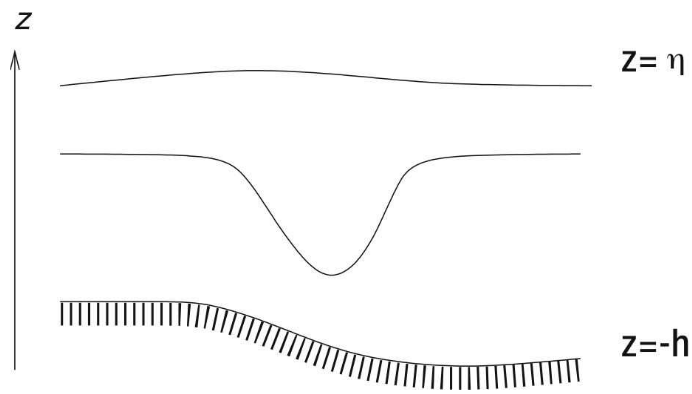


Figure 1: The basic coordinate system.

where we have moved to the new coordinate system, based on the travel time of the wave as introduced in Lecture 9, namely

$$\tau = \int^x \frac{\mathrm{d}x}{c}, \quad \xi = t - \tau\,, \tag{4}$$

and where the original amplitude A (defined by $\zeta = A(x - ct)\phi(z)$ where ζ is the vertical particle displacement) has been replaced by $\sqrt{Q}\,A$. Here Q is the linear magnification factor, defined so that QA^2 is the wave action flux. The linear long-wave speed c, and the coefficients $\alpha, \alpha_1, \lambda$ depend on x, and hence on the evolution variable τ. The coefficients $\alpha(\tau), \alpha_1(\tau), \lambda(\tau)$ and $Q(\tau)$ are given by

$$\alpha = \frac{\mu}{cQ^{1/2}}\,,\ \alpha_1 = \frac{\mu_1}{cQ}\,,\ \lambda = \frac{\delta}{c^3}\,,\ Q = c^2 I\,, \tag{5}$$

in terms of the coefficients in the extended KdV equation (6.42) (in different notations)

$$A_T + \mu A A_X + \mu_1 A^2 A_X + \delta A_{XXX} = 0\,, \tag{6}$$

and the definition (6.34)

$$I = 2\int_{-h}^{0} \rho_0 (c - u_0)\phi_z^2\,\mathrm{d}z\,. \tag{7}$$

Unlike the KdV or extended KdV equations, this variable coefficient equation (3) is not integrable in general, so we must seek a combination of asymptotic and numerical solutions.

2.1 Slowly-varying solitary waves

The veKdV equation (3) possesses two relevant conservation laws,

$$\int_{-\infty}^{\infty} A\,\mathrm{d}x = \text{constant}\,, \tag{8}$$

$$\int_{-\infty}^{\infty} A^2\,\mathrm{d}x = \text{constant}\,, \tag{9}$$

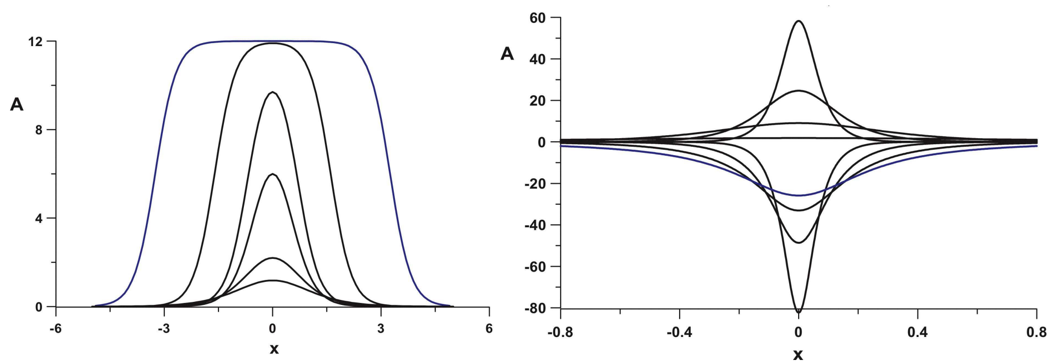


Figure 2: The family of solitary waves for (a) $\delta\mu_1 < 0$; (b) $\delta\mu_1 > 0$.

representing conservation of "mass" and "momentum" respectively (more strictly speaking, an approximate representation of the physical mass and wave action flux).

From Lecture 6, the solitary wave family of the eKdV equation (6) is given by

$$A = \frac{H}{1 + B\cosh K(X - VT)}\,, \tag{10}$$

$$\text{where } V = \frac{\mu H}{6} = \delta K^2\,,\ B^2 = 1 + \frac{6\delta\mu_1 K^2}{\mu^2}\,, \tag{11}$$

with a single parameter B. For $\delta\mu_1 < 0$ (Figure 2.1), $0 < B < 1$, and the family ranges from small-amplitude waves of KdV-type ("sech^2"-profile as $B \to 1$) to a limiting flat-topped wave of amplitude $-\mu/\mu_1$ ("table-top" wave as $B \to 0$). For $\delta\mu_1 > 0$ (Figure 2.1), there are two branches. One has $1 < B < \infty$ and ranges from small-amplitude KdV-type waves ($B \to 1$), to large waves with a "sech"-profile ($B \to \infty$). The other branch, $-\infty < B < 1$, has the opposite polarity and ranges from large waves with a "sech"-profile to a limiting algebraic wave of amplitude $-2\mu/\mu_1$. Waves with smaller amplitudes do not exist, and are replaced by breathers.

In the veKdV equation, we have a family of solitary waves as before, but its parameter $B(\tau)$ now varies slowly in a manner determined by conservation of momentum (9), which requires

$$\begin{aligned} G(B) &= \text{constant}\left|\frac{\alpha_1^3}{\lambda\alpha^2}\right|^{1/2}, \\ \text{where} \qquad G(B) &= |B^2 - 1|^{3/2}\int_{-\infty}^{\infty}\frac{\mathrm{d}u}{(1 + B\cosh u)^2}\,. \end{aligned} \tag{12}$$

The integral term in $G(B)$ can be explicitly evaluated, and so these expressions provide explicit formulas for the variation of $B(\tau)$ as the environmental parameters vary. However, since the conservation of momentum completely defines the slowly-varying solitary wave, total mass (8) is only conserved provided one adds a "trailing shelf" (linear long wave) whose amplitude A_{shelf} at the rear of the solitary wave is

$$VA_{\text{shelf}} = -\frac{\partial M_{\text{sol}}}{\partial\tau}\,, \qquad M_{\text{sol}} = \int_{-\infty}^{\infty} A_{\text{sol}}\,\mathrm{d}\xi\,, \tag{13}$$

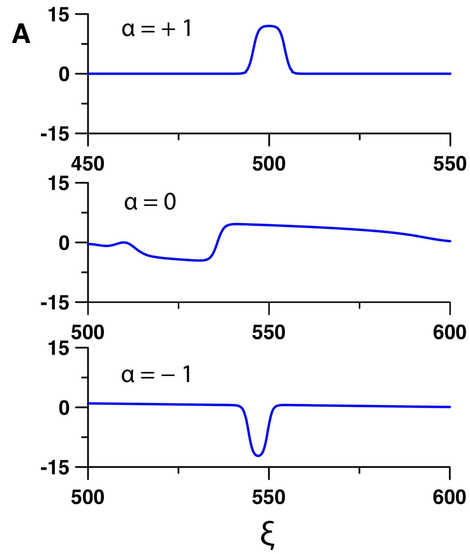


Figure 3: Critical point $\alpha = 0, \alpha_1 < 0$: eKdV case. Here $\lambda = 1, \alpha_1 = -0.083$ and α varies from 1 to -1 (that is, the variable coefficient eKdV equation, with a negative cubic nonlinear coefficient). This shows the conversion of an elevation "table-top" wave into a depression "table-top" wave, riding on a positive pedestal.

where $A_{\rm sol}$ is the solitary wave solution.

The adiabatic expressions (12, 13) show that the critical points where $\alpha = 0$ (or where $\alpha_1 = 0$) are sites where we may expect a dramatic change in the wave structure. There are two qualitatively different cases to consider.

2.2 Critical point $\alpha = 0, \alpha_1 < 0$

First, as α passes through zero, assume that $\alpha_1 < 0, 0 < B < 1$ at the critical point $\tau = 0$ where $\alpha = 0$. Then as $\alpha \to 0$, it follows from (12) that $B \to 0$ and the wave profile approaches the limiting "table-top" wave. But in this limit, $K \sim |\alpha|$, and so the amplitude approaches the limiting value $-\alpha/\alpha_1$. Thus the wave amplitude decreases to zero, the mass M_0 of the solitary wave grows as $|\alpha|^{-1}$ and the amplitude A_1 of the trailing shelf grows as $1/|\alpha|^4$. Essentially the trailing shelf passes through the critical point as a disturbance of the opposite polarity to that of the original solitary wave, which then being in an environment with the opposite sign of α, can generate a train of solitary waves of the opposite polarity, riding on a pedestal of the same polarity as the original wave (see Figure 3).

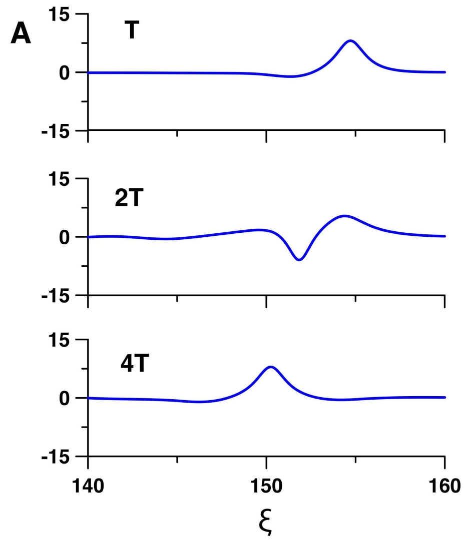


Figure 4: Critical point $\alpha = 0, \alpha_1 > 0$: eKdV case. Here $\lambda = 1, \alpha_1 = 0.3$ and α varies from 1 to -1 for $-T < \tau < T$ (that is, the variable coefficient eKdV equation, with a positive cubic nonlinear coefficient). This shows the adiabatic evolution of an elevation wave from $\tau = -T$ to $\tau = T$, where its amplitude is too small, and so the wave becomes a breather.

2.3 Critical point $\alpha = 0, \alpha_1 > 0$

Next, let us suppose that at the critical point where $\alpha = 0$, $\alpha_1 > 0$. In this case, $1 < |B| < \infty$ and there are the two sub-cases to consider, $B > 1$ or $B < -1$, when the the solitary wave has the same or opposite polarity to α. Then, as $\alpha \to 0, |B| \to \infty$ as $|B| \sim 1/|\alpha|$. It follows from (11) that then $K \sim 1$, $H \sim 1/|\alpha|, a \sim 1, M_0 \sim 1$. Therefore, the wave adopts the "sech"-profile, but has *finite* amplitude, and so can pass through the critical point $\alpha = 0$ without destruction. But the wave changes branches from $B > 1$ to $B < -1$ as $|B| \to \infty$, or *vice versa*. An interesting situation then arises when the wave belongs to the branch with $-\infty < B < -1$ and the amplitude is reducing. If the limiting amplitude of $-2\alpha/\alpha_1$ is reached, then there can be no further reduction in amplitude for a solitary wave, and instead a breather will form (Figure 4).

3 Wave propagation, deformation and disintegration

For real oceanic shelves, there can be wave paths along which the parameters in the veKdV equation may not vary sufficiently slowly, and they may also contain several critical points. As a result, an internal solitary wave loses its identity as a soliton within a finite lifetime. In this section, we describe direct numerical simulations performed using the veKdV equation

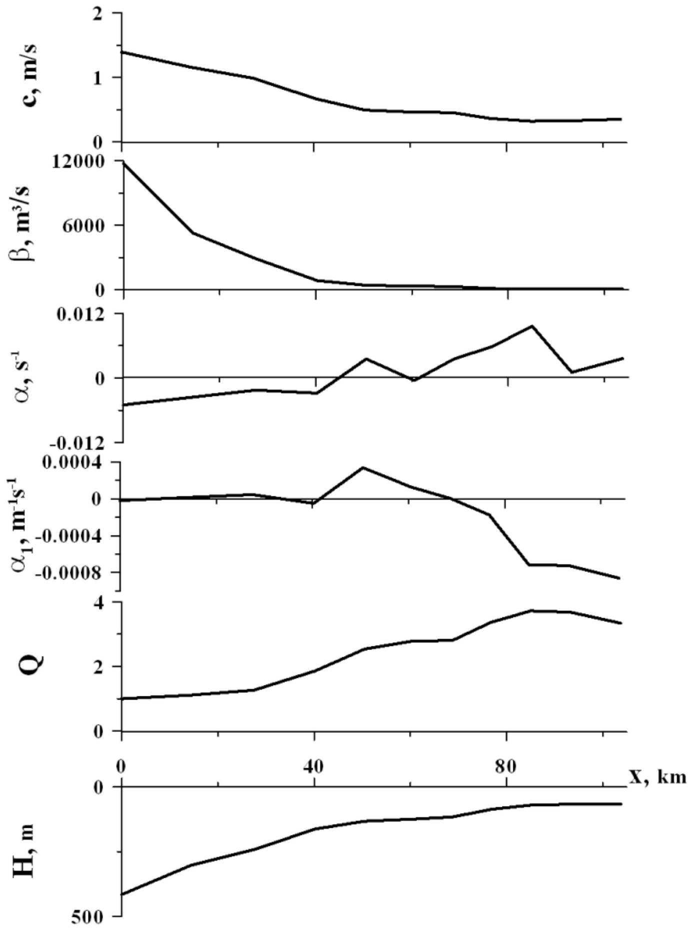


Figure 5: The coefficient of veKdV (3) for the passage of an initial solitary wave of depression across the North West Shelf of Australia.

(3) with the coefficients calculated for three oceanic shelves: the NWS of Australia, the MSE (west of Scotland), and the Arctic shelf (in the Laptev Sea). The initial condition is a typical solitary wave for each shelf. In all simulations, we only use the density stratification of the coastal zone, and ignore any background current. The notation for the quantities plotted in the figures is consistent with (3) with λ in (3) denoted as β in the figures.

3.1 North West shelf of Australia

The measured coefficients of the veKdV equation are presented in Figure 5. The hydrology can indeed be considered as slowly varying because the characteristic horizontal variation scale of the oceanic parameters (more than 10km) exceeds the characteristic soliton wavelength (about 1-2km). This shelf is distinguished by the property that the coefficient α of the quadratic nonlinear term has several sign changes.

A simulation for an initial wave amplitude of 15m is shown in Figure 6. The initial solitary wave has negative polarity, and transforms at a distance after 68 km into a nonlinear dispersive tail and a group of secondary solitons.

3.2 Malin shelf edge

The measured coefficients of the veKdV equation are presented in Figure 7. The coefficient of the quadratic nonlinear term is everywhere negative. There is only one critical point at a distance of about 8km associated with the sign change of the coefficient of the cubic nonlinear term.

A simulation for an initial wave amplitude of 21m is shown in Figure 8. The soliton-like shape is maintained for a distance of about 20km. Subsequently, the significant decrease of the dispersion parameter leads to the formation of a shock wave. For distances more than 25km, the borelike disturbance transforms into solitary waves.

3.3 Arctic shelf

The measured coefficients of the veKdV equation are presented in Figure 9. The nonlinear coefficients, as well as the dispersion parameter, vary significantly but quite slowly. The coefficient of the cubic nonlinear term is everywhere negative, and increases by three times. The critical point (a zero value of the coefficient of the quadratic nonlinear term) appears at the end of the wave path at a distance of 155km.

A simulation for an initial wave amplitude of 13m is shown in Figure 10. Before reaching the critical point, the solitary wave maintains its soliton-like shape for very long distances (140 km) as the background varies sufficiently slowly.

4 World map of eKdV coefficients

The previous simulations show the key role played by the coefficients of the veKdV equation. Thus in Figure 11, we display world maps of these coefficients (same notations as (6)). As expected, the linear phase speed and the linear dispersive coefficient scale respectively with $h^{1/2}$ and $h^{5/2}$. Hence, as is well known, the largest amplitude internal solitary waves will generally be found in the shallow seas of the coastal zones. However, the quadratic and cubic coefficients show considerable variability, with many sign changes, thus emphasizing again the importance of critical points.

References

[1] Grimshaw, R., *Internal solitary waves*, in *Environmental Stratified Flows*, (2001), ed. R. Grimshaw, Kluwer, Boston.

[2] Grimshaw, R., Pelinovsky, E., Talipova, T. and Kurkin, A. , *Simulation of the transformation of internal solitary waves on oceanic shelves*, J. Phys. Ocean., 34 (2004), pp. 2774-2779.

[3] Grimshaw, R., Pelinovsky, E. and Talipova, T. *Modeling internal solitary waves in the coastal ocean*, Surveys in Geophysics, 28 (2007), pp. 273-298.

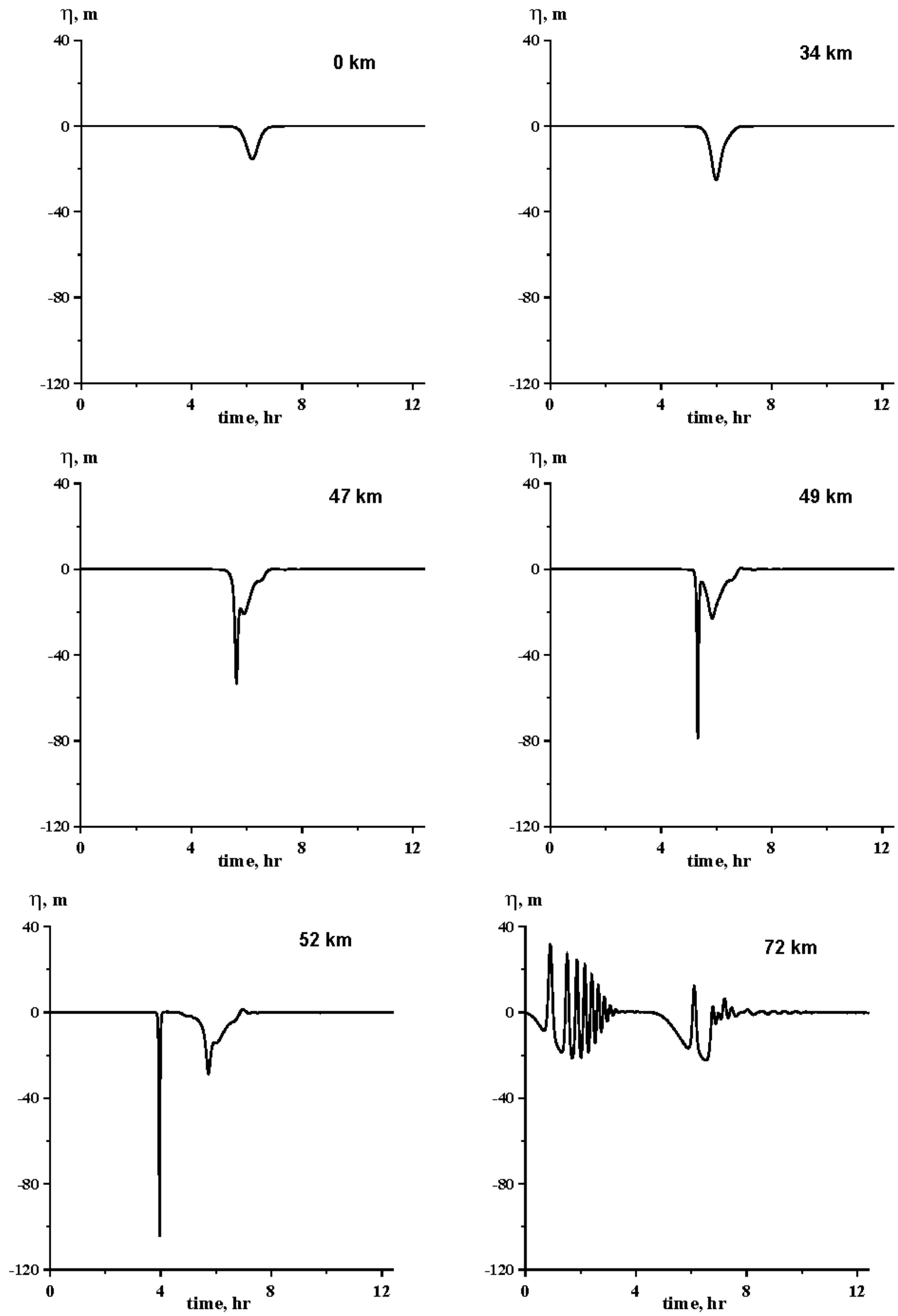


Figure 6: NWS, initial depression wave of 15m amplitude.

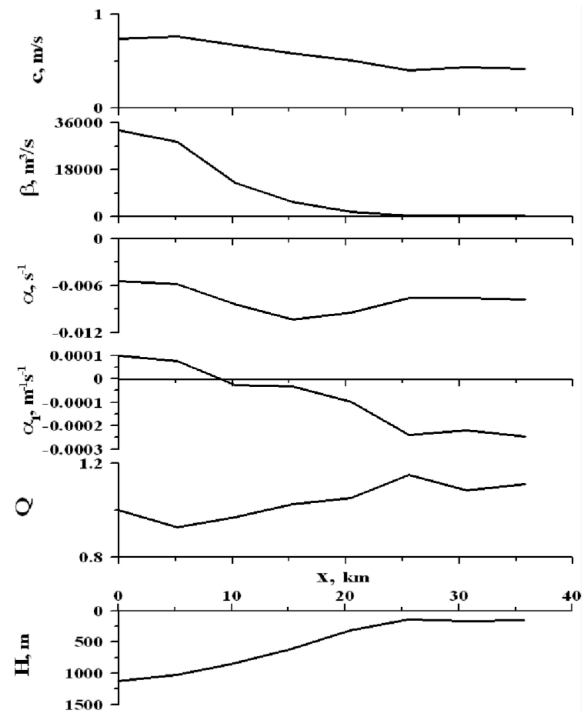


Figure 7: The coefficient of veKdV (3) for the passage of an initial solitary wave of depression across the Malin shelf off west coast of Scotland.

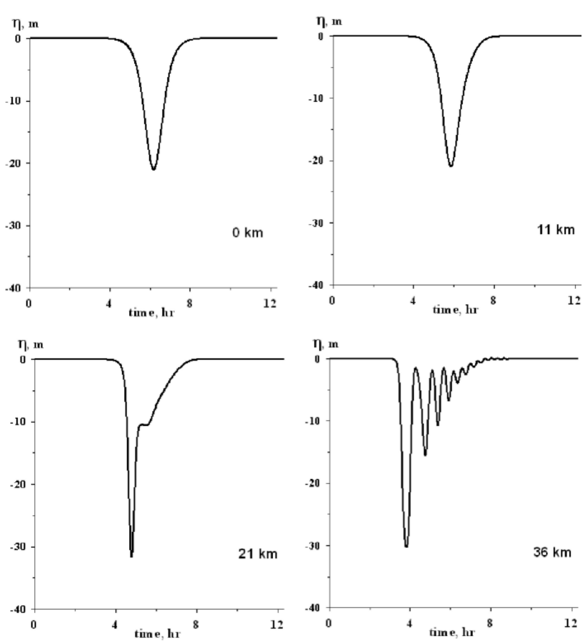


Figure 8: Malin Shelf, fission, initial amplitude of 21m.

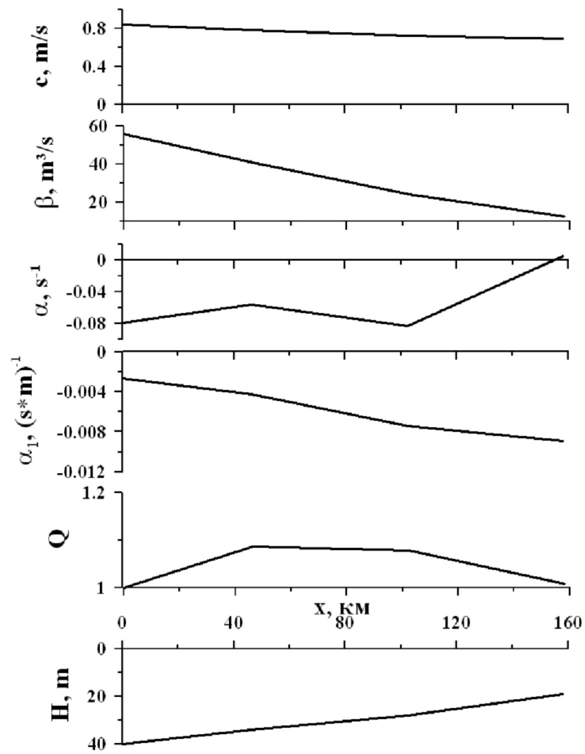


Figure 9: The coefficient of veKdV (3) for the passage of an initial solitary wave of depression across the Arctic shelf off north coast of Russia.

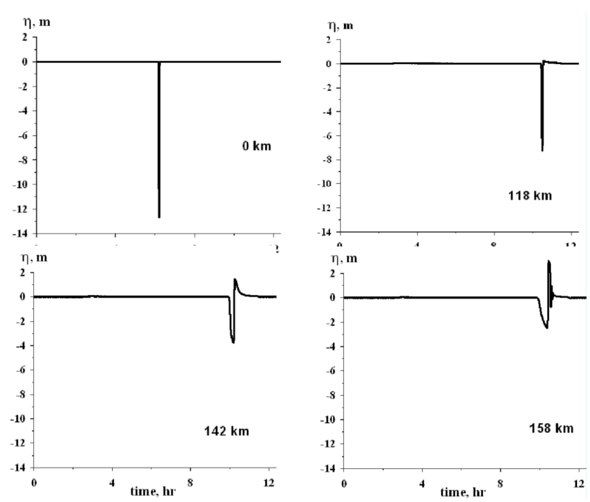
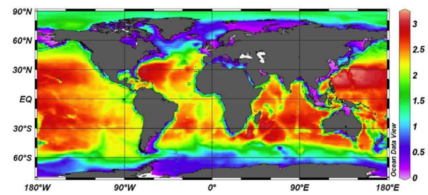
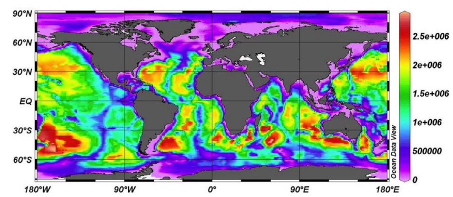


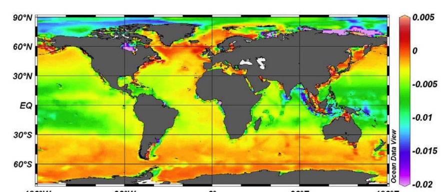
Figure 10: Arctic Shelf, adiabatic, initial amplitude of 13m.



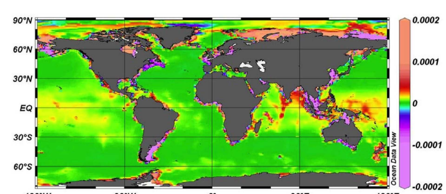
a) speed of propagation, c (m/s)



b) dispersion coefficient, δ (m^3/s)



c) coefficient of quadratic nonlinearity, μ (s^{-1})



d) coefficient of cubic nonlinearity, μ_1 ($m^{-1}s^{-1}$)

Figure 11: World map of eKdVcoefficients.